ASYMPTOTIC BEHAVIOR OF THE FINITE TIME RUIN PROBABILITY OF A GAMMA LÉVY PROCESS*

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In this paper we consider a jump-diffusion type approximation of the classical risk process by a gamma Lévy process. We derive here the asymptotic behavior (lower and upper bounds) of the finite time ruin probability for any gamma Lévy process.

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1. Introduction

In examining the nature of the risk associated with a portfolio of business in econophysics [1], it is often of interest to assess how the portfolio may be expected to perform over an extended period of time. One approach concerns the use of ruin theory [2, 3]. Ruin theory is concerned with the excess of the income (with respect to a portfolio of business) over the outgo, or claims paid. This quantity, referred to as insurer's surplus, varies in time. Specifically, ruin is said to occur if the insurer's surplus reaches a specified lower bound, *e.g.* minus of the initial capital. One measure of risk is the probability of such an event, clearly reflecting the volatility inherent in the business. In addition, it can serve as a useful tool in long range planning for the use of insurer's funds. The recent increasing interplay between actuarial and financial mathematics has led to surge of risk theoretic modelling, [4].

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Unfortunately, the ruin probabilities in infinite and finite time can only be calculated for a few special cases of the claim amount distribution. Thus, finding a reliable approximation, especially in the ultimate case when the straightforward Monte Carlo approach can not be utilized, is really important from a practical point of view, see [5]. Here we use another (jump-diffusion type) approximation of the classical risk process by a gamma Lévy process, [6] and [7] which permits to find asymptotic behavior of the finite time ruin probability. We give the exact forms of the constants C_1,C_2 and the function g where $C_1 \le \liminf_{u\to\infty} \mathbb{P}(\sup_{t\le T}(Z(t)-ct)>u)/g(u) \le \limsup_{u\to\infty} \mathbb{P}(\sup_{t\le T}(Z(t)-ct)>u)/g(u) \le C_2$ for any $T>0$ and $c>0$.

The finite time ruin probability is an important quantity in risk theory. Computing asymptotic, bounds and exact forms of ruin probability is the key task of risk theory (see *e.g.* [8] and [9]). Ruin probability of Brownian motion, stable processes, compound Poisson processes and Lévy processes is one of the most important problems of fluctuations theory in probability.

Let $\{Z(t) : t \in [0,1]\}$ be a gamma Lévy process that is a stochastic process starting from 0 with stationary, independent increments and $Z(1)$ having gamma distribution with shape parameter $a>0$ and scale parameter $b>0$. Precisely, random variable $Z(1)$ has the following density distribution function

$$f(y) = \begin{cases} 0 & \text{if } y \le 0\,, \\ \frac{1}{b^a \Gamma(a)}\, y^{a-1} \exp\left(-\frac{y}{b}\right) & \text{if } y > 0\,. \end{cases} \tag{1}$$

Shortly, we say that Z is a gamma Lévy process with shape parameter a and scale parameter b.

The aim of the paper is to find an asymptotic behavior of the following probability

$$\mathbb{P}(\sup_{t\le 1}(Z(t)-ct)>u)\,, \tag{2}$$

for any $c>0$ and $u\to\infty$. In our considerations a certain series representation will be crucial. For example, a result from [10] showing that a gamma random variable can be obtained as a shot noise variable, see also [11], gives the following representation

$$Z(t) = b\sum_{k=1}^{\infty} \exp\left(-\frac{\Gamma_k}{a}\right) V_k \boldsymbol{I}\{U_k \le t\}\,, \tag{3}$$

where $0\le t\le 1$ and $\{\Gamma_k\}_{k=1}^{\infty}$ is a sequence of arrival epochs in a Poisson process with unit arrival rate, $\{V_k\}_{k=1}^{\infty}$ is a sequence of iid standard (with parameter equal 1) exponential random variables and $\{U_k\}_{k=1}^{\infty}$ is a sequence of iid random variables uniformly distributed on $[0,1]$. These sequences are independent.

Our computation will rely on conditioning on Γ_1. It is easy to conclude that a gamma Lévy process Z under condition $\Gamma_1 = x$, where $x > 0$ (Γ_1 has exponential distribution with parameter equal 1) can be expressed as follows

$$Z_x(t) = A_x(t) + Y_x(t)\,, \tag{4}$$

where $A_x(t) = be^{-x/a} V_1 \boldsymbol{I}\{U_1 \le t\}$, $Y_x(t) \stackrel{d}{=} be^{-x/a} \sum_{k=1}^{\infty} \exp(-\Gamma_k/a) V_k \times \boldsymbol{I}\{U_k \le t\}$ is a gamma Lévy process with shape parameter a, scale parameter $be^{-x/a}$ and the processes A_x and Y_x are independent.

We write that $g(u) \cong h(u)$ for $u \to \infty$ if $\lim_{u\to\infty}[g(u)/h(u)] = 1$. We will need the following property of the incomplete gamma function

$$\int_u^\infty s^p e^{-s}\, ds = u^p e^{-u} \left(1 + O\left(\frac{1}{u}\right)\right), \tag{5}$$

for $u \to \infty$ where $p \in \mathbb{R}$ which implies that $\int_u^\infty s^p e^{-s}\, ds \cong u^p e^{-u}$.

2. Main result

We derive here the asymptotic properties (lower and upper bounds) of the finite time ruin probability of a gamma Lévy process.

Proposition 2.1 *Let Z be a gamma Lévy process with shape parameter a and scale parameter b. Then for any $c > 0$*

$$C_1 \le \liminf_{u\to\infty} \frac{\mathbb{P}(\sup_{t\le 1}(Z(t) - ct) > u)}{g(u)} \tag{6}$$

$$\le \limsup_{u\to\infty} \frac{\mathbb{P}(\sup_{t\le 1}(Z(t) - ct) > u)}{g(u)} \le C_2\,, \tag{7}$$

where

$$g(u) = u^{a-1} \exp(-u/b)\,, \tag{8}$$

$$C_1 = \frac{\exp(-c/b)}{b^{a-1}\Gamma(a)}\,, \tag{9}$$

and

$$C_2 = \frac{1 - \exp(-c/b)}{cb^{a-2}\Gamma(a)}\,. \tag{10}$$

Let us note that

$$\mathbb{P}(\sup_{t\le 1}(Z(t) - ct) > u) = \int_0^\infty \mathbb{P}(\sup_{t\le 1}(Z_x(t) - ct) > u) e^{-x}\, dx\,. \tag{11}$$

First we derive the upper bound. Since the process Y_x has non-decreasing trajectories we get the following upper bound

$$\begin{aligned}\mathbb{P}(\sup_{t\le 1}(Z_x(t)-ct)>u) &= \mathbb{P}(\sup_{t\le 1}(A_x(t)-ct+Y_x(t))>u)\\ &\le \mathbb{P}(\sup_{t\le 1}(A_x(t)-ct)+Y_x(1)>u)\,. \end{aligned}\tag{12}$$

The density distribution of the random variable $Y_x(1)$ is the following

$$f_x(y)=\begin{cases} 0 & \text{if } y\le 0\,,\\ \frac{e^x}{b^a\Gamma(a)}y^{a-1}\exp\left(-\frac{y}{b}e^{x/a}\right) & \text{if } y>0\,.\end{cases}\tag{13}$$

Thus using independence A_x and Y_x (12) can be computed as follows

$$\begin{aligned}\mathbb{P}(\sup_{t\le 1}(A_x(t)-ct)+Y_x(1)>u) &= \int_0^\infty \mathbb{P}(\sup_{t\le 1}(A_x(t)-ct)>u-y)f_x(y)dy\\ &= \int_0^u \mathbb{P}(\sup_{t\le 1}(A_x(t)-ct)>u-y)f_x(y)dy\\ &\quad+\int_u^\infty f_x(y)\,dy\,.\end{aligned}\tag{14}$$

First let us consider the integrand function in (14) for $y<u$

$$\begin{aligned}\mathbb{P}(\sup_{t\le 1}(A_x(t)-ct)>u-y) &= \mathbb{P}(be^{-x/a}V_1-cU_1>u-y)\\ &= \int_0^1 \mathbb{P}(be^{-x/a}V_1-cs>u-y)\,ds\\ &= \int_0^1 \mathbb{P}(V_1>\frac{u+cs-y}{b}\,e^{x/a})\,ds\\ &= \int_0^1 \exp\left(-\frac{u+cs-y}{b}\,e^{x/a}\right)\,ds\\ &= \exp\left(-\frac{u-y}{b}\,e^{x/a}\right)\int_0^1 \exp\left(-\frac{c}{b}\,e^{x/a}s\right)\,ds\\ &= \frac{b}{c}\,e^{-x/a}\exp\left(-\frac{u-y}{b}\,e^{x/a}\right)\left[1-\exp\left(-\frac{c}{b}\,e^{x/a}\right)\right]\,.\end{aligned}$$

Now we are in a position to calculate (14)

$$\int_0^u \mathbb{P}(\sup_{t\le 1}(A_x(t)-ct)>u-y)f_x(y)\,dy = \int_0^u \frac{b}{c}\,e^{-x/a}\exp\left(-\frac{u-y}{b}\,e^{x/a}\right)$$
$$\times\left[1-\exp\left(-\frac{c}{b}\,e^{x/a}\right)\right]\frac{e^x}{b^a\Gamma(a)}y^{a-1}\exp\left(-\frac{y}{b}e^{x/a}\right)\,dy$$
$$=\frac{e^x}{cb^{a-1}\Gamma(a)}\,e^{-x/a}\exp\left(-\frac{u}{b}\,e^{x/a}\right)\left[1-\exp\left(-\frac{c}{b}\,e^{x/a}\right)\right]\int_0^u y^{a-1}\,dy$$
$$=\frac{e^x u^a}{acb^{a-1}\Gamma(a)}\,e^{-x/a}\exp\left(-\frac{u}{b}\,e^{x/a}\right)\left[1-\exp\left(-\frac{c}{b}\,e^{x/a}\right)\right].$$

Thus by (11) we should integrate (14) with exponential density and using the last calculations we get

$$\int_0^\infty\int_0^u \mathbb{P}(\sup_{t\le 1}(A_x(t)-ct)>u-y)f_x(y)\,dy\,e^{-x}\,dx$$
$$=\frac{u^a}{acb^{a-1}\Gamma(a)}\int_0^\infty e^{-x/a}\exp\left(-\frac{u}{b}\,e^{x/a}\right)\left[1-\exp\left(-\frac{c}{b}\,e^{x/a}\right)\right]\,dx\,,$$

substituting $s=(u/b)\,e^{x/a}$ we continue

$$=\frac{u^{a+1}}{cb^a\Gamma(a)}\left[\int_{u/b}^\infty s^{-2}e^{-s}\,ds-\int_{u/b}^\infty s^{-2}e^{-s(1+c/u)}\,ds\right],$$

substituting $w=s(1+c/u)$ in the second integral we get

$$=\frac{u^{a+1}}{cb^a\Gamma(a)}\left[\int_{u/b}^\infty s^{-2}e^{-s}\,ds-\left(1+\frac{c}{u}\right)\int_{(u+c)/b}^\infty w^{-2}e^{-w}\,dw\right],$$

using (5) we obtain

$$\cong\frac{1-\exp(-c/b)}{cb^{a-2}\Gamma(a)}\,u^{a-1}\exp(-u/b)(1+O(1/u)).\tag{15}$$

Now let us integrate (14)

$$\begin{aligned}
\int_0^\infty \int_u^\infty f_x(y)\,dy\,e^{-x}\,dx &= \int_u^\infty dy \int_0^\infty f_x(y)e^{-x}\,dx \\
&= \frac{1}{b^a\Gamma(a)} \int_u^\infty dy\,y^{a-1} \int_0^\infty \exp\left(-\frac{y}{b}e^{x/a}\right)\,dx\,,
\end{aligned}$$

substituting $s=(y/b)\,e^{x/a}$ we proceed

$$\begin{aligned}
&= \frac{a}{b^a\Gamma(a)} \int_u^\infty dy\,y^{a-1} \int_{y/b}^\infty s^{-1}e^{-s}\,ds \\
&= \frac{a}{b^a\Gamma(a)} \int_{u/b}^\infty ds\,s^{-1}e^{-s} \int_u^{bs} y^{a-1}\,dy \\
&= \frac{1}{b^a\Gamma(a)} \int_{u/b}^\infty s^{-1}e^{-s}(b^a s^a - u^a)\,ds \\
&= \frac{1}{\Gamma(a)} \int_{u/b}^\infty s^{a-1}e^{-s}\,ds - \frac{u^a}{b^a\Gamma(a)} \int_{u/b}^\infty s^{-1}e^{-s}\,ds \\
&= \frac{1}{b^{a-1}\Gamma(a)}\,u^{a-1}e^{-u/b}(1+O(1/u)) - \frac{1}{b^{a-1}\Gamma(a)}\,u^{a-1}e^{-u/b}(1+O(1/u)) \\
&= \frac{1}{b^{a-1}\Gamma(a)}\,u^{a-1}\exp(-u/b)\,O(1/u)\,, \qquad (16)
\end{aligned}$$

where in the second last equality we use (5). Combining (15) and (16) we get (7).

Now we consider the following lower bound for the finite time ruin probability

$$\begin{aligned}
\mathbb{P}(\sup_{t\le 1}(Z(t)-ct)>u) &\ge \mathbb{P}(Z(1)-c>u) \\
&= \mathbb{P}(Z(1)>u+c) \\
&= \frac{1}{b^a\Gamma(a)} \int_{u+c}^\infty y^{a-1}\,e^{-y/b}\,dy
\end{aligned}$$

$$\begin{aligned} &= \frac{1}{\Gamma(a)} \int\limits_{(u+c)/b}^{\infty} s^{a-1}\, e^{-s}\, ds \\ &\cong \frac{1}{\Gamma(a)} \left(\frac{u+c}{b}\right)^{a-1} e^{-(u+c)/b} \\ &\cong \frac{\exp(-c/b)}{b^{a-1}\Gamma(a)}\, u^{a-1} \exp(-u/b)\,, \end{aligned}$$

where in the third equality we substitute $s = y/b$ and in the second last we use (5) which gives (6).

3. Conclusions

Observe that if we admit $c \downarrow 0$, then $C_1 \to \frac{1}{b^{a-1}\Gamma(a)}$ and $C_2 \to \frac{1}{b^{a-1}\Gamma(a)}$ and Proposition 2.1 gives the exact asymptotic probability because

$$\begin{aligned} \mathbb{P}(\sup_{t\le 1} Z(t) > u) &= \mathbb{P}(Z(1) > u) \\ &= \frac{1}{b^a \Gamma(a)} \int\limits_{u}^{\infty} y^{a-1} e^{-y/b}\, dy \\ &= \frac{1}{\Gamma(a)} \int\limits_{u/b}^{\infty} s^{a-1} e^{-s}\, ds \\ &\cong \frac{1}{b^{a-1}\Gamma(a)}\, u^{a-1} \exp\left(-u/b\right)\,, \end{aligned}$$

where in the second last equality we substitute $s = y/b$ and in the last one we use (5).

If we consider gamma Lévy process Z on the non-negative half-line that is $\{Z(t) : t \in [0, \infty)\}$ it is easy to conclude the following result.

Let Z be a gamma Lévy process with shape parameter a and scale parameter b. Then for any $T > 0$ and $c > 0$

$$C_1 \le \liminf_{u\to\infty} \frac{\mathbb{P}(\sup_{t\le T}(Z(t) - ct) > u)}{g(u)} \tag{17}$$

$$\le \limsup_{u\to\infty} \frac{\mathbb{P}(\sup_{t\le T}(Z(t) - ct) > u)}{g(u)} \le C_2\,, \tag{18}$$

where

$$g(u) = u^{aT-1} \exp(-u/b)\,, \tag{19}$$

$$C_1 = \frac{\exp(-cT/b)}{b^{aT-1}\Gamma(aT)} \tag{20}$$

and

$$C_2 = \frac{1-\exp(-cT/b)}{cTb^{aT-2}\Gamma(aT)}\,. \tag{21}$$

Indeed, since

$$\mathbb{P}(\sup_{t\le T}(Z(t)-ct)>u) \;=\; \mathbb{P}(\sup_{t\le 1}(Z(Tt)-cTt)>u)$$

and the process $Z'(t) = Z(Tt)$ is a gamma Lévy process with shape parameter aT and scale parameter b we need to put $a := aT$ and $c := cT$ in Proposition 2.1.

Let us notice finally that $C_1 \downarrow 0$ and $C_2 \downarrow 0$ as $c \to \infty$. The plot of the functions $C_1 = C_1(c)$ and $C_2 = C_2(c)$ for $a = 2$, $b = 1$ and $T = 1$ is given in Fig. 1.

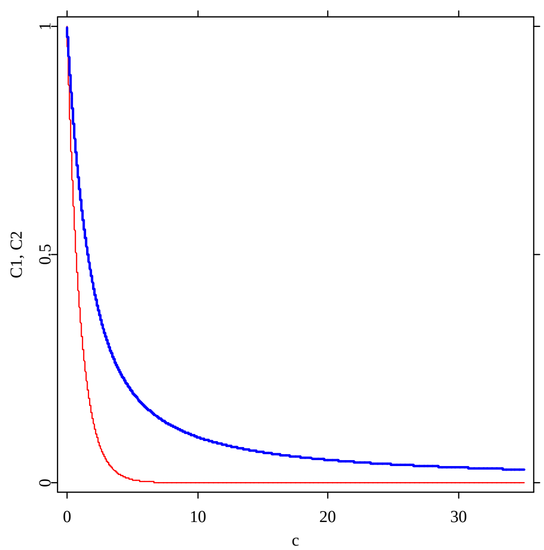

Fig. 1. Plot of the constants C_1 and C_2 as a function of c for $a = 2$, $b = 1$ and $T = 1$ (C_1 — thin line, C_2 — thick line).

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