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NUCLEAR TETRAHEDRAL SYMMETRY AND COLLECTIVE ROTATION*

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There is an increasing number of theoretical predictions suggesting that many nuclear states should be characterised by spatial tetrahedral and/or octahedral symmetries. One of the most crucial points in this domain of research is how to demonstrate the existence of this new exotic quantum mechanism through experiment. We discuss in some detail how the rotational properties of tetrahedral nuclei can be used to pin down the presence of the tetrahedral symmetry in sub-atomic universe.

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1. Introduction: Nuclear point-group symmetries

The studies of nuclear geometrical symmetries have been focused over many years on the spherical-prolate-oblate shape coexistence, on triaxial and on mass asymmetric deformations such as pear-shape superposed with the axial-quadrupole-deformations. The point-group symmetries associated with those studies belong to the poorest in symmetry elements: D_2-group for the tri-axial nuclear forms and C_∞-group for the axially symmetric ones. A part of this evolution was the discovery of the nuclear super-deformation. Yet from the symmetry point of view, the latter discovery brought no news about possibly richer symmetries.

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The point-group symmetries play a very important role in molecular and condensed matter physics and so far much less so in sub-atomic physics. One of the greatest successes of the point-group symmetries in molecular spectroscopy is to provide the understanding of the characteristic energy-level degeneracies, the latter serving in fact as signals of the symmetries [1]. Indeed, suppose that group $G = \{\hat{g}_1, \hat{g}_2, \ \ldots \hat{g}_f\}$ is composed of symmetry elements $\hat{g}_i$ of Hamiltonian $\hat{H}$ so that

$$\left[\hat{H}, \hat{g}_i\right] = 0 \quad i = 1, 2, \ \ldots \ f \,. \tag{1}$$

If this happens we say that G is the symmetry group of the studied system. Suppose that G has irreducible representations, say, $\mathcal{R}_1, \mathcal{R}_2, \ \ldots \ \mathcal{R}_r$ of dimensions $d_1, d_2, \ \ldots \ d_r$, respectively. Under these conditions one can show a property of primordial importance in spectroscopy: The eigen-values of the problem

$$\hat{H}\Psi^{(i)}_{\nu_i} = E^{(i)}_{\nu_i}\,\Psi^{(i)}_{\nu_i} \quad \text{for} \quad i = 1, 2, \ldots \ r \,, \tag{2}$$

form multiplets, above enumerated with the index "i", with degeneracies equal to one of the dimensions $d_1, d_2, \ \ldots \ d_r$, respectively. Since the number of point-groups of potential importance for sub-atomic physics is finite and since all corresponding mathematical properties are very well known, the degeneracy pattern predicted by the point group theory can indeed be used to help identifying the symmetry in question through experiment. In molecular physics such a comparison can be seen as a direct one, Ref. [1]; in sub-atomic physics the situation is quite different.

In molecular quantum systems there are *several symmetry aspects* that intervene *simultaneously* and have direct impact on the implied spectroscopic properties. For instance, molecules containing identical nuclei have extra symmetries related to their exchange properties; odd-A nuclei if present introduce the effects related to the magnetic moments; the rotational and vibrational degrees of freedom often mix directly; the electronic spectra are subject to sub-structures related to the presence of magnetic moments on the atomic and nuclear scales *etc.*

In nuclear physics, in contrast, the nucleonic mass distributions are compact and extremely uniform as compared to the molecular systems, the anti-symmetrisation involves 'the only like objects' (nucleons rather than nuclei *and* electrons) and the characteristic excitation energies related to rotation, vibration and the nucleonic excitations are close, within a factor of ten of the discrepancy range. In the following we address the typically nuclear aspects of the symmetry problem.

2. Expected rotational properties of tetrahedral nuclei

Let us begin with a qualitative discussion of a number of geometrical properties related to classical objects (rotors) with uniform mass distribution — the properties relevant from the point of view of the tetrahedral symmetry. These simplified considerations will be followed by gradually more realistic ones.

2.1. A model of static, rigid, classical two-component rotors

Consider a two-component rotor originally composed of two concentric spherical mass distributions. Such a system resembles a classical model of an atomic nucleus composed of Z protons and N neutrons with the two centres of mass placed at the origin of the coordinate frame. Now let us gradually introduce tetrahedral deformation, say α_{32}, in terms of the spherical harmonic expansion of the two mass distributions. It is easy to demonstrate that the positions of the associated centres-of-mass remain unchanged for both subsystems, thus implying that there is no induced *dipole moment* caused by an increase in the tetrahedral deformation. We conclude that *static classical distributions* of protons and neutrons that are tetrahedral-deformed generate no dipole moments. Similar can be said about the quadrupole moments: rigid objects with exact static tetrahedral deformation generate no quadrupole moments, the first non-vanishing ones being Y_{32} octupole one. Consequently, in the case of the tetrahedral symmetric shapes there is, to a leading order, neither dipole nor quadrupole radiation predicted for the rotating rigid two-component objects discussed here. The only possible multipole radiation must have the octupole character.

The above result contrasts with an analogous one for a two-component rotor whose axially symmetric pear-shape octupole deformations are often described with the help of Y_{30}-spherical harmonic. Indeed, the corresponding two pear-shape distributions of the un-equal proton- *versus* neutron-number systems will generate the centre-of-mass positions that do not coincide. It then follows that the non-zero overall electric dipole moment is induced. Generally the stronger the octupole deformation the stronger the implied dipole-moment polarisation. There is yet another difference when comparing the properties of the simplified models of the octupole and tetrahedral nuclei discussed so far. The octupole deformation can be superposed with an arbitrary axial deformation (such as *e.g.* the quadrupole one) still preserving the octupole symmetry of the resulting objects. This is not true for the tetrahedral-symmetric shapes for which it is possible to superpose tetrahedral symmetry components coming from various multipolarities that are represented by very precise proportions of certain spherical harmonics, *cf. e.g.* Ref. [3].

2.2. A model of quantum rotor-vibrator like nuclei

Let us now relax the condition for the systems in question being rigid, classical two-component objects and consider instead two quantum mass distributions ("quantum liquids") moving while preserving the same shape symmetries as those discussed so far. In the case of modelling the nuclear behaviour the two components are expected to move together as an expression of the short-range character of the nuclear forces what implies that the proton and neutron deformation parameters can be kept the same. Let us consider one multipolarity at a time for simplicity. We will shorten the notation[1] by writing $\alpha_{\lambda\mu} \to \alpha$. The corresponding systems have been discussed by Bohr and Mottelson within harmonic approximation [4]. These authors consider the classical energy and the corresponding Hamiltonian of the form

$$E(\alpha,\dot{\alpha}) = \frac{1}{2}\mathcal{B}\,\dot{\alpha}^2 + \frac{1}{2}\mathcal{C}\,\alpha^2 \quad \to \quad \hat{H} = \frac{1}{2\mathcal{B}}\frac{\partial^2}{\partial\alpha^2} + \frac{1}{2}\mathcal{C}\,\alpha^2\,, \tag{3}$$

where constants $\mathcal{B}$ and $\mathcal{C}$ denote the collective inertia parameter and the so called "stiffness" coefficient, respectively, the latter determining the harmonic-oscillation potential. The energy solutions are

$$E_n = \left(n+\tfrac{1}{2}\right)\hbar\omega \quad \leftrightarrow \quad \omega \overset{\text{df.}}{=} \sqrt{\mathcal{C}/\mathcal{B}}\,. \tag{4}$$

The normalised wave functions are given by

$$\varphi_n(\alpha) = \frac{1}{\sqrt{\sqrt{2\pi}2^n n!}}\,\frac{1}{\mathcal{A}}\,e^{-\alpha^2/2\sigma^2}H_n(\alpha/\sigma), \quad \sigma \overset{\text{df.}}{=} \sqrt{2}\mathcal{A}, \tag{5}$$

where H_n denotes the Hermite polynomials and the zero-point amplitude $\mathcal{A}$ is defined by

$$\mathcal{A} \overset{\text{df.}}{=} \left[\langle\varphi_{n=0}|\alpha^2|\varphi_{n=0}\rangle\right]^{1/2} = \left[\hbar^2/(4\mathcal{B}\mathcal{C})\right]^{1/4}\,. \tag{6}$$

The above relations reveal qualitatively a couple of important tendencies: 1. The larger the collective inertia the smaller the corresponding energy excitations while at the same time. 2. A decrease in either $\mathcal{B}$ or $\mathcal{C}$ (or both) will result in an *increase of the collective zero-point vibration amplitude.*

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[1] Here and in the following we assume that the nuclear shapes and associated surfaces, Σ, can be represented in terms of the usual spherical harmonic expansion

$$\Sigma: \quad R(\vartheta,\varphi) = R_0 c(\{\alpha_{\lambda\mu}\})[1+\sum_\lambda\sum_\mu \alpha_{\lambda\mu}\,Y^*_{\lambda m}(\vartheta,\varphi)]\,,$$

where R_0 denotes the nuclear radius parameter and function $c(\{\alpha_{\lambda\mu}\})$ assures the constant volume condition.

2.3. Dynamical perturbations of the nuclear tetrahedral symmetry

The results of the previous section bring us to yet another, the most relevant at this point, the dynamical aspects presented shortly below.

Tetrahedral Symmetry Distortions Caused by Dynamical Effects: There always exists, however small, a non-zero *dynamical deformation* in any multipolarity. We can introduce a measure of such a dynamical deformation in the form of

$$\langle|\alpha|\rangle \sim \int |\alpha|\, \varphi_n^2(\alpha)\, d\alpha\,. \tag{7}$$

As a consequence, even though in a tetrahedrally-symmetric, statically deformed nucleus many types of vibrations (*e.g.* quadrupole ones) take place about the null values of static equilibria, *the zero point vibrations average out leading to a non-zero dynamical equilibrium deformation*[2].

Tetrahedral Symmetry Distortions Caused by Valence Nucleons: It has been discussed in preceding publications (*cf. e.g.* Refs. [3,5]) that there exist optimal 'magic numbers' corresponding to the strongest tetrahedral shape stability manifested by the deepest tetrahedral energy minima. Conversely, adding (or taking away) the nucleons to (from) the doubly-magic tetrahedral configurations destabilises the configurations in question, by gradually bringing in new 'symmetry polluting' deformations.

Tetrahedral Symmetry Distortions Caused by Alignment: In rotating nuclei, as it is very well known, the Coriolis (or equivalently cranking) terms in the effective Hamiltonians

$$\hat{H}_{\text{Coriolis}} \sim -\vec{R}\cdot\hat{\jmath} \quad \text{or} \quad \hat{\mathrm{H}}_{\text{Cranking}} \sim -\vec{\omega}\cdot\hat{\mathrm{j}} \tag{8}$$

lead to a decrease in the energy of the system if the individual nucleonic angular momenta align with the collective rotation $\vec{R}$ (alternatively with the collective cranking-frequency vector $\vec{\omega}$). In Eq. (8) operator $\hat{\jmath}$ represents all the three components of the nucleonic angular momentum. As a consequence of such an alignment, shape polarisations with the oblate-quadrupole type components with respect to the collective rotation axis will be expected, thus bringing yet another contribution to the non-zero quadrupole moment that is *increasing with the total nuclear spin.*

To summarise: The non-zero (small) quadrupole deformations are to be expected in the tetrahedral-deformed nuclei. In the corresponding rotational bands the quadrupole type polarisation is predicted to increase with increasing spin due to the gradual alignment effects.

[2] The above observations apply generally to any shape vibration; here we are of course particularly interested in the low-energy low-multipolarity oscillations since they give the most important contribution to the electro-magnetic decay radiation.

2.4. Consequences of the quadrupole polarisation in tetrahedral nuclei

The corresponding conclusion related directly to the quadrupole deformation has important implications for the induced electric *dipole radiation*. Indeed, it can be argued [2] that using a simple modelling of the nuclear surfaces in terms of the spherical harmonics leads to an estimate for the electric dipole moment

$$Q_{10} \approx 9/\sqrt{21\pi}\,\rho\,R_0^4\,\alpha_{2\pm 2}\,\alpha_{3\pm 2}\,, \tag{9}$$

valid for small deformations.

Now we only need to connect the result in Eq. (9) with the standard expressions for the electromagnetic transition probabilities. These are very well known; here we quote them for reader's convenience. The probability of the gamma-emission in transitions per second can be expressed by

$$T(E_\gamma; L) = \frac{8\,\pi}{\hbar c}\,\frac{c}{[\hbar c]^{(2L+1)}}\,\frac{(L+1)}{L[(2L+1)!!]^{\,2}}\,E_\gamma^{(2L+1)}B(L)\,, \tag{10}$$

where L denotes the transition multipolarity, E_γ transition energy, and $B(L)$ the reduced transition probability. Expressing the numerical factor in front of the product $E_\gamma^{(2L+1)} \cdot B(L)$ gives the following result

$$T(E_\gamma; L=1) \;=\; 1.590206 \times 10^{15} E_\gamma^{\,3} B(L=1)\,, \tag{11}$$

$$T(E_\gamma; L=2) \;=\; 1.225184 \times 10^{9}\;\; E_\gamma^{\,5} B(L=2)\,, \tag{12}$$

$$T(E_\gamma; L=3) \;=\; 5.707943 \times 10^{2}\;\; E_\gamma^{\,7} B(L=3)\,. \tag{13}$$

The above shows that any perturbation of the tetrahedral deformation, be it statical or dynamical, resulting in a non-zero quadrupole deformation, the latter inducing both the quadrupole and dipole moments, will immediately generate the quadrupole and dipole radiation due to the huge factors of the order of 10^6 or 10^{12}. They originate from the natural-constant factors that are about six orders of magnitude stronger for the quadrupole transitions as compared to the octupole ones, *cf.* Eqs. (12) and (13) and yet additional six orders of magnitude stronger when comparing the dipole transitions with the quadrupole ones, *cf.* Eqs. (11) and (12).

Consider a rotational band generated by a tetrahedral-deformed nucleus. At relatively high spins, say $I \sim 20\,\hbar$, we should expect that Coriolis induced quadrupole polarisation whose effect is proportional to I and the implied dipole moments take relatively 'large' values (in relative terms, the Coriolis polarisation expressed *e.g.* in terms of polarised quadrupole moment, $\delta Q_{2\mu}$ at spin $I \sim 20\,\hbar$ can be considered significantly larger as compared to the similar quantity at spin $I \sim 3 - 5\,\hbar$).

All the above observations bring us to the following scenario that can be associated with the rotation of tetrahedral-symmetric nuclei. Since the tetrahedral shapes have to a first approximation a non-axial Y_{32} *octupole structure* we expect that the associated bands carry negative parity. According to calculations in Ref. [2] the tetrahedral minima, and thus the related band-heads, are expected at several hundreds of keV to a couple of MeV above the yrast line. The higher this excitation energy the stronger the *disadvantages* when trying to populate those bands. In particular, the heavy-ion xn reactions that populate preferentially high-spin low energy range near the yrast line are likely *not* to populate such states that are at relatively high energies and moderate spins. Reactions induced by α- and other types of light particles are in this respect privileged.

On the other hand, the stronger the excitation of the tetrahedral band above the yrast, usually ground-state band, the stronger the *advantages* in terms of the relatively high-energy *electric dipole* transitions [E_γ in Eq. (11)] possibly connecting the tetrahedral-excited and the ground-state bands. At the highest spins, the quadrupole polarisation induced through zero-point vibrations but first of all through angular momentum alignment is expected to give rise to non-vanishing quadrupole and dipole moments. Their presence should imply a competition between intra-band E2-transitions within the tetrahedral band against the inter-band E1-transitions between the tetrahedral and the ground-state band. With decreasing spin the intra-band E2 transition energies decrease linearly with (thus the corresponding probabilities as the fifth power of) the transition energy while the inter-band E1 transitions remain of the same order or possibly even increase[3]. As a consequence the E2 quadrupole transition rates may become so low that the only transitions seen by the detectors may be those of the inter-band dipole character.

The scenario described above is precisely what is observed in numerous experiments on ^{156}Gd nucleus whose two respective bands are given in Fig. 1 — and also in several other nuclei of the Rare Earth range in which theoretical predictions suggest the presence of the tetrahedral symmetry.

The above observations bring us to the very important conclusion about the *qualitative difference* between the electro-magnetic transition properties of the 'traditional axial octupole nuclei' described by Y_{30} plus axial quadrupole deformation and the 'new non-axial octupole nuclei' described by Y_{32} supplemented with the induced polarisation effects. Polarisation due to the zero-point motion take of course place also for those pear-shape octupole nuclei as well, yet their role there is to renormalise the (already large)

[3] These energy proportions are in fact to be seen in the experimental spectrum of Fig. 1 presented below; the latter demonstrates that the discussed relations give a realistic representation of the situation encountered in nuclei.

quadrupole and octupole moments — not leading to any quantitative change. In tetrahedral nuclei instead, the polarisation effects lead to the entire modification of the radiation pattern: otherwise absent dipole and quadrupole transitions — are most likely winning with the octupole ones thanks to the huge factors in Eqs. (11–13) as discussed above.

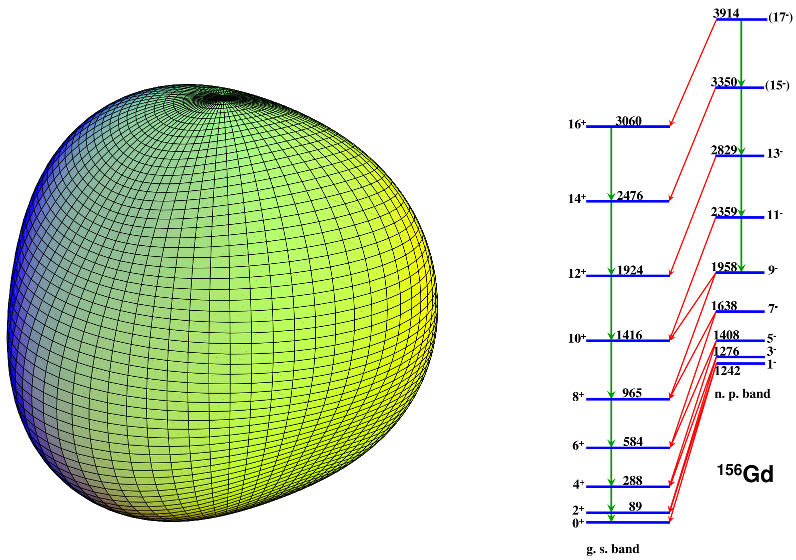

Fig. 1. According to theoretical predictions of Ref. [2] the negative parity band (right) is expected to correspond to a tetrahedral symmetry nuclear shape (left).

As it turns out there are several nuclei close to the ones predicted in Ref. [2] that actually do manifest the properties discussed qualitatively. An example of ^{156}Gd nucleus has been studied in about twenty various experiments with varying target-beam combinations and detection systems. Invariably, the intra-band E2-transitions escape the detectors below $I^{\pi} = 9^{-}$ state in all experiments to a surprise of the authors. For instance, already in the early eighties it was noticed by Konijn and co-workers, Ref. [6], an odd-behaviour of the negative-parity bands in ^{156}Gd, at that time interpreted as aligned octupole bands: "A striking feature is that the $B(\mathrm{E1})/B(\mathrm{E2})$ ratios are about a factor of 50 higher for the odd-spin negative-parity bands than those for the even-spin negative-parity bands". This phenomenon seen already before in various other nuclei (see references quoted in [6]) received no adequate explanation at that time and was simply forgotten since then. A simple explanation of this lack of interest is most likely the rush towards high spin state studies: in the eighties and nineties, heavier and heavier ion beams were used with the result of no longer directly populating the single excited negative parity states of interest in the present study. This are these states, as we suggest, that could have been the first experimental signature of the tetrahedral symmetry in nuclei.

The qualitative predictions of the tetrahedral band behaviour as far as the electromagnetic transitions are concerned can be confronted in addition with the microscopic cranking model prediction. The corresponding results in the form of comparison between the calculated and measured angular momentum alignment (total spin) are presented in Fig. 2. The curves were calculated at the deformation points corresponding closely to the predicted equilibrium deformations by using the Strutinsky method with the deformed Woods–Saxon Hamiltonian.

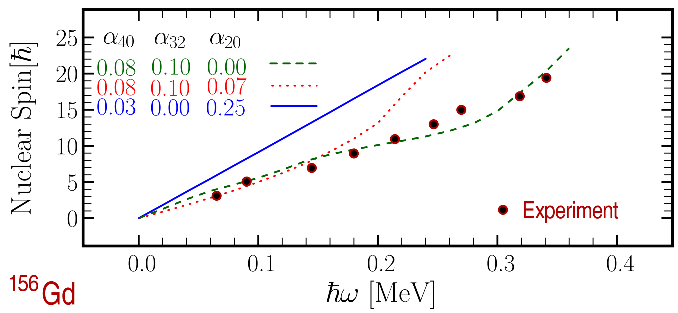

Fig. 2. Cranking-model results with the deformed Woods-Saxon potential, using universal parametrisation, as the one used in Ref. [2]. The experimental results are from Ref. [7]. The three curves represent three deformations: tetrahedral, tetrahedral with a small quadrupole component of $\alpha_{20} = 0.07$ and the ground-state deformation of $\alpha_{20} = 0.25$ and a small hexadecapole deformation.

All these results can be seen as strongly encouraging. However, in order to be able to address the problems of symmetry based on predicted properties of the rotational spectra yet another method will be needed — the one that allows to study directly both symmetries of the Hamiltonian and the corresponding transition probabilities. Such methods have been studied mainly in the molecular physics; below we wish to discuss the possible extension of those methods to the nuclear case.

3. Nuclear rotation and tetrahedral nuclei as quantum rotors

There exist well known methods designed to treat collective rotation in nuclear physics such as *nuclear cranking model* directly related to the mean-field approach in both self- and non-self-consistent versions and the so-called *Bohr model* that allows in a natural way to treat at the same time a coupling between collective-rotational and vibrational motion (although in a rather limited fashion as far as the number of degrees of freedom is concerned).

3.1. Generalised rotor Hamiltonians

Despite the fact that advanced quantum methods have been developed to describe the rotational motion on the molecular and subatomic levels, some opinions found occasionally in the literature stress formal difficulties related, among others, to the very definition of this concept. An example of a statement: 'the concept of rotational motion of an n-body system can, strictly speaking, be attributed to rigid bodies only' gives an illustration. Not entering into the most general context and philosophical background of the concept of rotation of the interacting many-body systems we limit ourselves to the nuclear systems whose bulk properties can be described within the mean-field approach with stationary solutions. Let the Hamiltonian of an A-body nuclear system be given. It is constructed using the canonically conjugated nucleonic momentum and position operators, $\hat{p}_i$ and $\hat{x}_i$, respectively, and the nucleonic spin operators, $\hat{s}_i$. This mean-field Hamiltonian, denoted $H_{\rm mf}$, can be defined in a given laboratory-reference frame[4].

Discussion of the concept of a nuclear collective rotor is facilitated by the adiabaticity of the nuclear rotation. Here we are concerned with low-lying nuclear collective excitations, $E_{\rm rot}$, say at spins that do not exceed $I \sim 20$. They depend somewhat on the nuclear mass and vary between a few keV for the lowest-lying rotational states in heavy nuclei to the excitations of the order of some MeV at highly excited states belonging to well developed rotational bands in lighter nuclei. For a nucleus with a mass A, this implies still a very low contributions of the rotational energy *per nucleon*, $E_{\rm rot}/A$, that varies from some dozens of eV/nucleon to some dozens of keV/nucleon depending on the mass range. These can be compared to average energies of the individual-nucleonic motion in the mean-field potential that amount to about 25 MeV/nucleon, assuming that the effective mean-field potential depth is of the order of 60 MeV. This highly pronounced disproportion between the individual and collective-rotational kinetic energies allows to treat the rotational degrees of freedom as perturbations and encourages the formulation of the basic Ansatz of the rotor model

$$\hat{H}_{\rm nucl} = \hat{H}_{\rm rot} + \hat{H}_{\rm mf} \tag{14}$$

— the splitting of the nuclear Hamiltonian into the part that focuses on the individual-nucleonic degrees of freedom, $\hat{H}_{\rm mf}$, and $\hat{H}_{\rm rot}$ responsible for the energy changes caused by the varying *orientation of the system with respect to the laboratory reference frame.*

[4] In fact for the ground-states of the even–even systems studied here these two types of reference frames can be made coincide. For the concept of rotating body-fixed frame in the case stationary solutions of the mean-field Hamiltonian — see below.

The mean-field theory framework leads to a natural description of the spontaneous symmetry-breaking phenomena and in particular to a non-zero nuclear deformation that allows to introduce the notion of the system's *orientation in space*. When this happens for a stationary solution of $\hat{H}_{\rm mf}$, we have a possibility to define the nuclear surface *e.g.* using the concept of the equi-density surfaces. By selecting appropriately four position-points on such a surface we have natural way of defining the associated body-fixed reference frame and thus the orientation of the body-fixed reference frame with respect to the laboratory frame. This orientation will be identified with the orientation of the nucleus with respect to the laboratory frame. The adiabaticity of the nuclear collective rotational motion makes us to expect that the rotating density distribution remains stationary, if not exactly then at least to a good approximation.

The main idea in constructing the related quantum formalism is to focus the description on those degrees of freedom that are responsible for the *orientation of the nuclear system in space*. Since an orientation of the system as a whole with respect to a given reference frame is conveniently described with the help of orientation angles and the associated generators are the total angular momentum operators the corresponding Hamiltonians can be constructed using those two classes of objects. A method that is perfectly suited to examine the effects of point-group symmetries on the rotational properties of many-body systems has been designed in molecular physics in the form of what we refer to as *generalised rotor Hamiltonian*. Here we wish to present this particular approach in a version adapted to the nuclear physics applications.

Let us begin by observing that Eq. (14) hides an apparent conflict of the formulation: the system has $6A$ degrees of freedom and both the Hamiltonian $\hat{H}_{\rm nucl}$ and the mean-field Hamiltonian $\hat{H}_{\rm mf}$ depend on all of them. Therefore, strictly speaking, there is no room for the Euler-angles as variables or extra degrees of freedom describing the orientation. On the other hand, the *ex-post success of the rotor approximation* fully justifies[5] the interest in further exploration of the method in question. Consequently, when using this type of Hamiltonians we have to assume that the rotation and thus the implied time evolution of the Euler angles is the well defined *result* of (perhaps very complicated) interplay among the individual degrees of freedom. The adiabatic character of the rotational motion as well as the short-range property of the nuclear interactions that allow to see the nuclei as relatively compact objects of well defined shape of the density distribution makes us to expect that such a relation should exist. Since we cannot, at least for the time being, obtain any formal derivation of the time-evolution

[5] Indeed, the particle-rotor coupling models as *e.g.* in the formulation of Mayer-ter-Vehn are among those numerically most successful in nuclear structure physics.

of the Euler angles as functionals of the intrinsic degrees of freedom we may use another choice: to assume a phenomenological form of the corresponding Hamiltonian. Its form must be compatible with the above consideration of the number of degrees of freedom. Since as argued above the rotation can be treated as perturbation of the individual-nucleonic motion, it is natural to expect that in most cases its effect will be too weak to cause an extra spontaneous symmetry breaking of $\hat{H}_{\rm mf}$. In other words: *it should be assumed that symmetries of all the three operators in Eq. (14) are the same.*

3.2. Symmetry compatibility between rotor and mean-field Hamiltonians

In the earlier publications, *cf. e.g.* [2] and references therein, the exotic tetrahedral and octahedral symmetries have been predicted to be rather abundant throughout the Periodic Table. These predictions were based on the nuclear mean-field approach what implies, according to compatibility principle among the members of relation (14) that the rotor Hamiltonian suited for our purposes must be tetrahedrally-symmetric as well.

This aspect of high-rank symmetries *e.g.* tetrahedral or octahedral ones of the quantum rotor brings us to yet another element of possible confusion that is found occasionally in the literature. Some generally incorrect statements such as 'the nuclear rotor Hamiltonian must be a quadratic expression of angular momentum components (operators)' and 'the rotor Hamiltonian is a function of the inertia tensor' are a few examples. In nuclear as in molecular quantum mechanics the rotor Hamiltonian is the energy operator associated with the varying orientation of the system with respect to the laboratory coordinate frame. There is *a priori* no limitation of the form of this operator as far as dependence in terms of the angular momentum operators are concerned while the inertia tensor can simply not be defined in general for numerous situations of interest[6].

To present what we call the generalised rotor formalism let us introduce a basis of the n^{th} order tensor operators

$$\hat{T}_{\lambda\mu}(n;\underbrace{\lambda_2,\lambda_3,\,\ldots\,\lambda_{n-1}}_{\{\Lambda\}_n}) \stackrel{\rm df.}{=} \left[\Big(((\hat{I}\otimes\hat{I}\,)_{\lambda_2}\otimes\hat{I}\,)_{\lambda_3}\otimes\,\ldots\,\otimes\hat{I}\,\Big)_{\lambda_{n-1}}\right]_{\lambda\mu}, \quad (15)$$

where the symbol "$\otimes$" refers to the standard Clebsch–Gordan couplings *e.g.*

[6] For many classes of symmetries that bypass the simple ellipsoidal shapes (*e.g.* tetrahedral or octahedral ones) the standard inertia tensor becomes a multiple of the unit operator and does not allow to distinguish between various geometries of interest here. In molecular physics, for instance, the description of the degeneracies of the rotational levels can be seen as a valid motivation for the assumed forms of the Hamiltonians, *cf.* Eq. (17) below.

$$(\hat{I}\otimes\hat{I}\,)_{\lambda_2\mu_2} = \sum_{\mu=-1}^{1}\sum_{\mu'=-1}^{1}(1,\mu;\;1,\mu'\mid\lambda_2,\mu_2)\;\hat{I}_{1\mu}\;\hat{I}_{1\mu'};\quad \lambda_2=0,1 \text{ or } 2\,, \tag{16}$$

and where $\hat{I}_{1\mu}$ are the spherical components of the nuclear angular momentum operator. As a technical remark: in Eq. (15) we assume what we call 'stretched coupling' *i.e.* $\lambda_2 = 2$, $\lambda_3 = 3$, *etc.*; it can be shown that this introduces no loss of generality. In such a case the dependence on the symbol $\{\Lambda\}_n$ becomes redundant and the notation can be simplified. With the simplified notation, Eq. (15) can serve to expand an arbitrary operator of the n^{th} order constructed of the angular momentum components

$$\hat{h}(n,\,\lambda) \equiv \sum_{\mu=-\lambda}^{\lambda} c_{\lambda\mu}(n)\;\hat{T}_{\lambda\mu}\,(n) \quad\to\quad \hat{H}_{\rm rot} = \sum_{n}\sum_{\lambda}\hat{h}(n,\lambda)\,, \tag{17}$$

using the numerical expansion coefficients $c_{\lambda\mu}(n)$; above the nuclear rotation Hamiltonian has been denoted by $\hat{H}_{\rm rot}$. The coefficients must satisfy certain additional relations assuring that the final expansion gives an Hermitian operator. In its form of Eq. (17), the Hamiltonian of the rotational motion is perfectly adapted to building in the required symmetry. Indeed, as discussed in Ref. [8] the point-group symmetries of the mean-field Hamiltonian can be conveniently represented with the help of the spherical tensors — here the same tensor structure of the *rotor* Hamiltonian is obtained. This allows to parametrise the symmetry of the mean-field and of the collective rotor Hamiltonian using the same language of the spherical tensors of the underlying SO(3) group.

4. Expected rotational properties of tetrahedral nuclei

It will not be possible for us to discuss the present status development related to the symmetry properties of the nuclear quantum rotor Hamiltonian and its solutions. We will limit ourselves to a few comments only. One of the most important ones is related to the general observation that the group-symmetry studies related to this particular subject practically do not exits in the nuclear physics literature with the notable exception of Ref. [4] where the discussion of some selected properties of the irreducible representations of the D_2 group of the triaxial nuclear rotor can be found. The rotational properties of the quantum rotor Hamiltonians of the quadrupole-deformed (D_2-symmetry) nuclei are particularly simple — not to say trivial — and there seemed to be very little interest in the literature in developing the group-theory aspects of the problem — neither that of the selection rules of the related electromagnetic transitions. Indeed, for well deformed nuclei the main decay sequences are composed of the strong stretched quadrupole intra-band E2-transitions with only very marginal inter-band transitions on the level of 10^{-2} to 10^{-3} relative intensities.

But for those ‘simple’ quadrupole-deformed nuclei the higher-excited bands were to our knowledge never analysed from the symmetry point of view; even the ‘elementary’ tri-axial quadrupole deformation in nuclei has not been studied, neither properly identified. Let us make this point a little bit more explicit.

According the additivity represented in (14) to each rotational band there exists a contribution from the intrinsic (*e.g.* mean-field) Hamiltonian.

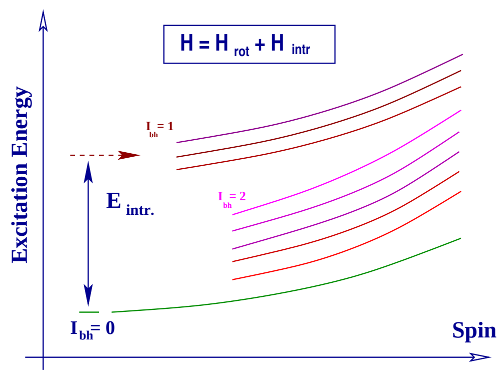

Fig. 3. A schematic representation of the energy-spin sequences based on the solutions of the nuclear rotor-plus-mean-field Hamiltonian of Eq. (14).

This brings us to the energy shifts associated with each band, intrinsic band-head energies, as illustrated schematically in Fig. 3. These energy shifts will play an essential role in the interpretation of the experimental results — yet for our purposes they are a disturbing factor and will be ignored. The illustrations presented below contain only the pure rotor Hamiltonian eigen-energies.

The first of the two illustrations of the quantum rotor properties presents the solutions to the triaxial rotor Hamiltonian with the rotor parameters corresponding to the quadrupole deformation $\beta_2 = 0.25$ and $\gamma = 30°$ of a nucleus with $A \sim 80$ mass range. There are four families of solutions marked with different colours corresponding to four one-dimensional irreducible representations of the D_2 group. The fact that they are one-dimensional implies that strictly speaking none of the energies is degenerate with any other. Yet, the solutions appear nearly degenerate on a few eV level for many energies! A remarkable difference of the energy pattern is visible in the transitional region of the energies in the middle of the diagram, Fig. 4. Below this transitional region these are the states belonging to the irreducible representations $A2$ and $B3$ on the one hand and $A1$ and $B4$ on the other hand that are very close in energy (degenerate in the scale of the figure). Above this transitional region the partnership relations change and $B3$ and $B4$ on the one hand and $A1$ and $A2$ on the other hand are nearly degenerate. The transitional region itself presents a dramatic change in the rotational properties. These can be seen *e.g.* by analysing the 3D distributions of the spin-orientation probabilities (not shown here *cf.* forthcoming publication) — neither there were any attempts undertaken to examine the corresponding energy regime in experiments.

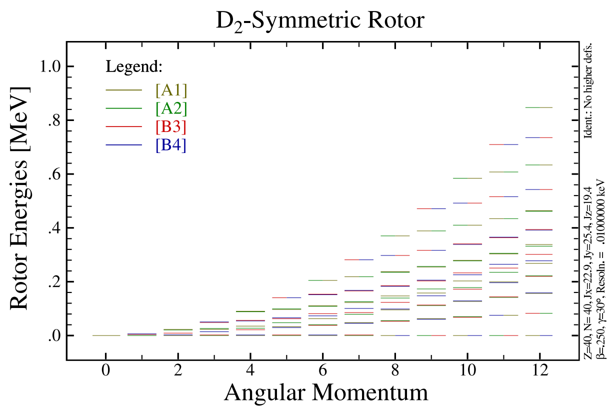

Fig. 4. The energy-spin sequences from the solutions of the nuclear rotor-plus-mean-field Hamiltonian of Eq. (14); the intrinsic energies $E_{\rm intr}$ of Fig. 3 are set equal to zero. The yrast line energies are subtracted for convenience.

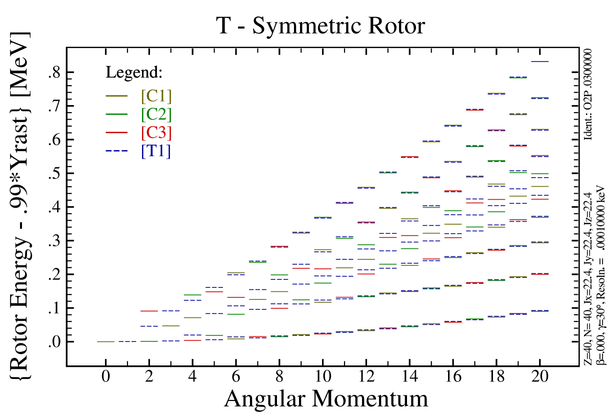

Fig. 5. Similar to the above but for the tetrahedral-symmetric rotor.

Fig. 5 illustrates the results for the quantum rotor with tetrahedral symmetry. The presence of very different energy pattern deserves noticing. In particular a three-dimensional irreducible representation gives rise to the exact three-fold degeneracy in the rotor spectra. The energy spectrum of the tetrahedral rotor presents in addition various types of the so-called energy staggering; here we wish to mention only the visible from the figure 'oscillation pattern': three-fold degenerate energies above and below the neighbouring one-dimensional irreducible representation partner.

5. Perspectives

We believe that the energy staggering pattern presented shortly above needs further exploration from the experimental point of view. After subtracting the average trend the stagerring oscillations amount to a few keV — the regime that is accessible within the present day multi-detection system techniques. Moreover, the electromagnetic transition selection rules have been already calculated by us (*cf.* a forthcoming publication) leading to a clear-cut selection patterns that are very difficult to confuse with the competing decay schemes coming from simpler (*i.e.* not involving high-rank symmetries) types of nuclear rotation.

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