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ACTIVE BROWNIAN MOTION OF PAIRS AND SWARMS OF PARTICLES*

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Following the route of Smoluchowski we continue the study of single active Brownian particles by investigations of the motion of pairs. After studying free motion we consider the relative motion of bound pairs. We study the attractor structure in a space of five dynamical variables. In particular we investigate the translational motion and analyze the bifurcations between the translational and the rotational modes of the pairs. The influence of noise is studied. Finally, we investigate extensions to the dynamics of N-particle swarms with harmonic interactions.

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1. Introduction

A decade after his first investigations on Brownian motion Smoluchowski started to work on the dynamics of pairs. His motivation was based on some interest in chemical kinetics and in particular in the kinetics of coagulation processes [1]. In this connection Smuluchowski studied the relative Brownian motion of pairs of particles. He quickly realized that the kinetics of pairs may be reduced to the kinetics of single particles and stated: "In this connection it can be easily proven that the relative displacement of two particles which move independently of each other is again given by the ordinary Brownian motion formula. The only difference is that the diffusion coefficient D is equal to the sum of the coefficients of the two particles D_a+D_b" [1]. Another decade later Onsager formulated equations for the Brownian motion of pairs of ions and developed a new conductivity theory of electrolytes on the basis of the pioneering work of Debye and Hückel [2].

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Active Brownian motion is an extension of the concept of Brownian motion to self-propelling particles which was developed mainly by Klimontovich and his school [3, 4]. Here we concentrate on the Brownian motion of pairs of active (self-propelling) particles. This may be considered as a first step to studies of the dynamics of moving swarms of animals. We introduce the general notion of "swarms" for confined systems of particles (or more general objects) driven to states far from equilibrium. The study of living objects like swarms of animals is a rather young field of physical studies (see *e.g.* Refs. [5–7]).

Since the dynamics of swarms of driven particles has attracted the interest of theorists, several interesting effects have been revealed and in part already explained. We mention the early studies of Niwa [12] and the comprehensive survey of Okubo and Levine [13] on swarm dynamics in biophysics and ecology. In the survey of Okubo and Levine we find a classification of the modes of collective motions of swarms of animals. It is pointed out that animal groups have three typical modes of motion:

1. translational motions,
2. rotational excitations and
3. amoeba-like motions.

This classification is confirmed by several experimental studies. For example, Ordemann, Balazsi and Moss [8] studied the modes of swarming of daphnia. Depending on the existence of an external light source in the center, a swarm of daphnia switches from translational motion to rotations around a light shaft in the center.

At present it seems to be impossible to describe all the complex collective motions observed in nature. Instead we study in the following the dynamical modes and the distribution functions of a simple model based on the concept of active Brownian motion. We investigate pairs of Brownian particles confined by linear attracting forces which are self-propelled by active friction. Then we will show that in the case of linear forces, the study of pairs may be easily extended to N-particle systems. This model is considered as a rough picture for the collective motion of non-equilibrium clusters and of swarms of cells and organisms as well [9–11]. For alternative models based *e.g.* on velocity–velocity interactions see Refs. [5–7]. Our special interest is the influence of noise, since noise is responsable for many interesting effects and, in particular, noise may lead to transitions between the deterministic attractors [14–16]. By extending these studies we will investigate here the stochastic transition from translations to rotations analytically.

We begin with the simplest case $N = 2$ and will show that the relative motion of pairs of driven Brownian particles reproduces already some of the typical modes of motions of swarms and in particular the transition from translation to rotation. Needless to say that the study of the motion of pairs may also be useful for studies in chemical kinetics and in other fields where relative motion is of relevance as shown already by Smoluchowski. Finally we extend, within the framework of linear forces, the studies to arbitrary N considering this way global coupling.

2. Stochastic dynamics of free motion

We postulate a dynamics of Brownian particles which is determined by the Langevin equation:

$$\dot{\boldsymbol{r}}_i = \boldsymbol{v}_i\,, \qquad m\dot{\boldsymbol{v}}_i = \boldsymbol{F}_i + \sqrt{2D}\boldsymbol{\xi}(t)\,, \tag{1}$$

where $\xi(t)$ is a stochastic force with strength D and a δ-correlated time dependence:

$$\langle \xi_i(t)\rangle = 0\,, \qquad \langle \xi_i(t)\xi_j(t')\rangle = \delta(t-t')\delta_{ij}\,. \tag{2}$$

The dissipative forces are expressed in the form

$$\boldsymbol{F}_i = -m\gamma(\boldsymbol{v}_i^2)\boldsymbol{v}_i\,. \tag{3}$$

The function $\gamma(x)$ denotes a velocity-dependent friction, which in our model has a negative part. The second term expresses a (small) tendency to synchronize the individual velocity with the swarm velocity $\boldsymbol{V}(t)$ which is the velocity of the center of mass of the swarm. This way the dynamics of our Brownian particles is determined by the Langevin equation with dissipative contributions. In the case of thermal equilibrium systems we have $\gamma(\boldsymbol{v}) = \gamma_0 = \text{const}$. In the general case where the friction is velocity dependent we will assume that the friction is monotonically increasing with the velocity and converges to γ_0 at large velocities. In previous work we often used an ansatz based on the depot model for the energy supply [9, 11]

$$\gamma(\boldsymbol{v}^2) = \left(\gamma_0 - \frac{dq}{c + dv^2}\right)\,, \tag{4}$$

where c, d, q are certain positive constants characterizing the energy flows from the depot to the particle. Dependent on the parameters γ_0, c, d, and q the dissipative force function may have one zero at $\boldsymbol{v} = 0$ or two more zeros with

$$\boldsymbol{v}_0^2 = \frac{d}{c}\zeta\,, \qquad \zeta = \frac{qd}{c\gamma_0} - 1\,. \tag{5}$$

Here ζ is a bifurcation parameter. In the case $\zeta > 0$ a finite characteristic velocity v_0 exists which determines an attractor of motion. Then we speak about active particles. For $|\boldsymbol{v}| < v_0$, the dissipative force is positive, *i.e.*

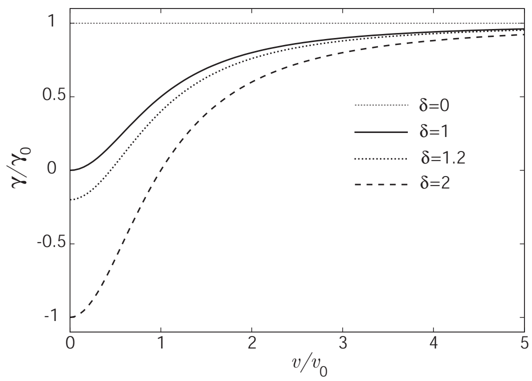

Fig. 1. The typical form of a friction function with active (negative) part at small velocities (parameter $\delta = \zeta + 1$).

the particle is provided with additional free energy. Hence, slow particles are accelerated, while the motion of fast particles is damped (see Fig. 1). Sometimes we will use a simpler expression for the the friction which is valid near to the bifurcation point $\zeta \ll 1$. Formally we obtain this expression by a Taylor expansion cut after the second term which leads to Rayleigh formula containing only 2 constants

$$\gamma(\boldsymbol{v}^2) = \left(-\alpha + \beta v^2\right) . \tag{6}$$

The stationary velocity is then

$$v_0^2 = \frac{\alpha}{\beta} . \tag{7}$$

Let us study now the free (independent) motion of active particles in a two-dimensional space, $\vec{r} = \{x_1, x_2\}$. The stationary solution of the corresponding Fokker–Planck equation for 1 particle reads [11]

$$P_0(\vec{v}_1) = C \left(1 + d v_1^2\right)^{(q/2D_v)} \exp\left[-\frac{\gamma_0}{2D_v} v_1^2\right] , \tag{8}$$

where $D_v = D/m^2$. In the following we will use units with $m = 1$ and $D = D_v$. The mean square displacement is in the limit of strong driving

$$\langle(\vec{r}_1(t) - \vec{r}_1(0))^2\rangle = \frac{v_0^4}{D_v} (2t) . \tag{9}$$

For two particles we introduce the coordinates of the center of mass and the relative motion with respect to the center of mass

$$\vec{R} = \frac{(\vec{r}_1 + \vec{r}_2)}{2} , \qquad \vec{x} = (\vec{r}_1 - \vec{r}_2) . \tag{10}$$

The center of mass shows exactly the same displacement as for one particle. However, the relative motion of pairs shows the double displacement in full agreement with Smoluchowski's findings for the usual Brownian motion.

$$\langle(\vec{x}(t) - \vec{x}(0))^2\rangle = \frac{2v_0^4}{D_v}\,(2t)\,. \tag{11}$$

So far the usual Brownian motion and the active Brownian motion seem to behave in a quite similar way up to the completely different expression for the effective diffusion coefficient (the factor in front of $2t$).

3. Stochastic dynamics of pairs with linear radial attraction

Let us consider two Brownian particles which are pairwise bound by an attracting radial pair potential $U(r_1 - r_2)$. The pair of particles will form dumb-bell like configurations. Then the motion consists of two independent parts: The free motion of the center of mass and the relative motion. The center of mass $\boldsymbol{R} = (\boldsymbol{r}_1 + \boldsymbol{r}_2)/2$ moves with the mass velocity $\boldsymbol{V} = (\boldsymbol{v}_1 + \boldsymbol{v}_2)/2$. The corresponding coordinates are $X_1 = (x_{11} + x_{21})/2$ and $X_2 = (x_{12}+x_{22})/2$. The relative motion under the influence of the forces is described by the relative radius vectors $\boldsymbol{r} = (\boldsymbol{r}_1 - \boldsymbol{r}_2)/2$ and the relative velocity $\boldsymbol{v} = (\boldsymbol{v}_1 - \boldsymbol{v}_2)/2$. The relative coordinates are $x_1 = (x_{11} - x_{12})/2$ and $x_2 = (x_{12} - x_{21})/2$. The deterministic motion of the center of mass is described by the equations

$$m\frac{d}{dt}\boldsymbol{V} = \frac{1}{2}\left[\boldsymbol{F}\left(\boldsymbol{V}+\boldsymbol{v}\right) + \boldsymbol{F}\left(\boldsymbol{V}-\boldsymbol{v}\right)\right]. \tag{12}$$

The relative motion is described by

$$m\frac{d}{dt}\boldsymbol{v} + U'(r)\frac{\boldsymbol{r}}{r} = \frac{1}{2}\left[\boldsymbol{F}\left(\boldsymbol{V}+\boldsymbol{v}\right) - \boldsymbol{F}\left(\boldsymbol{V}-\boldsymbol{v}\right)\right]. \tag{13}$$

For the case of Rayleigh-driving, the dynamical equations have a simpler form

$$m\frac{d}{dt}\boldsymbol{V} \;=\; \left[\alpha - \beta V^2 - \beta v^2\right]\boldsymbol{V} - 2\beta(\boldsymbol{V}\cdot\boldsymbol{v})\,\boldsymbol{v}\,, \tag{14}$$

$$m\frac{d}{dt}\boldsymbol{v} + U'(r)\frac{\boldsymbol{r}}{r} \;=\; \left[\alpha - \beta v^2 - \beta V^2\right]\boldsymbol{v} - 2\beta(\boldsymbol{V}\cdot\boldsymbol{v})\boldsymbol{V}\,. \tag{15}$$

This system possesses two types of attractors. The first attractor corresponds to a translation of the dumb-bell

$$\boldsymbol{V} = v_0\boldsymbol{n}\,, \qquad \boldsymbol{R}(t) = v_0\boldsymbol{n}t + \boldsymbol{R}(0)\,. \tag{16}$$

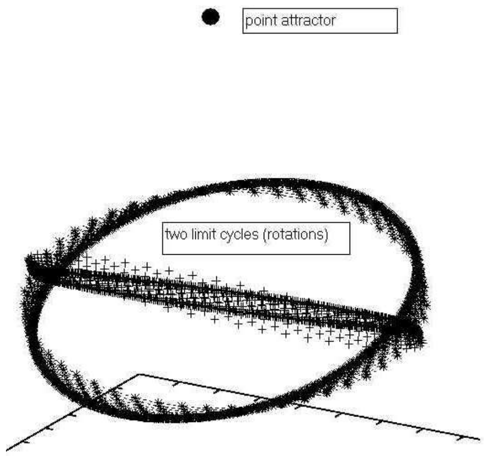

Fig. 2. Schema of the attractor structure: There exist one stable point (above) corresponding to the translational motion of the (not rotating) pair, and two limit cycles (below) corresponding to left/right rotations of the pair at rest.

The second and the third attractors correspond to left and right rotations of the dumb-bell at rest. The principal schema of the attractor structures is shown in Fig. 2. In order to study the dynamical problem in more detail we consider the special case of linear attracting forces: $U = m\omega_0^2 r^2/2$. We introduce further the coordinates $z = V^2$, corresponding to the velocity of the center of mass squared, and the relative coordinates and relative velocities x_1, x_2, v_1, v_2, parallel and perpendicular to the velocity of the center of mass. Then we get the following 5 differential equations

$$\dot{z} = 2z(\alpha - \beta z - 3\beta v_1^2 - \beta v_2^2)\,, \tag{17}$$
$$\dot{v}_1 = v_1(\alpha - 3\beta z - \beta v_1^2 - \beta v_2^2) - \omega_0^2 x_1\,, \tag{18}$$
$$\dot{v}_2 = v_2(\alpha - \beta z - \beta v_1^2 - \beta v_2^2) - \omega_0^2 x_2\,, \tag{19}$$
$$\dot{x}_1 = v_1\,, \tag{20}$$
$$\dot{x}_2 = v_2\,. \tag{21}$$

As a first result of analysis we see that the system has indeed a stable point attractor at $z = \alpha/\beta, v_1 = v_2 = x_1 = x_2 = 0$. The stability analysis shows that this point is linearly stable in all directions except in the direction v_2 corresponding to the motion perpendicular to the translation. In other words, the fixed point corresponding to translation is stable only in the second (quadratic) approximation. The parabola

$$\beta z = (\alpha - \beta v_2^2)\,, \qquad v_1 = 0\,, \tag{22}$$

which connects the stable fixed point at $z = \alpha/\beta$ with the limit cycles at $z = 0$ plays a very special role in the dynamics. This curve has in the first order neutral stability, and correspondingly it shows the character of a saddle. The saddle character facilitates transitions between the attractors along the path (22).

In Figs. 3 and 4 we show several numerical solutions corresponding to initial conditions in the attractor region We see from the figures that typ-

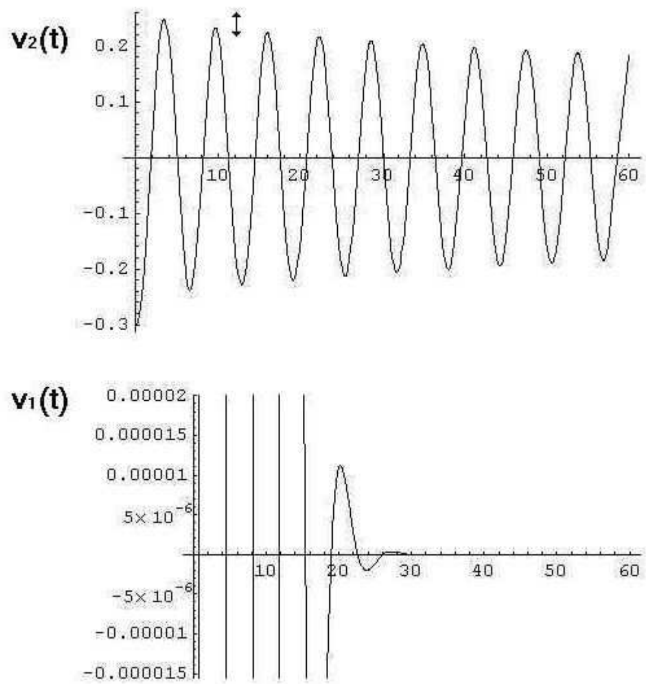

Fig. 3. Solutions of the o.d.e. (17)–(21) for $\alpha = \beta = 1$ and initial conditions corresponding to the region of the translational attractor. We show the transversal velocity $v_2(t)$ (above) and the longitudinal velocity $v_1(t)$ (below) *versus* time. Note the weak damping of the transversal oscillation and the strong damping (and very small amplitudes) of the longitudinal mode.

ically the relative velocity perpendicular to the translation $v_2(t)$ decays to zero very slowly, however the relative velocity $v_1(t)$ is strongly damped. Further the velocity of the center of mass goes to the attractor value α/β in a slow oscillatory way. The attracting region belonging to the stable point $z = z_0 = \alpha/\beta$ is rather complicated, in general it corresponds to the region $|z - z_0| \ll z_0$ but this is not generally true. For example, along the saddle-like curve $\beta z = \alpha - \beta v_2^2, v_1 = x_1 = x_2 = 0$ the attracting region goes far to

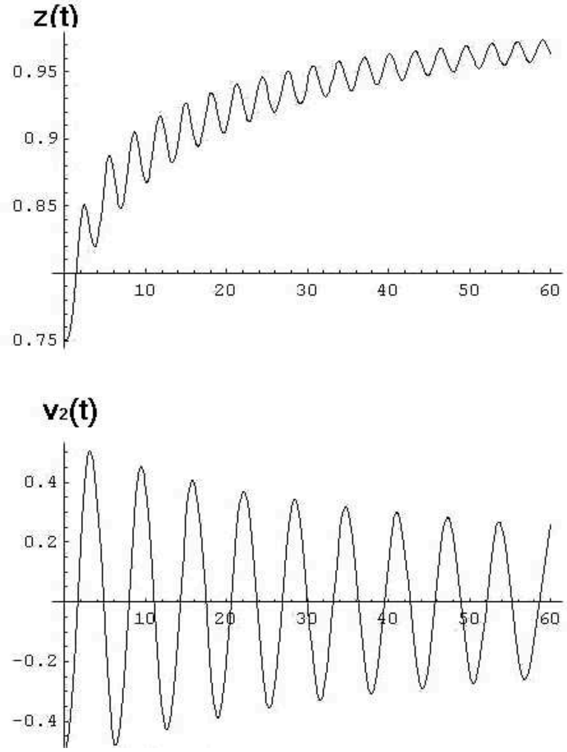

Fig. 4. Solutions of the o.d.e. (17)–(21) for $\alpha = \beta = 1$ in the region of the translational attractor: We show above the center of mass motion $z(t)$, and below the corresponding transversal oscillations $v_2(t)$.

the neighborhood of $z = 0$. Beside the attractor of translational motion at $z = \alpha/\beta$ we find other attractors in the plane $z = 0$ corresponding to rest of the center of mass. Indeed there are two attractors representing left/right rotations (see our schema). The two limit cycles in question have both the projections

$$v_1^2 + v_2^2 = v_0^2\,, \qquad x_1^2 + x_2^2 = (v_0/\omega_0)^2\,, \qquad z = 0\,. \tag{23}$$

Why this attractor is getting unstable if we introduce a translational motion. In order to understand this we introduce a small but finite translation $z_1 \ll v_0^2$. Due to the structure of Eqs. (21) this leads immediately to a destruction of the rotational symmetry of the limit cycles, to an elliptic deformation with the longer axis in the direction perpendicular to the translation. As shown by Erdmann *et al.* [15] the loss of rotational symmetry leads to leaving an Arnold tongue of stability and consequently to a destruction of the limit cycles. We see that the rotations are indeed stable only in and near to the plane $z = 0$ *i.e.* for swarms at rest or near to the resting state.

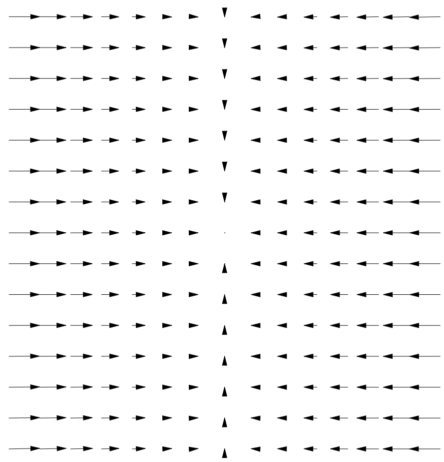

Fig. 5. The vectorfield $v_1 - v_2$ for 2 driven particles with harmonic interaction. We observe the high stability with respect to elongations in z_1-direction and low stability in z_2-direction.

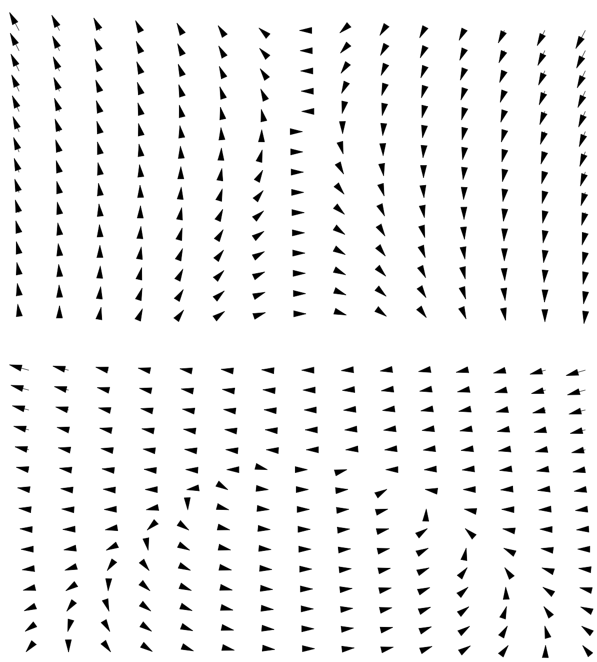

Fig. 6. Vectorfields for 2 driven particles with harmonic interaction: (above: plane v_1 — z, below: plane v_2 — z). The fields illustrate the special role of the parabolic saddle curve.

In order to understand the attractor structure in more detail we will give also several representations of the vector fields corresponding to the d.e. (21). The result of our analysis is shown in Figs. 5 and 6. In order to find the physical meaning of the different attractors we may use a different way of writing the basic dynamical equation

$$\frac{d}{dt}\frac{V^2}{2} = V^2(\alpha - \beta V^2 - \beta v^2 - 2\beta v_1^2)\,, \tag{24}$$

$$\frac{d}{dt}\left[\frac{v^2}{2} + \omega_0^2\frac{r^2}{2}\right] = v^2\left(\alpha - \beta V^2 - \beta v^2 - 2\beta v_1^2\right)\,, \tag{25}$$

$$\frac{d}{dt}\left[\frac{v_1^2}{2} + \omega_0^2\frac{x_1^2}{2}\right] = v_1 2\left(\alpha - \beta V^2 - \beta v^2 - 2\beta V^2\right)\,, \tag{26}$$

$$\frac{d}{dt}\frac{\boldsymbol{r}^2}{2} = \boldsymbol{r}\cdot\boldsymbol{v}\,. \tag{27}$$

Writing the dynamical equations in this form of energy balances — which by the way is valid also in the 3-dimensional case — shows clearly that the most important physical processes are connected with the exchange of energy. In the first attractor state which corresponds to translation of the pair, all the energy is concentrated in the kinetic energy of the center of mass. In the alternative sttractors which correspond to the limit cycles, all the energy is sitting in the rotational motion, the center of mass does not move and does not require energy.

4. The influence of noise on the dynamics

Including noise we expect some distribution around the attractors. Let us estimate the distributions in the simplest case. Decoupling the stochastic equations by replacing v_1^2, v_2^2 by averages we get for the mean velocity

$$\dot{V} = V\left(\alpha - \beta V^2 - 3\beta\langle v_1^2\rangle - \beta\left\langle v_2^2\right\rangle\right) + \sqrt{2D}\xi(t)\,. \tag{28}$$

This leads to the distribution of the center of mass velocity

$$f^{(0)}(\boldsymbol{V}) = C\exp\left[-\frac{1}{D}\left(\alpha_1 V^2 - \beta V^4\right)\right]\,. \tag{29}$$

According to this distribution the most probable velocity is given by

$$V_1^2 = \frac{\alpha_1}{\beta}\,, \qquad \alpha_1 = \alpha - 3\beta\left\langle v_1^2\right\rangle - \beta\left\langle v_2^2\right\rangle\,. \tag{30}$$

This distribution as well as the most probable velocity contain still an unknown constant α_1 which is determined by the distributions of the longitudinal and translational velocities. First, we linearize the equation for the relative longitudinal motion

$$\dot{v}_1 = -2\alpha_1 v_1 - \omega_0^2 x_1 + \sqrt{2D}\xi_1(t)\,. \tag{31}$$

With respect to the transversal dispersion we have some difficulties since strictly speaking they are infinite due to the neutral stability of the transversal velocities [16]. We approximate the Langevin equation for the transversal fluctuations by

$$\dot{v}_2 = -\beta v_2^3 - \omega_0^2 x_2 + \sqrt{2D}\xi_2(t)\,. \tag{32}$$

The corresponding distribution for the velocity fluctuations (which are uncoupled in our approximation) is given by

$$f(v_1, v_2) = C \exp\left[-\frac{1}{2D}\left(2\alpha_1 v_1^2 + \frac{\beta}{2} v_2^4\right)\right]\,. \tag{33}$$

We remind that this distribution corresponds to a driven motion of the center of mass supplemented by a small oscillatory relative motion against the center of mass. The corresponding dispersions are

$$\langle v_1^2 \rangle \simeq \frac{D}{2\alpha_1}. \tag{34}$$

$$\langle v_2^2 \rangle \simeq 2\frac{\Gamma(3/4)\sqrt{D}}{\Gamma(1/4)\sqrt{\beta}} \simeq 0.676\frac{\sqrt{D}}{\sqrt{\beta}}\,. \tag{35}$$

We see in agreement with the findings in [16] that the dispersion of the longitudinal velocity fluctuations with D and the dispersion of the transversal fluctuations goes with $\sqrt{D}$. The dispersions are connected by the relations

$$\alpha_1 = \alpha - \frac{3D}{2\alpha_1}\beta - 0.676\sqrt{\frac{D}{\beta}}\,, \tag{36}$$

$$\alpha_1^2 - \alpha_2\alpha_1 + \frac{3}{2}\beta D = 0; \qquad \alpha_2 \simeq \alpha - 0.676\sqrt{\frac{D}{\beta}}\,. \tag{37}$$

The quadratic equation has the solution

$$\alpha_1 = \frac{1}{2}\alpha_2 + \sqrt{\frac{\alpha_2^2}{4} - \frac{3}{2}\beta D}\,. \tag{38}$$

We see that the dispersion of V^2 is maximal for the critical noise strength

$$D_{\text{cr}} = \frac{\alpha_2^2}{6\beta}\,. \tag{39}$$

For the case $\alpha = \beta = 1$ this gives the critical value $D_{\rm cr} \simeq 0.01$. This is in quite good agreement with the value $D_{\rm cr} \simeq 0.07$ found in a simulation for this choice of parameters [16].

The solutions for the rotational mode are distributed around two limit cycles corresponding to left or right rotations.

Summarizing our findings we may state: For two interacting active particles there exist a translational and a rotational mode. In the rotational mode the center of the "dumb-bell" is at rest and the system is driven to rotate around the center of mass. Only the internal degrees of freedom are excited and we observe driven rotations. In the translational mode of the dumb-bell the center of mass of the dumb-bell makes a driven Brownian motion similar to a free motion of the center of mass.

5. Harmonic swarms with global coupling

This section is devoted to an extension of the previous results to $N > 2$. We consider two-dimensional systems of N point masses m with the numbers $1, 2, \ldots, i, \ldots, N$ and assume that the masses m are confined by linear pair forces $m\omega_0^2 (\boldsymbol{r}_i - \boldsymbol{r}_j)$. This model was first proposed and investigated by Schweitzer *at al.* [14]. The dynamics of the system is given by the following equations of motion

$$\frac{d}{dt}\boldsymbol{r}_i = \boldsymbol{v}_i \,, \qquad m\frac{d}{dt}\boldsymbol{v}_i + m\omega_0^2 (\boldsymbol{r}_i - \boldsymbol{R}(t)) = \boldsymbol{F}_i(\boldsymbol{v}_i) + \sqrt{2D}\boldsymbol{\xi}_i(t) \,. \tag{40}$$

As in the case $N = 2$ we start with an investigation of the translational mode of this system. For the mean velocity we find by summation and expanding around $\boldsymbol{V}$ in a symbolic representation

$$\frac{d}{dt}\boldsymbol{V} = \boldsymbol{F}(\boldsymbol{V}) + \frac{1}{2}(\delta\boldsymbol{v}) \cdot \boldsymbol{F}^{''}(\boldsymbol{V}) \cdot (\delta\boldsymbol{v}) + \ldots \,. \tag{41}$$

In the translational mode of this system all the particles form a noisy flock which moves with nearly constant velocity modulus

$$\boldsymbol{V}(t) = \dot{\boldsymbol{R}}(t) = v_0\boldsymbol{n} \,, \qquad \boldsymbol{r}_i(t) - \boldsymbol{R}(t) = 0; \qquad i = 1, \ldots, N \,. \tag{42}$$

The direction $\boldsymbol{n}$ may change from time to time due to stochastic influences. In order to find explicite results we simplify the equation for the mean momentum $\boldsymbol{V}$ similar as in the previous section for $N = 2$ assuming

$$\frac{d}{dt}\boldsymbol{V} = (\alpha - \beta\boldsymbol{V}^2)(\boldsymbol{V}) - \frac{\beta}{N}\Big\langle \sum_i (\delta v_i)^2 \Big\rangle - 2\frac{\beta}{N}\Big\langle \sum_i (\boldsymbol{V}\delta\boldsymbol{v}_i)\, \delta\boldsymbol{v}_i \Big\rangle + \sqrt{2D}\xi(t) \,. \tag{43}$$

In this way we decouple the center of mass motion from the relative motion. By averaging with respect to δv_i and neglecting the tensor character of the coupling to the relative motion we get

$$\frac{d}{dt}\boldsymbol{V} = (\alpha_1 - \beta \boldsymbol{V}^2)\boldsymbol{V} + \sqrt{2D}\xi(t)\,. \tag{44}$$

Here the effective driving strength α_1 is approximated as in the case $N = 2$ by

$$\alpha_1 = \alpha - \frac{\beta}{N}\sum_i \left[(\delta v_i 1)^2 + (\delta v_i 2)^2\right]\,. \tag{45}$$

The factor α_1 has still to be estimated. The corresponding velocity distribution is

$$f^{(0)}(\boldsymbol{V}) = C\exp\left[\frac{1}{2D}\left(\alpha_1 \boldsymbol{V}^2 - \beta \boldsymbol{V}^4\right)\right]\,. \tag{46}$$

This way we find the most probable velocity

$$\boldsymbol{V}_1{}^2 = \frac{\alpha_1}{\beta}\,. \tag{47}$$

The most probable velocity of the swarm is shifted to values smaller than for the free motion. The shift with respect to the free mode $V_0 = v_0\boldsymbol{n}$ is proportional to the noise strength D. As in the case $N = 2$ this solution breaks down if the dispersion δv^2 is so large that the linearization around V is no more possible. With increasing noise we find a bifurcation. This corresponds to the findings of Erdmann *et al.* [16]. The dispersion in the direction of the flight $\boldsymbol{V}$ is smaller than perpendicular to it. Here we modify this approach and include an analytical study of the bifurcation following same route as in the case $N = 2$.

For the longitudinal fluctuations around the center om mass of the swarm we find

$$\frac{d}{dt}\delta v_{i1} + \omega^2 \delta x_{i1} = -\alpha_1 \delta v_{i1} + \sqrt{2D}\xi_{i1}(t)\,. \tag{48}$$

For the transversal fluctuations against the center om mass of the swarm we find similar as in the previous section

$$\frac{d}{dt}\delta v_{i2} + \omega^2 \delta x_{i2} = -\beta(\delta v_{i2})^3 + \sqrt{2D}\xi_{i2}(t)\,. \tag{49}$$

In this way the distribution of the relative velocities can be approximated as

$$f(\boldsymbol{v}_i) = C\exp\left[-\frac{1}{2D}\left(\alpha_1(\delta v_{i1})^2 + \frac{\beta}{2}(\delta v_{i2})^4\right)\right]\,. \tag{50}$$

Following now the same route as for $N = 2$ we find the same quadratic equation for α_1 as for $N = 2$. This way we find again that the translational mode of V^2 breaks down beyond the critical noise strength

$$D_{\mathrm{cr}} = \frac{(\alpha - 0.676\sqrt{D/\beta})^2}{6\beta}\,. \tag{51}$$

This is connected with the fact that the dispersion of V^2 is real only for strength of noise below this critical value. Above the critical value the roots are complex what is a hint to the existence of the rotational solutions. Our result is quite similar but not identical to earlier findings for $N \gg 1$ [16]. In simulations Erdmann *et al.* found for $\alpha = \beta = 1$ a critical noise strength $D_{\mathrm{cr}} \approx 0.07$ [16]. This is near to our theoretical estimate with which gives $D_{\mathrm{cr}} \simeq= 0.1$. This way we confirmed the numerical findings for $N \ll 1$ by a theoretical estimate.

6. Conclusions

We studied here the active Brownian dynamics of pairs of self-propelled particles with velocity-dependent friction and attracting interactions. Confinement was created by pair-wise linear attracting forces. Our basic results may be summarized as follows:

> By analysis of the simple models for $N = 2$ and $N-$ arbitrary, we could identify analytically two qualitative modes of movement: rotational and translational modes. Due to the noise the Brownian particles may switch from a translational mode to one of two rotational modes. Also transitions between the two rotational modes (limit cycles) are possible, this means inversion of the angular momentum (direction of rotation) [11, 15].

The situation which is for $N = 2$ described by 5 Langevin equations may be extended — at least in the harmonic approximation — to an arbitrary number of particles N. Summarizing our findings we may state the existence of two limit states:

1. Translational motions of the swarm with fluctuating relative positions, the energy is concentrated in the translational degree of freedom.
2. Rotations of the swarm as a whole around the common center of mass, there are no translations, the energy is concentrated in the internal angular momentum.

In spite of the little success we achieved for N-particle systems, the way to develop a more general theory in the spirit of statistical mechanics is still very long [17].

We note however, that the study of dynamic modes of collective movement of pairs or swarms may be of some importance for the understanding of many biological and social collective motions. To support this view we refer again to the book of Okubo and Levin [13] where the modes of collective motions of swarms of animals are classified in way which reminds very much the theoretical finding for the model investigated here. In particular we mention also the motion of animals in water, for example the collective motion of daphnia [8]. Let us conclude with a quotation from Smoluchowsk's work [1]: "One can hope that this theory will prove useful as a guide for more advanced investigations in this area, which up till now has been quite inaccessible to mathematics".

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