ANOMALOUS DIFFUSION APPROXIMATION OF RISK PROCESSES IN OPERATIONAL RISK OF NON-FINANCIAL CORPORATIONS*

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We introduce an approximation of the risk processes by anomalous diffusion. In the paper we consider the case, where the waiting times between successive occurrences of the claims belong to the domain of attraction of α-stable distribution. The relationship between the obtained approximation and the celebrated fractional diffusion equation is emphasised. We also establish upper bounds for the ruin probability in the considered model and give some numerical examples.

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1. Introduction

Since certain operational risk data are in many ways akin to insurance losses, it is clear that methods from the field of non-life insurance can play a fundamental role in their quantitative analysis, [1]. The key problem in the classical Cramér–Lundberg model of the collective risk theory concerns finding the ruin probability, *i.e.* the probability that the risk process

$$R(t) = u + ct - \sum_{i=1}^{N_t} X_i \tag{1}$$

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falls down below zero level. Here u is assumed to be an initial risk capital, $c > 0$ a premium paid by insurer in time unit and X_i losses that happen in random moment modelled by the counting process N_t. The commonly used method of finding the ruin probability is first to determine the distribution of claims X_i, then to derive the differential equations which are satisfied by the unknown ruin probability and finally to obtain the properties of the ruin probability directly from solutions of the equations [2].

Unfortunately, the ruin probabilities in infinite and finite time can only be precisely calculated for a few claim distributions. Additionally, most methods use the standard Poisson process as the counting process, which is the additional restriction. Thus, from a practical point of view, finding a reliable approximation, especially in the case when the Monte Carlo techniques cannot be used, is of great interest. A survey of the most popular and effective numerical approximations of ruin probability can be found in [3].

An alternative method consists in replacing the continuous-time random walk (CTRW), *i.e.* the random part of the risk process $R(t)$, with the diffusion process. The basic idea is to make the claim sizes small and simultaneously to let the number of claims grow in such a way that the risk process converges weakly to a diffusion [4].

In the case, when the claims indicate the heavy-tailed distributional properties, it is also possible to approximate the risk process with an α-stable Lévy motion with drift. The detailed discussion of such approach can be found in [5].

In this paper we consider a more general case, when both the waiting times between consecutive claims and the claims themselves are heavy-tailed. In such a setting the random part of $R(t)$ can be replaced with the anomalous diffusion process (see next section for details).

An interesting application of our results is the recently developed promising technique of modelling the operational risk of non-financial corporations [7]. In this approach, every random variable X_i in (1) represents different loss type and the whole sum yields the total operational loss of the corporation. Since the concept of CTRW has found its widespread applications in many other scientific fields, our theoretical and numerical results can be applied in each of these fields. Particularly, results presented in the paper allow for the use of heavy-tailed distributions to model claim severities as well as waiting times between their occurrences. This is especially important when considering the operational risk of a large company, like KGHM, that can include such high, but not very frequent, risks as natural disasters and fraud.

Our considerations and results are related to the methodology of the CTRW. Thus, they can be applied not only in the field of insurance and collective risk theory, but also in physics, for example in the Cole–Cole relaxation responses, see [6].

2. Convergence of risk processes to anomalous diffusion

Let us begin with the basic concepts and definitions from the classical collective risk theory (see *e.g.* [2]). Given a sequence T_i, $i = 1, 2, \ldots$, of independent, identically distributed (i.i.d.) and positive random variables representing the time intervals between the consecutive occurrences of the claims, we denote by

$$T(n) = \sum_{i=1}^{n} T_i \,, \qquad T(0) = 0 \,, \tag{2}$$

the time interval of n appearances of the claims. In this setting, the *counting process* describing the number of claims in the interval $(0, t]$ takes the form

$$N_t = \max\{n : T(n) \le t\} \,. \tag{3}$$

The process N_t is also referred to as the *renewal process*.

The successive claims X_i, $i = 1, 2, \ldots$, are assumed to be a sequence of i.i.d. random variables, independent of T_i, $i = 1, 2, \ldots$. Consequently, the cumulative value of n successive claims is given by

$$X(n) = \sum_{i=1}^{n} X_i, \quad X(0) = 0 \,. \tag{4}$$

Now, the classical *risk process* has the form

$$R(t) = u + ct - X(N_t) = u + ct - \sum_{i=1}^{N_t} X_i \,, \tag{5}$$

where $u > 0$ is the initial risk reserve of the company, and $c > 0$ is the risk premium per unit time paid by the policyholders. Let us notice that the random part of $R(t)$ is the well-known CTRW model which is fundamental for the understanding of the diffusion phenomenon. The notion of CTRW was first introduced in pioneering works by Scher, Montroll and Weiss [8,9]. Since then CTRW became a widely used tool in modelling various real-life phenomena (see *e.g.* [10–12]).

Let us now assume that the time intervals T_i fulfil the following requirement $P(T_i > t) \sim t^{-\alpha}$ as $t \to \infty$, where $0 < \alpha < 1$. It implies that they belong to the domain of attraction of a totally skewed α-stable distribution $S_{\alpha,1}$(t) (we use the notation $S_{\alpha,\beta}(t)$ for a α-stable distribution, where $0 < \alpha \le 2$ is the index of stability and $|\beta| < 1$ is the skewness parameter, see [13,14]). Thus, from the generalised central limit theorem [14] we obtain for the sum (2)

$$n^{-1/\alpha}\, T([n\tau]) \xrightarrow[n\to\infty]{d} U(\tau) \,, \tag{6}$$

where $U(\tau)$ is the α-stable subordinator, *i.e.* the strictly increasing α-stable Lévy motion. Here "$\xrightarrow{d}$" denotes convergence in probability and "$[x]$" is the integer part of x. In a similar manner, if we assume that the claims X_i belong to the domain of attraction of a γ-stable distribution $S_{\gamma,\beta}(x)$ with $0 < \gamma \leq 2$, then we obtain for (4)

$$n^{-1/\gamma} X([n\tau]) \xrightarrow[n\to\infty]{d} Y(\tau) \,. \tag{7}$$

Here $Y(\tau)$ is the γ-stable Lévy motion. Note that for $\gamma = 2$ we get the classical Brownian motion.

Using the fact that the following relationship between $T(n)$ and the counting process N_t holds

$$\{T([x]) \leq t\} = \{N_t \geq x\}$$

and applying the limit result (6) we have

$$n^{-\alpha} N_{nt} \xrightarrow[n\to\infty]{d} V_t \,, \tag{8}$$

where V_t is the inverse α-stable subordinator [15] defined as

$$V_t = \inf\{\tau : U(\tau) > t\} \,. \tag{9}$$

It is not difficult to show that the subordinator V_t is self-similar with index $H = \alpha$. Since we are able to compute the moments of the random variable V_1, the series expansion of the Laplace transform $\widehat{g}(u,t) = \langle \exp(-uV_t) \rangle$ of V_t, (we use the notation $\langle X \rangle$ for the expected value of X), yields the following result

$$\widehat{g}(u,t) = E_\alpha(-ut^\alpha) \,, \tag{10}$$

where

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + 1)} \tag{11}$$

is the Mittag–Leffler function [16].

Now, we construct a sequence $Q_n(t)$ of risk processes

$$Q_n(t) = u_n + c_n t - \sum_{i=1}^{N_{nt}} X_i \,.$$

If the following four assumptions are fulfilled:

- time intervals T_i belong to the domain of attraction of a totally skewed α-stable distribution $S_{\alpha,1}(\mathrm{t})$;
- claims X_i belong to the domain of attraction of a γ-stable distribution $S_{\gamma,\beta}(x)$ with $0<\gamma\leq 2$;
- $n^{-\alpha/\gamma}\, u_n \xrightarrow[n\to\infty]{} u\,;$
- $n^{-\alpha/\gamma}\, c_n \xrightarrow[n\to\infty]{} c\,;$

then, taking advantage of (7) and (8), we get

$$\begin{aligned} n^{-\alpha/\gamma}Q_n(t) &= n^{-\alpha/\gamma}u_n + n^{-\alpha/\gamma}c_n t - n^{-\alpha/\gamma}X(N_{nt}) \\ &\approx n^{-\alpha/\gamma}u_n + n^{-\alpha/\gamma}c_n t - (n^{\alpha})^{-1/\gamma}X([n^{\alpha}V_t]) \\ &\xrightarrow[n\to\infty]{d} u + ct - Y(V_t)\,. \end{aligned}$$

Thus, we have shown that the limit of the sequence of risk processes $Q_n(t)$ is given by $Q(t)=u+ct-Y(V_t)$. The subordinate process $Y(V_t)$, as the only random part of $Q(t)$, plays the key role in the considered model. Let us remind that $Y(\tau)$ belongs to the family of γ-stable Lévy motions and V_t is the inverse α-stable subordinator. Basing on the recent paper [15], we call $Y(V_t)$ *anomalous diffusion* (see Fig. 1). Therefore,

$$Q(t)=u+ct-Y(V_t) \tag{12}$$

is called the *anomalous diffusion approximation* of the risk processes.

As a first special case, let us consider the situation, when $Y(\tau)$ is the classical Brownian motion, *i.e.* $\gamma=2$. Since $Y(\tau)$ and V_t are assumed to be independent stochastic processes, the probability density function (p.d.f.) $p(x,t)$ of $Y(V_t)$ is given by the formula

$$p(x,t)=\int_0^{\infty} f(x,\tau)g(\tau,t)d\tau\,, \tag{13}$$

where $f(x,\tau)$ and $g(\tau,t)$ are the p.d.f.s of $Y(\tau)$ and V_t, respectively. Next, the Fourier transform $\widetilde{p}(k,t)=\langle\exp(ikY(V_t))\rangle$ takes the form

$$\widetilde{p}(k,t) \;=\; \int_0^{\infty} \widetilde{f}(k,\tau)g(\tau,t)d\tau\,. \tag{14}$$

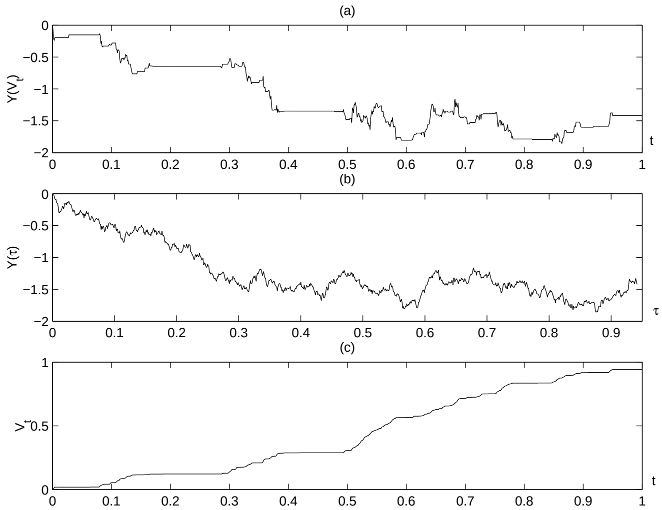

Fig. 1. An exemplary realization of: (a) anomalous diffusion $Y(V_t)$, (b) γ-stable Lévy motion $Y(\tau)$, (c) inverse α-stable subordinator V_t. The parameters are $\alpha = 0.7$ and $\gamma = 2$ (Brownian motion). The constant intervals of $Y(V_t)$ indicate the heavy-tailed waiting times between successive claims. Note the similarities between the constant intervals of $Y(V_t)$ and V_t, and the similarities between $Y(V_t)$ and $Y(\tau)$ in the remaining domain.

Taking advantage of the fact that the Fourier transform of the Brownian motion is given by

$$\widetilde{f}(k,t) = \langle \exp(ikY(\tau)) \rangle = \exp(-\tau k^2)\,,$$

and using results (10) and (14), we see that the Fourier transform of the anomalous diffusion $Y(V_t)$ is given by

$$\widetilde{p}(k,t) = \langle \exp(ikY(V_t)) \rangle = E_\alpha(-k^2 t^\alpha)\,.$$

The above result proves that the p.d.f. of $Y(V_t)$ is the solution of the celebrated fractional diffusion equation [17]

$$\frac{\partial p(x,t)}{\partial t} = {}_0D_t^{1-\alpha} \frac{\partial^2}{\partial x^2}\, p(x,t).$$

Since in this case the second moment of $Y(V_t)$ is given by

$$\langle Y^2(V_t) \rangle = \frac{2}{\Gamma(1+\alpha)} t^\alpha,$$

we obtain the typical anomalous-diffusion behaviour of the considered model.

The second special case can be obtained by letting the parameter $\alpha \to 1$. In this setting, the inverse α-stable subordinator V_t becomes deterministic linear function. Then, the anomalous diffusion $Y(V_t)$ is the standard γ-stable Lévy motion and $Q(t)$ becomes the γ-stable approximation of risk processes. This is the case discussed in detail in [5].

3. The ruin probability

Considering the ruin probability problem, let us begin with defining the ruin time T as the first time the company has a negative risk reserve. For the model with risk process $Q(t)$, the ruin time can be expressed as

$$T = \inf\{t > 0 : Q(t) < 0\}\,.$$

Finding the ruin probability $\psi(u)$ defined as

$$\psi(u) = P(T < \infty | Q(0) = u)\,,$$

i.e. the probability that the risk process becomes negative, is one of the major problems of the risk theory. However, an insurance company is mostly interested in finding the ruin probability in finite time

$$\psi(u,t) = P(T \leq t | Q(0) = u)\,,$$

which is the probability that the risk process drops below zero level before time t. A comprehensive study concerning this subject can be found in [2]. In what follows, we find the upper bounds for the ruin probability in finite time, in the model with risk process $Q(t) = u + ct - Y(V_t)$ given by the previously introduced anomalous diffusion approximation (12).

We begin with the case, when the γ-stable Lévy motion $Y(\tau)$ is symmetric. Then, we can use the following inequality [5]

$$P\left(\sup_{0\leq s\leq \tau} Y(s) \geq z\right) \leq 2P(Y(\tau) > z)\,. \tag{15}$$

Formula (15) together with some standard arguments yields

$$\begin{aligned} P\left(\sup_{0\leq s\leq t} Y(V_s) \geq z\right) \;&\leq\; P\left(\sup_{0\leq s\leq V_t} Y(s) \geq z\right) \\ &\leq\; 2P(Y(V_t) > z)\,. \end{aligned} \tag{16}$$

Now, using the above result, we are able to establish the following upper bound for the ruin probability in finite time horizon

$$\begin{aligned}\psi(u,t) &= P(T \le t | Q(0) = u) = 1 - P\left(\inf_{0\le s\le t}(u + cs - Y(V_s)) \ge 0\right)\\ &= 1 - P\left(\sup_{0\le s\le t}(-u - cs + Y(V_s)) \le 0\right)\\ &= P\left(\sup_{0\le s\le t}(-u - cs + Y(V_s)) > 0\right)\\ &\le P\left(\sup_{0\le s\le t}(Y(V_s)) \ge u\right) \le 2P(Y(V_t) > u)\,.\end{aligned}$$

Finally, we have obtained the following result

$$\psi(u,t) \le 2P(Y(V_t) > u)\,. \tag{17}$$

It is worth mentioning that the anomalous diffusion $Y(V_t)$, as a subordination of two independent stochastic processes $Y(\tau)$ and V_t, can be effectively simulated with the help of some standard numerical techniques for α-stable processes (see [18] for details). An exemplary realizations of the risk process $Q(t)$ are presented in Fig. 2. Since $Y(V_t)$ is α/γ-selfsimilar, we have that

$$Y(V_t) \stackrel{d}{=} t^{\alpha/\gamma} Y(V_1)\,.$$

Thus, it is enough to employ the Monte Carlo methods only for calculating the distribution of $Y(V_1)$, in order to compute the probability on the righthand side of (17).

For the more general case, when $Y(\tau)$ is a γ-stable Lévy motion with arbitrary $\gamma \ne 1$ and $|\beta| \le 1$, we can use the following inequality [5]

$$P\left(\sup_{0\le s\le \tau} Y(s) \ge z\right) \le \frac{1}{q}\, P(Y(\tau) > z)\,. \tag{18}$$

Here

$$q = P(Y(\tau) > 0) = \frac{1}{2} + \frac{1}{\pi\alpha}\arctan(\beta\tan(\pi\alpha/2))\,.$$

Then, the similar arguments as those used for the symmetric case, yield the following upper bound for the ruin probability in finite time

$$\psi(u,t) \le \frac{1}{q} P(Y(V_t) > u)\,. \tag{19}$$

Also in this case, we can employ the same numerical techniques, as in the symmetric case, in order to simulate sample paths and to calculate the distribution of $Y(V_t)$.

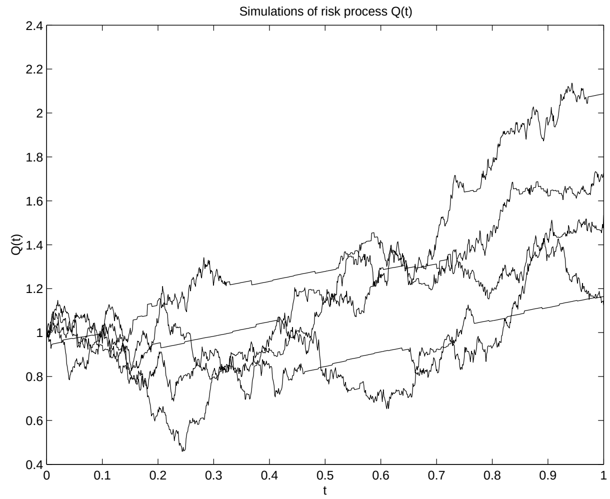

Fig. 2. Five trajectories obtained by simulations of the risk process $Q(t) = u + ct - Y(V_t)$ with parameters $u = 1$ and $c = 1/2$. Here V_t is the inverse α-stable subordinator with $\alpha = 0.9$, and $Y(\tau)$ is the standard Brownian motion.

4. Conclusions

We have constructed the sequence of risk processes that converges in probability to the risk process, whose random part is the well-known anomalous diffusion. The key assumption in our construction was that both the waiting times between successive claims as well as the claim severities are in the domain of attraction of stable distributions. Our considerations and results provide a promising link between the collective risk theory and the anomalous diffusion. In particular, we have presented in detail the relation between our model and the celebrated fractional diffusion equation. We have also derived the upper bounds for the ruin probability in the discussed model and explained, how the problem can be illustrated numerically. We hope that anomalous diffusion approximation of risk processes developed here provides a new tool which may be useful towards the statistical analysis of operational loss data.

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