SUPERBOSONIZATION*

Hans-Jürgen Sommers

Fachbereich Physik, Universität Duisburg-Essen
47048 Duisburg, Germany

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We give a constructive proof for the superbosonization formula for invariant random matrix ensembles, which is the supersymmetry analog of the theory of Wishart matrices. Formulas are given for unitary, orthogonal and symplectic symmetry, but worked out explicitly only for the orthogonal case. The method promises to become a powerful tool for investigating the universality of spectral correlation functions for a broad class of random matrix ensembles of non-Gaussian type.

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1. Introduction

I will report on superbosonization, a technique in random matrix theory (RMT), the appellation given by Efetov *et al.* [1] which we want to make a rigorous tool. A rather mathematical paper from SFB/TR12 is in preparation [2]. The method is the supersymmetry analog of the theory of Wishart matrices. Other people have done related work connected with universality properties of invariant ensembles [1, 3–5]. But nowhere so far the precise integration domain for the supermatrices under consideration have been specified. Here I want to present a constructive proof of the superbosonization formula on a level, which can be understood by a physicist.

One may generate correlations of random Green functions from some Gaussian integrals over a set of N-dimensional vectors building a $N \times p$ dimensional matrix Φ, which after averaging are no longer Gaussian, but the integrand is a function of the matrix $Q = (\Phi^\dagger \Phi)/N$ of invariants. The transition to this new set of variables is called bosonization and it makes *e.g.* large-N evaluations simpler using saddle-point techniques. This kind of order parameter was previously introduced in the theory of spin glasses

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and the theory of Anderson localization. These theories involve the so called replica trick, where at the end of calculation, after averaging over the ensemble, one has to take the limit of number of replicas to zero, a procedure that is very difficult to make mathematically rigorous.

A more rigorous method is to introduce supervectors Ψ (if possible) with commuting and anticommuting components. Then no replica limit is needed, in some sense the dimension of Grassmannian (anticommuting) vectors can be considered as negative. The change of variables from a set of supervectors Ψ to a supermatrix $Q = (\Psi^\dagger \Psi)/N$ is called superbosonization. It is useful for invariant ensembles and non-Gaussian ensembles and avoids the sometimes cumbersome Hubbard–Stratonovic transformation.

2. Superbosonization formula

Let us assume, that we want to integrate a function F of invariants (with respect to a symmetry group)

$$I_{pq} = \int D\zeta\, D\Phi\, F \begin{pmatrix} \zeta^\dagger\zeta & \zeta^\dagger\Phi \\ \Phi^\dagger\zeta & \Phi^\dagger\Phi \end{pmatrix} . \tag{1}$$

Here ζ are anticommuting variables building a rectangular $N \times p$ matrix (p even), Φ are commuting variables building a $N \times q$ matrix (and we need later on $N \geqslant q$). Originally one starts with an invariant function of ζ and Φ and ends up with a function of invariants, which is not unique for Grassmannians, nevertheless the result of integration is unique. $D\zeta$ is the Berezin integration form and $D\Phi$ is the flat measure of matrix elements.

We want to transform to integrations over supermatrices $Q = \left(\begin{smallmatrix} A & \sigma^\dagger \\ \sigma & B \end{smallmatrix}\right)$ with bosonic (commuting) entries A, B and fermionic (anticommuting) entries σ, $\sigma^\dagger$. In this way we reduce considerably the number of bosonic and fermionic integration variables especially in the case where N goes very large. It is not surprising that B runs over positive Hermitian matrices and σ and $\sigma^\dagger$ run over Grassmannians. But the amazing thing is that the entry in the Fermi–Fermi sector, which is originally nilpotent, is replaced by a matrix A which runs over a manifold of unitary matrices. There is a compact way of writing the integral (1) as super-integral containing some power M of the so called superdeterminant $S\det Q$ of the supermatrix Q:

$$\begin{aligned} I_{pq} &= \mathcal{N}\int DA \int DB \int D(\sigma,\sigma^\dagger) \left[\frac{\det B}{\det(A - \sigma^\dagger \frac{1}{B}\sigma)}\right]^M F\begin{pmatrix} A & \sigma^\dagger \\ \sigma & B \end{pmatrix} \\ &= \mathcal{N}\int DQ\, (S\det Q)^M\, F(Q)\,. \end{aligned} \tag{2}$$

The appearing measures are the flat measures and $\mathcal{N}$ is a normalization constant. This is the superbosonization formula and I want to specify it for different symmetry groups.

For the real orthogonal group O_N the dagger † just means the transposed $^{\mathrm{T}}$, A is skew-symmetric ($A=-A^{\mathrm{T}}$) and unitary, and B is real symmetric ($B=B^{\mathrm{T}}$) and positive. The power M is given by $M=(N+p-q-1)/2$. For the unitary symplectic group $U\mathrm{Sp}_N$ (N even) the dagger † means the dual $^{\mathrm{D}}$, with *e.g.* $A^{\mathrm{D}}=ZA^{\mathrm{T}}Z^{\mathrm{T}}$ where Z is the symplectic unit which is quasi-diagonal with quaternion elements $\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$ on the diagonal. A is anti-self dual ($A=-A^{\mathrm{D}}$) and unitary and B is Hermitian, self dual ($B=B^{\mathrm{D}}$) and positive. The power M is given by $M=(N+p-q+1)/2$. In the case of unitary group U_N the dagger means just the usual adjoint = transposed and complex conjugate. The Grassmannian $\sigma^\dagger$ can be any independent set $\widetilde{\sigma}$ of Grassmannians. Here $M=N+p-q$.

Let us say a bit on the manifolds. $A=-A^{\mathrm{T}}$ implies that $(ZA)^{\mathrm{D}}=(ZA)$, *i.e.* for orthogonal symmetry ZA belongs to the set of self dual unitary matrices, which is called in physics the circular symplectic ensemble CSE (not only meant as an invariant measure but also as a manifold). Similarly $A=-A^{\mathrm{D}}$ implies that ZA is symmetric. Thus for symplectic symmetry ZA belongs to the circular orthogonal ensemble COE. Finally for unitary symmetry ZA belongs to the circular unitary ensemble CUE.

On the other hand the matrices B are positive Hermitian and symmetric or self dual in the orthogonal or symplectic case. All those manifolds are Riemannian symmetric spaces, which were under special consideration in our SFB/TR12, related to the so called ten-fold way of universalities in RMT for fermionic systems [6]. Moreover, the supermatrices $\widetilde{Q}=\left(\begin{smallmatrix}Z&0\\0&1\end{smallmatrix}\right)Q$ belong to Riemannian symmetric superspaces introduced by Zirnbauer [7].

The measures which we have indicated so far are the flat ones and they are related to the invariant measures on the corresponding manifolds by some power of determinant or superdeterminant. *E.g.* the invariant measures on the supermanifolds are (up to normalization)

$$d\mu(Q)=DA\ DB\ D(\sigma,\sigma^\dagger)(S\det Q)^R\,, \tag{3}$$

with $R=p-q$ for $\beta=2$, $R=(p-q-1)/2$ for $\beta=1$, $R=(p-q+1)/2$ for $\beta=4$. The invariant measures on the manifolds of A and B can be read off from these expressions by putting $q=0$ or $p=0$.

Let us say a little bit about the normalization constant

$$\mathcal{N}=C(N,q)\cdot D(N-q,p)\,. \tag{4}$$

It factorizes in two constants coming from the Bose–Bose sector $C(N,q)$ and from the Fermi–Fermi sector $D(N-q,p)$. They are essentially ratios of volumes of the corresponding symmetry groups. They are separately only defined for $N \geqslant q$ because $N-q$ is the smallest dimension of symmetry group that appears. And the proof shows that this is actually needed. However, the product $\mathcal{N} = C \cdot D$ is even defined for $N+p-q \geqslant 0$, it is not clear if this has physical relevance.

3. Idea of proof

Let me now give an idea of the proof of the superbosonization formula. We start with $p=0$, no Grassmannians, this is just the case of Wishart matrices. In this case one has to integrate some function $H(\Phi^\dagger\Phi)$ over bosonic vectors Φ:

$$\begin{aligned}\int D\Phi H(\Phi^\dagger\Phi) &= \int DBD\Phi\, \delta(B-\Phi^\dagger\Phi)H(B) \\ &= C(N,q)\int DB\, H(B)(\det B)^M\,. \end{aligned}\tag{5}$$

The last equation can simply be found by rescaling $\Phi = \Phi'\sqrt{B}$ with $B>0$ for $N \geqslant q$. Since $C(N,q)$ is independent of the function $H(B)$ we may choose $H(B) = \exp(-\mathrm{Tr}\ B)$ and find for example with $\beta = 1$ using Selberg's integral [8] and diagonalization of B with orthogonal matrices:

$$\begin{aligned}\frac{\pi^{Nq/2}}{C(N,q)} &= \int\limits_{B>0} DB\, \mathrm{e}^{-\mathrm{Tr}B}\det B^{(N-q-1)/2} \\ &= \frac{1}{2^{q/2}}\prod_{j=0}^{q-1}\pi^{j/2}\Gamma((N-q+1+j)/2)\,. \end{aligned}\tag{6}$$

This constant is related to the ratio of volumes of orthogonal groups $V(O_{N-q})/V(O_N)$. Since the manifold $B>0$ is noncompact there appear noncompact integrals over eigenvalues of B: $b_i > 0$.

More interesting is the case $q=0$, only Grassmannians, the case of Grassmann Wishart matrices ζ: $(N-q)\times p$, p even. In this case the naive introduction of a δ-function for $A = \zeta^{\mathrm{T}}\zeta$ leads to inconsistencies and we have to be more careful. We have to integrate a function $G(\zeta^{\mathrm{T}}\zeta)$ which we write with the help of a shift operator

$$\begin{aligned}\int D\zeta\, G(\zeta^{\mathrm{T}}\zeta) &= \int D\zeta\, \exp\left(\mathrm{Tr}\ \zeta^{\mathrm{T}}\zeta\frac{\delta}{\delta A}\right)G(A)\bigg|_{A=0} \\ &= \sqrt{\det\left(2\frac{\delta}{\delta A}\right)^{N-q}}\, G(A)\Bigg|_{A=0}\,. \end{aligned}\tag{7}$$

Here A is an antisymmetric matrix and the ζ-integration yields the $(N-q)$-th power of the Pfaffian $\sqrt{\det\left(2\frac{\delta}{\delta A}\right)}$.

The superbosonization formula to be proven says:

$$\int D\zeta\, G(\zeta^{\mathrm{T}}\zeta) = D(N-q,p)\int \frac{DA}{(\det A)^{\frac{N-q+p-1}{2}}}\, G(A)\,, \tag{8}$$

where A runs over the unitary antisymmetric matrices. $G(A)$ may be any polynomial or analytic function. Since A is unitary there is no problem at $A=0$ or $\det A = 0$ for this integral. Obviously it is enough to prove this formula for any exponential function $G(A) = \exp(\mathrm{Tr}\ AB/2)$ where B is again antisymmetric. In this case the Pfaffian action (7) can simply be calculated and the last equation can again be proven by rescaling choosing ZB a self dual unitary matrix. To calculate the constant $D(N-q,p)$ one may choose $B=Z$. Then one finds diagonalizing ZA with symplectic matrices

$$\begin{aligned}\frac{(-1)^{(N-q)p/2}}{D(N-q,p)} &= \frac{V(U\mathrm{Sp}_p)}{V(U\mathrm{Sp}_2)^{p/2}\left(\frac{p}{2}\right)!}\cdot\oint\prod_{i<j}(a_i-a_j)^4\prod_{k=1}^{p/2}da_k\, a_k^{-(N-q+p-1)}\mathrm{e}^{a_k}\\ &= \left(\frac{2\pi i}{2^{N-q}}\right)^{p/2}\prod_{j=0}^{p-1}\frac{\pi^{j/2}}{\Gamma((N-q+1+j)/2)}\,.\end{aligned} \tag{9}$$

This is again related to the ratio of volumes of real orthogonal groups $V(O_{N-q+p})/V(O_{N-q})$. Since the manifold $ZA \in CSE$ is compact there appear integrals over eigenvalues a_i of ZA along the unit circle. Interestingly, these compact integrals in the complex plane are related to the non-compact integrals with $b_i > 0$ along the real axis.

Finally let me consider the full supermatrix Q. I only report here on the orthogonal case. In the integral

$$I_{pq} = \int D\zeta\int D\Phi\ F\begin{pmatrix}\zeta^{\mathrm{T}}\zeta & \zeta^{\mathrm{T}}\Phi\\ \Phi^{\mathrm{T}}\zeta & \Phi^{\mathrm{T}}\Phi\end{pmatrix} \tag{10}$$

we have to go from $N\times(p,q)$ variables (ζ,Φ) to the $(p,q)\times(p,q)$ supermatrix Q. We choose an orthogonal matrix O which rotates Φ to a quadratic $q\times q$ matrix $\sqrt{B}$

$$\Phi = O\Phi_0\,,\quad \Phi_0 = \begin{pmatrix}0\\ \sqrt{B}\end{pmatrix}. \tag{11}$$

Then the Grassmannians are rotated correspondingly

$$O^{\mathrm{T}}\zeta = \widetilde{\zeta} = \begin{pmatrix}\zeta_1\\ \zeta_0\end{pmatrix}, \tag{12}$$

which has a $q \times p$ component ζ_0 which transforms to σ

$$\sigma = \Phi^{\mathrm{T}}\zeta = \Phi_0^{\mathrm{T}}\widetilde{\zeta} = \sqrt{B}\zeta_0 \tag{13}$$

and a remaining $(N-q)\times p$ component ζ_1 which can be integrated out using (8) and

$$\zeta^{\mathrm{T}}\zeta = \widetilde{\zeta}^{\mathrm{T}}\widetilde{\zeta} = \zeta_1^{\mathrm{T}}\zeta_1 + \zeta_0^{\mathrm{T}}\zeta_0 \,. \tag{14}$$

Similarly the integration over B follows from (5). The result is almost what we want

$$I_{pq} = \mathcal{N}\int DA \int DB \int D\sigma \left[\frac{\det B}{\det A}\right]^{\frac{(N-q+p-1)}{2}} F\begin{pmatrix} A+\sigma^{\mathrm{T}}\frac{1}{B}\sigma & \sigma^{\mathrm{T}} \\ \sigma & B \end{pmatrix} . \tag{15}$$

Now shift $A \to A - \sigma^{\mathrm{T}}\frac{1}{B}\sigma$, which is possible since the manifold CSE has no boundary (compact symmetric space) and we end up with

$$I_{pq} = \mathcal{N}\int DQ\ (S\det Q)^{\frac{(N-q+p-1)}{2}}\ F(Q) \tag{16}$$

with $S\det Q = \det B/\det(A - \sigma^{\mathrm{T}}\frac{1}{B}\sigma)$. This completes the proof for the orthogonal case. Similar considerations lead to the corresponding results for the symplectic and unitary cases.

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