# LANGEVIN APPROACH TO LÉVY FLIGHTS IN FIXED POTENTIALS: EXACT RESULTS FOR STATIONARY PROBABILITY DISTRIBUTIONS*

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The functional method to derive the fractional Fokker–Planck equation for probability distribution from the Langevin equation with Lévy stable noise is proposed. For the Cauchy stable noise we obtain the exact stationary probability density function of Lévy flights in different smooth potential profiles. We find confinement of the particle in the superdiffusion motion with a bimodal stationary distribution for all the anharmonic symmetric monostable potentials investigated. The stationary probability density functions show power-law tails, which ensure finiteness of the variance. By reviewing recent results on these statistical characteristics, the peculiarities of Lévy flights in comparison with ordinary Brownian motion are discussed.

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## 1. Introduction

Anomalous diffusion in the form of Lévy flights appears in many physical, chemical, biological, and financial systems [1–4]. Lévy flights are stochastic processes characterized by the occurrence of extremely long jumps. The

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length of these jumps is distributed according to a Lévy stable statistics with a power law tail and divergence of the second moment. This peculiar property strongly contradicts the ordinary Brownian motion, for which all moments of the particle coordinate are finite. The presence of anomalous diffusion can be explained as a deviation of real statistics of fluctuations from Gaussian law, that it has lead to the generalization of the central limit theorem by Lévy and Gnedenko [5–7]. The divergence of the variance of Lévy flights poses some problems as regards to the physical meaning of these processes. However, recently the relevance of Lévy motions appeared in many physical, natural and social complex systems. The Lévy type statistics, in fact, is observed in various scientific areas. Among many interesting examples we cite here the subrecoil laser cooling [8–10], the diffusion by flows in porous media [11], the fluctuations in plasmas [12], the molecular collisions [13], the spatial gazing patterns of bacteria [14], the flights of an albatross [15], the long paleoclimatic time series of the Greenland ice core measurements [16], and the financial time series [17–19]. Experimental evidence of Lévy processes was also observed in the motion of single ion in a one-dimensional optical lattice [20] and in the particle evolution along polymer chains [21, 22].

The problem of the barrier crossing in a bistable potential, the particle escape from a metastable state, and the first passage time density have been analyzed, recently, for Lévy flights [23–35]. The main focus in these papers is to understand how the barrier crossing behavior, according to the Kramers law [36], is modified by the presence of the Lévy noise.

Lévy flights are a special class of Markovian processes, therefore the powerful methods of the Markovian analysis are in force in this case. We mean a possibility to investigate the stationary probability distributions of superdiffusion, the first passage time and the residence time characteristics, the spectral characteristics of stationary motion, *etc.* Of course, this type of diffusion has a lot of peculiarities different from those observed in normal Brownian motion. The main difference from ordinary diffusion consists in replacing the white Gaussian noise source in the underlying Langevin equation with a Lévy stable noise.

In this paper we use functional approach to derive the Fokker–Planck equation, with fractional space derivative, directly from Langevin equation with a Lévy stable noise source. Starting from this equation we find the exact stationary probability distribution (SPD) of fast diffusion in symmetric smooth monostable potentials for the case of Cauchy stable noise. Specifically, we consider symmetric potential profiles $U(x) = \gamma x^{2m}/(2m)$ (with odd $m = 2n + 1$ and even $m = 2n$), describing the dynamics of overdamped anharmonic oscillator driven by Lévy noise. We find that for Lévy flights in steep potential well, with steepness greater or equal to four, the variance of

the particle coordinate is finite. This gives rise to a confined superdiffused motion, characterized by a bimodal stationary probability density, as previously reported in Refs. [37–39]. However, in previous Ref. [39] the authors analyzed the properties of stationary probability distribution for nonlinear Lévy oscillators and its bimodality as a function of the Lévy index $\alpha$, by finding that it is more pronounced for $\alpha = 1$ (Cauchy stable noise) and becomes the Boltzmann stationary distribution in the limit of $\alpha \to 2$. Here we analyze the SPDs as a function of a dimensionless parameter $\beta$, which is the ratio between the noise intensity $D$ and the steepness of the potential profile $\gamma$. We find that the SPDs remain bimodal with increasing $\beta$ parameter, that is with decreasing the steepness $\gamma$ of the potential profile, or by increasing the noise intensity $D$.

## 2. Functional method to derive the fractional Fokker–Planck equation from Langevin equation with Lévy stable noise

Ditlevsen and Yanovsky with co-authors for the first time obtained the fractional Fokker–Planck equation directly from Langevin equation, by replacing the white Gaussian noise with Lévy stable noise [23, 40] (see also [41, 42]). However, some attempts were undertaken before in Ref. [43,44].

The theory of Lévy processes is closely linked to that of infinitely divisible distributions [43–44]. Therefore, starting from this link, we have recently developed a more general approach, based on the theory of infinitely divisible distributions and functional analysis, to derive the generalized Kolmogorov equation for arbitrary non-Gaussian white noise source [48]. Here we obtain the equation for probability distribution from Langevin equation with Lévy stable noise, by a different approach with respect to that reported in Ref. [40].

Let us consider the anomalous overdamped motion in the potential profile $U(x)$

$$\frac{dx}{dt} = -U'(x) + L(t).  \tag{1}$$

Here $x(t)$ is the displacement of particle and $L(t)$ is the symmetric $\alpha$-stable Lévy noise with the characteristic function of increments

$$\langle \exp\{ik[\eta_L(t+\Delta t) - \eta_L(t)]\}\rangle = \left\langle \exp\left\{ik \int_t^{t+\Delta t} L(\tau)\,d\tau\right\}\right\rangle$$
$$= \exp\{-D|k|^\alpha \Delta t\},  \tag{2}$$

where $\eta_L(t)$ is a generalized Wiener process [48, 49] which derivative is the Lévy stable noise $(d\eta_L/dt = L(t))$. Here $\alpha$ is the Lévy exponent $(0 < \alpha < 2)$

and $D$ is the intensity of Lévy noise. The case $\alpha = 1$ corresponds to Lévy noise $L(t)$ with symmetric Cauchy distribution. First of all, we calculate the characteristic functional of the noise $L(t)$.

According to the definitions of the characteristic functional of the random process $L(t)$ and the Stiltjes integral we have

$$
\begin{aligned}
\Theta_t[u] &= \left\langle \exp\left\{i\int_0^t u(\tau) L(\tau) d\tau\right\}\right\rangle = \left\langle \exp\left\{i\int_0^t u(\tau) d\eta_L(\tau)\right\}\right\rangle \\
&= \left\langle \exp\left\{i \lim_{\delta_\tau \to 0} \sum_{k=1}^n u(\vartheta_k)[\eta_L(\tau_k) - \eta_L(\tau_{k-1})]\right\}\right\rangle \\
&= \lim_{\delta_\tau \to 0} \left\langle \prod_{k=1}^n \exp\{iu(\vartheta_k)[\eta_L(\tau_k) - \eta_L(\tau_{k-1})]\}\right\rangle,
\end{aligned}
$$

where $\vartheta_k$ is some internal point of the time interval $(\tau_{k-1}, \tau_k)$, $\delta_\tau = \max_k \Delta\tau_k$, $\Delta\tau_k = \tau_k - \tau_{k-1}$ ($\tau_0 = 0$, $\tau_n = t$). Taking into account that the increments of non-overlapping time intervals of the generalized Wiener process $\eta_L(t)$ are statistically independent and using Eq. (2) we obtain

$$
\begin{aligned}
\Theta_t[u] &= \lim_{\delta_\tau \to 0} \prod_{k=1}^n \langle \exp\{iu(\vartheta_k)[\eta_L(\tau_k) - \eta_L(\tau_{k-1})]\}\rangle \\
&= \lim_{\delta_\tau \to 0} \prod_{k=1}^n \exp\{-D|u(\vartheta_k)|^\alpha \Delta\tau_k\} \\
&= \exp\left\{-D \lim_{\delta_\tau \to 0} \sum_{k=1}^n |u(\vartheta_k)|^\alpha \Delta\tau_k\right\}.
\end{aligned}
\tag{3}
$$

By using the definition of the Riemann integral we finally get

$$
\Theta_t[u] = \exp\left\{-D \int_0^t |u(\tau)|^\alpha d\tau\right\}.
\tag{4}
$$

To derive the fractional Fokker–Planck equation from Langevin equation (1) we need the functional correlational formula for symmetric $\alpha$-stable Lévy noise $L(t)$. We start from the generalization of Furutsu–Novikov formula [50,51] for arbitrary non-Gaussian random process $\xi(t)$, obtained previously in [52],

$$
\langle \xi(t) R_t[\xi + z]\rangle = \left.\frac{\dot\Phi_t[u]}{iu(t)}\right|_{u=\frac{\delta}{i\delta z}} \langle R_t[\xi + z]\rangle,
\tag{5}
$$

where $R_t [\xi]$ is a functional of noise $\xi(t)$, defined on the observation interval $(0, t)$, $z(t)$ is a deterministic function, and $\Phi_t [u] = \ln \Theta_t [u]$. Following Klyatskin, we use the translation functional operator and taking into account that the function $z(t)$ is deterministic we have

$$\langle \xi(t) R_t [\xi + z] \rangle = \left\langle \xi(t) \exp \left\{ \int_0^t \xi(\tau) \frac{\delta}{\delta z(\tau)} d\tau \right\} \right\rangle R_t [z] . \tag{6}$$

For the average entering in Eq. (6), after evident rearrangements, we find

$$\left\langle \xi(t) \exp \left\{ i \int_0^t \xi(\tau) u(\tau) d\tau \right\} \right\rangle = \frac{1}{iu(t)} \frac{d}{dt} \Theta_t [u]$$

$$= \frac{\Theta_t [u]}{iu(t)} \frac{d}{dt} \ln \Theta_t [u]$$

$$= \frac{\dot{\Phi}_t [u]}{iu(t)} \left\langle \exp \left\{ i \int_0^t \xi(\tau) u(\tau) d\tau \right\} \right\rangle. \tag{7}$$

Substituting Eq. (7) in Eq. (6) and using again the functional translation formula we obtain Klyatskin result (5). By using the following integral representation for $|u|^\alpha$

$$|u|^\alpha = \frac{\Gamma(\alpha + 1) \sin(\pi\alpha/2)}{\pi} \int_{-\infty}^{+\infty} \frac{1 - \cos(xu)}{|x|^{1+\alpha}} dx , \tag{8}$$

we rewrite Eq. (4) as

$$\Theta_t [u] = \exp \left\{ -Q \int_0^t d\tau \int_{-\infty}^{+\infty} \frac{1 - \cos(xu(\tau))}{|x|^{1+\alpha}} dx \right\} , \tag{9}$$

where

$$Q = \frac{D\Gamma(\alpha + 1) \sin(\pi\alpha/2)}{\pi} . \tag{10}$$

We obtain, therefore, the following expression for the variational operator in Eq. (5) for Lévy stable noise $L(t)$

$$\frac{\dot{\Phi}_t [u]}{iu(t)} = Q \int_{-\infty}^{+\infty} \frac{e^{ixu(t)} - 1}{iu(t) |x|^{1+\alpha}} dx = Q \int_{-\infty}^{+\infty} \frac{dx}{|x|^{1+\alpha}} \int_0^x e^{iu(t)y} dy . \tag{11}$$

Substituting this expression in Eq. (5) we get

$$\langle L(t) R_t [L + z] \rangle = Q \int_{-\infty}^{+\infty} \frac{dx}{|x|^{1+\alpha}} \int_0^x \exp\left\{ y \frac{\delta}{\delta z(t)} \right\} \langle R_t [L + z] \rangle \, dy. \quad (12)$$

By inserting the operator of functional differentiation into the average and by putting $z = 0$, we get finally

$$\langle L(t) R_t [L] \rangle = Q \int_{-\infty}^{+\infty} \frac{dx}{|x|^{1+\alpha}} \int_0^x \left\langle \exp\left\{ y \frac{\delta}{\delta L(t)} \right\} R_t [L] \right\rangle dy. \quad (13)$$

Now we are ready to derive the fractional Fokker–Planck equation for Lévy flights using the functional approach. By differentiating, with respect to time $t$, the expression for probability density of random process $x(t)$

$$W(x, t) = \langle \delta (x - x(t)) \rangle, \quad (14)$$

and taking into account Eq. (1), we obtain

$$\frac{\partial W}{\partial t} = \frac{\partial}{\partial x} \left[ U'(x) W \right] - \frac{\partial}{\partial x} \langle L(t) \, \delta (x - x(t)) \rangle. \quad (15)$$

To evaluate the average in Eq. (15) we apply the formula (13)

$$\langle L(t) \, \delta (x - x(t)) \rangle = Q \int_{-\infty}^{+\infty} \frac{dz}{|z|^{1+\alpha}} \int_0^z \left\langle \exp\left\{ y \frac{\delta}{\delta L(t)} \right\} \delta (x - x(t)) \right\rangle dy.$$

$$(16)$$

Using functional differentiation rules, from Eq. (1) we get

$$\frac{\delta}{\delta L(t)} \delta (x - x(t)) = -\frac{\partial}{\partial x} \delta (x - x(t)) \frac{\delta x(t)}{\delta L(t)} = -\frac{\partial}{\partial x} \delta (x - x(t)). \quad (17)$$

Thus, the variational operator $\delta/\delta L(t)$ with respect to the functional $\delta(x - x(t))$ is equivalent to the ordinary differential operator $-\partial/\partial x$. As a result, we have

$$\langle L(t) \, \delta (x - x(t)) \rangle = Q \int_{-\infty}^{+\infty} \frac{dz}{|z|^{1+\alpha}} \int_0^z \exp\left\{ -y \frac{\partial}{\partial x} \right\} dy \, W(x, t). \quad (18)$$

After substitution of Eq. (18) in Eq. (15) and evaluation of the internal integral we arrive at

$$\frac{\partial W}{\partial t} = \frac{\partial}{\partial x} \left[ U'(x)W \right] + Q \int\limits_{-\infty}^{+\infty} \left[ \exp\left\{ -z\frac{\partial}{\partial x} \right\} - 1 \right] W(x,t) \frac{dz}{|z|^{1+\alpha}} . \quad (19)$$

By using the property of the translation operator

$$\exp\left\{ -z\frac{d}{dx} \right\} f(x) = f(x-z), \quad (20)$$

we arrive at the following Kolmogorov equation for the probability density of nonlinear systems (1) driven by a symmetric $\alpha$-stable Lévy noise

$$\frac{\partial W}{\partial t} = \frac{\partial}{\partial x} \left[ U'(x)W \right] + Q \int\limits_{-\infty}^{+\infty} \frac{W(x-z,t) - W(x,t)}{|z|^{1+\alpha}} \, dz . \quad (21)$$

The Eq. (21) represents the well-known Fokker–Planck equation with fractional space derivative, which describes superdiffusion in the form of Lévy flights

$$\frac{\partial W}{\partial t} = \frac{\partial}{\partial x} \left[ U'(x)W \right] + D\frac{\partial^\alpha W}{\partial |x|^\alpha} . \quad (22)$$

### 3. Stationary probability distributions for Lèvy flights

First of all, we can try to evaluate the stationary probability distribution $W_{\mathrm{st}}(x)$ from Eq. (22), if it exists. Of course, this evaluation is impossible for any potential profile, but the potential $U(x)$ should satisfy some constraints. It is better to apply Fourier transform to the integro-differential equation (21) and to write the equation for the characteristic function

$$\vartheta(k,t) = \left\langle e^{ikx(t)} \right\rangle = \int\limits_{-\infty}^{+\infty} e^{ikx} W(x,t) \, dx . \quad (23)$$

After simple manipulations we find

$$\frac{\partial \vartheta}{\partial t} = -ik \int\limits_{-\infty}^{+\infty} e^{ikx} U'(x) W(x,t) \, dx - D\,|k|^\alpha\, \vartheta . \quad (24)$$

For smooth potential profiles $U(x)$, expanding in power series near the point $x = 0$, we can rewrite this equation in the operator form

$$\frac{\partial \vartheta}{\partial t} = -ikU'\left(-i\frac{\partial}{\partial k}\right)\vartheta - D\,|k|^{\alpha}\,\vartheta\,. \tag{25}$$

In particular, for stationary characteristic function, from Eq. (25) we get

$$U'\left(-i\frac{d}{dk}\right)\vartheta_{\mathrm{st}} - iD\,|k|^{\alpha-1}\,\mathrm{sgn}\,k\,\vartheta_{\mathrm{st}} = 0\,, \tag{26}$$

where $\mathrm{sgn}\,k$ is the sign function. Unfortunately, one cannot solve Eq. (26) for arbitrary potential $U(x)$ and arbitrary Lévy exponent $\alpha$.

Let us consider, as in [39], the symmetric smooth monostable potential $U(x) = \gamma x^{2m}/(2m)$ $(m = 1, 2, \ldots)$. The Eq. (26), therefore transforms into the following differential equation of $(2m-1)$-order

$$\frac{d^{2m-1}\vartheta_{\mathrm{st}}}{dk^{2m-1}} + (-1)^{m+1}\beta^{2m-1}\,|k|^{\alpha-1}\,\mathrm{sgn}\,k\,\vartheta_{\mathrm{st}} = 0\,, \tag{27}$$

where $\beta = \sqrt[2m-1]{D/\gamma}$. As it was proved by analysis of Eq. (27) in [37], the stationary probability distribution $W_{\mathrm{st}}(x)$ has non-unimodal shape and power tails

$$W_{\mathrm{st}}(x) \sim \frac{1}{|x|^{2m+\alpha-1}}\,, \qquad |x| \to \infty\,. \tag{28}$$

In Ref. [37], the estimation of bifurcation time for transition from unimodal initial distribution to bimodal stationary one and the existence of a transient trimodal state for $m > 2$ were found.

Exact solution of Eq. (27) can be only obtained for the case of Cauchy noise: $\alpha = 1$. Due to the symmetry of the characteristic function $\vartheta_{\mathrm{st}}(-k) = \vartheta_{\mathrm{st}}(k)$ we can reduce Eq. (27) to the linear differential equation with constant parameters

$$\frac{d^{2m-1}\vartheta_{\mathrm{st}}}{dk^{2m-1}} - (-1)^{m}\,\beta^{2m-1}\vartheta_{\mathrm{st}} = 0 \qquad (k > 0)\,. \tag{29}$$

From the corresponding characteristic equation

$$\lambda^{2m-1} = (-1)^{m}\,\beta^{2m-1}\,, \tag{30}$$

we select the roots with negative real part, which are meaningful from physical point of view. The general solution of Eq. (29), therefore, reads

$$\vartheta_{\mathrm{st}}(k) = \sum_{l=0}^{[(m-1)/2]} A_l \exp\left\{-\beta\,|k|\cos\frac{\pi(m-2l-1)}{2m-1}\right\}$$

$$\times \cos\left(\beta\,|k|\sin\frac{\pi(m-2l-1)}{2m-1} - \varphi_l\right)\,, \tag{31}$$

where the quadratic brackets in the upper limit of the sum denote the integer part of the expression. The unknown constants $A_l$ and $\varphi_l$ can be calculated from the obvious conditions

$$\vartheta_{\text{st}}(0) = 1, \quad \vartheta_{\text{st}}^{(2j-1)}(+0) = 0 \quad (j = 1, 2, \ldots, m-1). \qquad (32)$$

Substituting Eq. (31) in Eq. (32) we have

$$\sum_{l=0}^{[(m-1)/2]} A_l \cos \varphi_l = 1, \qquad (33)$$

$$\sum_{l=0}^{[(m-1)/2]} A_l \cos \left[ \frac{\pi(2j-1)(m+2l)}{2m-1} - \varphi_l \right] = 0 \quad (j = 1, 2, \ldots, m-1).$$

Making the reverse Fourier transform of Eq. (31) we obtain the stationary probability distribution (SPD) of the particle coordinate

$$W_{\text{st}}(x) = \frac{\beta}{\pi} \sum_{l=0}^{[(m-1)/2]} A_l \frac{x^2 \cos \left[ \frac{\pi(m-2l-1)}{2m-1} + \varphi_l \right] + \beta^2 \cos \left[ \frac{\pi(m-2l-1)}{2m-1} - \varphi_l \right]}{x^4 - 2x^2\beta^2 \cos \frac{\pi(4l+1)}{2m-1} + \beta^4}. \qquad (34)$$

The parabolic potential profile $U(x) = \gamma x^2/2$ corresponds to a linear system (1). In this situation, from Eqs. (33) and (34) we easily obtain the following obvious result

$$W_{\text{st}}(x) = \frac{\beta}{\pi(x^2 + \beta^2)}, \qquad (35)$$

*i.e.* due to the stability of the Cauchy distribution (35), the probabilistic characteristics of driving noise increments (see Eq. (2)) and Markovian process $x(t)$ are similar.

For quartic potential ($m = 2$), from the set of Eq. (33), we find $A_0 = 2/\sqrt{3}$, $\varphi_0 = \pi/6$. Substituting these parameters in Eq. (34) we obtain

$$W_{\text{st}}(x) = \frac{\beta^3}{\pi(x^4 - x^2\beta^2 + \beta^4)}, \qquad (36)$$

which coincides, for $\beta = 1$, with the result obtained in Ref. [39]. The plots of stationary probability distributions (36) for Lévy flights in symmetric quartic potential for different values of parameter $\beta$ are shown in Fig. 1. The superdiffusion in the form of Lévy flight gives rise to a bimodal stationary probability distribution when the particle moves in a monostable potential,
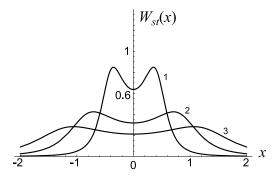
Fig. 1. Stationary probability distributions for Lévy flights in symmetric quartic potential $U(x) = \gamma x^4/4$ for different values of dimensionless parameter $\beta : 1 - \beta = 0.5, 2 - \beta = 1, 3 - \beta = 1.5$.

differently from the ordinary diffusion of the Brownian motion characterized by unimodal SPD.

The SPD of superdiffusion has two maxima at the points $x = \pm\beta/\sqrt{2}$, with the value $(W_{st})_{max} = 4/(3\pi\beta)$. Since the value of the minimum is $W_{st}(0) = 1/(\pi\beta)$, the ratio between maximum and minimum value is constant and equal to $4/3$. The width of probability density increases with increasing parameter $\beta = \sqrt[3]{D/\gamma}$, i.e. with decreasing the steepness $\gamma$ of the quartic potential profile, or with increasing the noise intensity $D$.

Carrying out analogous procedure we obtain the stationary probability distributions for the cases $m = 3, 4, 5$

$$W_{st}(x) = \frac{\beta^5}{\pi(x^2 + \beta^2)(x^4 - 2\beta^2 x^2 \cos\pi/5 + \beta^4)},$$

$$W_{st}(x) = \frac{\beta^7}{\pi(x^4 - 2\beta^2 x^2 \cos\pi/7 + \beta^4)(x^4 + 2\beta^2 x^2 \cos 2\pi/7 + \beta^4)}, \quad (37)$$

$$W_{st}(x) = \frac{\beta^9}{\pi(x^2 + \beta^2)(x^4 - 2\beta^2 x^2 \cos\pi/9 + \beta^4)(x^4 + 2\beta^2 x^2 \cos 4\pi/9 + \beta^4)}.$$

The plots of distributions (37), for different values of parameter $\beta$, are respectively shown in Figs. 2–4. It must be emphasized that according to Figs. 2–4, these distributions remain bimodal and have the same tendency with increasing $\beta$, but the ratio between maximum and minimum increases with increasing $m$. From Eqs. (36) and (37) we see that the second moment of the particle coordinate is finite for $m \geq 2$. This means that there is a confinement of the particle motion due to the steep potential profile, even if the particle moves according to a superdiffusion in the form of Lévy flights [37]. The presence of two maxima is a peculiarity of the superdiffusion motion. Because of the fast diffusion due to Lévy flights, the particle
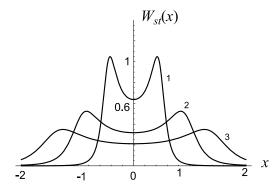
Fig. 2. Stationary probability distributions for Lévy flights in symmetric potential $U(x) = \gamma x^6/6$ for different values of dimensionless parameter $\beta : 1 - \beta = 0.5, 2 - \beta = 1, 3 - \beta = 1.5$.
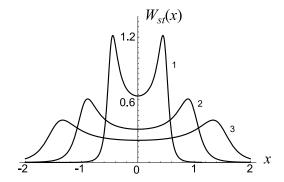


Fig. 3. Stationary probability distributions for Lévy flights in symmetric potential $U(x) = \gamma x^8/8$ for different values of dimensionless parameter $\beta : 1 - \beta = 0.5, 2 - \beta = 1, 3 - \beta = 1.5$.

reaches very quickly regions near the potential walls on the left or on the right with respect to the origin $x = 0$. Then the particle diffuses around this position, until a new flight moves it in the opposite direction to reach the other potential wall. As a result, the particle spends a large time in some symmetric areas with respect to the point $x = 0$, differently from the Brownian diffusion in monostable potential profiles. These symmetric areas lie near the maxima of the bimodal SPD. For fixed $D$ and $m$, these maxima are closer or far away from the point $x = 0$ depending on the greater or smaller steepness $\gamma$ of the potential profile. This corresponds to a greater or smaller confinement of the particle motion. Of course, such a confinement is more pronounced for greater $m$, that is for steeper potential profiles.

On the basis of Eqs. (35)–(37) and the known behavior of density tails (28), we can write the general expressions for stationary probability distri-
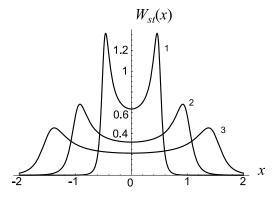
Fig. 4. Stationary probability distributions for Lévy flights in symmetric potential $U(x) = \gamma x^{10}/10$ for different values of dimensionless parameter $\beta : 1 - \beta = 0.5, 2 - \beta = 1, 3 - \beta = 1.5$.

bution in the case of potential $U(x) = \gamma x^{2m}/(2m)$ with odd $m = 2n + 1$

$$W_{\text{st}}(x) = \frac{\beta^{4n+1}}{\pi(x^2 + \beta^2)} \prod_{l=0}^{n-1} \frac{1}{x^4 - 2\beta^2 x^2 \cos[\pi(4l+1)/(4n+1)] + \beta^4}, \quad (38)$$

and even $m = 2n$

$$W_{\text{st}}(x) = \frac{\beta^{4n-1}}{\pi} \prod_{l=0}^{n-1} \frac{1}{x^4 - 2\beta^2 x^2 \cos[\pi(4l+1)/(4n-1)] + \beta^4}, \quad (39)$$

which are, together with Eqs. (36)–(37), the main result of this paper.

## 4. Conclusions

We used functional analysis approach to derive the fractional Fokker–Planck equation directly from Langevin equation with symmetric $\alpha$-stable Lévy noise. This approach allows to describe anomalous diffusion in the form of Lévy flights. We obtained the general formula for stationary probability distribution of superdiffusion in symmetric smooth monostable potential for Cauchy driving noise. All distributions have bimodal shape and become more narrow with increasing steepness of the potential or with decreasing noise intensity. We found that the variance of the particle coordinate is finite for quartic potential profile and for steeper potential profiles, that is a confinement of the particle in a superdiffusion motion in the form of Lévy flights. As a result, we can evaluate the power spectral density of a stationary motion. Calculations of residence times for the case of Lévy flights in bistable

potential with steep potential wells, and anomalous diffusion in periodic ratchet-like potentials are the subjects of forthcoming investigations.

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