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INVERSE PROBLEM OF VARIATIONAL CALCULUS FOR NONLINEAR EVOLUTION EQUATIONS

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We couple a nonlinear evolution equation with an associated one and derive the action principle. This allows us to write the Lagrangian density of the system in terms of the original field variables rather than Casimir potentials. We find that the corresponding Hamiltonian density provides a natural basis to recast the pair of equations in the canonical form. Amongst the case studies presented the KdV and modified KdV pairs exhibit bi-Hamiltonian structure and allow one to realize the associated fields in physical terms.

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1. Introduction

In the calculus of variations one is concerned with two types of problems, namely, the direct and inverse problems. The direct problem is essentially the conventional one in which one first assigns a Lagrangian and then computes the equation of motion through Euler–Lagrange equations. As opposed to this, the inverse problem begins with the equation of motion and then constructs a Lagrangian consistent with the variational principle [1]. The object of the present work is to derive an uncomplicated method for the Lagrangian representation of nonlinear evolution equations. We shall see that our results for the Lagrangian densities provide a natural basis to recast these equations in the Hamiltonian form [2, 3].

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Studies in the Hamiltonian structure of nonlinear evolution equations are based on a mathematical formulation that does not make explicit reference to Lagrangians [4]. We feel that the Lagrangian formulation of these equations should be quite interesting because Lagrangian densities, via the Legendre map, will give us a direct route to construct the expressions for Hamiltonian densities that characterize the equation of Zakharov, Faddeev [2] and Gardner [3]. To gain some weightage for the physical and mathematical motivation of our work we proceed by noting the following.

Let $P[v] = P(x, v^{(n)}) \in \mathcal{A}^r$ be an r-tuple of differentiable function. The Fréchet derivative of P is the differential operator D_P: $\mathcal{A}^q \to \mathcal{A}^r$ defined by

$$D_P(Q) = \frac{d}{d\in}\bigg|_{\in=0} P[v + \epsilon Q[v]] \tag{1}$$

for any $Q \in \mathcal{A}^q$. If $\mathcal{D} = \sum\limits_J P[u] D_J$, $P_J \in \mathcal{A}$ is a differential operator, its adjoint $D^\star$ is given by

$$D^\star = \sum_J (-D)_J \,.\, P_J \,. \tag{2}$$

Helmholtz theorem for inverse variational problem [5] asserts that any nonlinear evolution equation $u_t = P[u]$ will have a Lagrangian representation only if D_P is self-adjoint. When the self-adjointness is guaranteed, a Lagrangian density $\mathcal{L}$ for $P[u]$ can be explicitly constructed using the homotopy formula $\mathcal{L}[v] = \int_0^1 vP[\lambda v] d\lambda$. A single evolution equation is never an Euler–Lagrange expression. One common trick to put a single evolution equation into the variational form is to replace u by a potential function w with $u = -w_x$. This yields $w_{xt} = P[w_x]$.The function w is often called the Casimir potential. For many nonlinear evolution equations the Fréchet derivative of $P[w_x]$ is self adjoint, while that of $P[u]$ is not. In view of this, the Lagrangian densities for single evolution equations are always written in terms of partial derivatives of w. There are some equations for which even $P[w_x]$ is not self-adjoint. These are often referred to as non-Lagrangian. The well-known Burgers equations and nonlinear evolution equations with nonlinear dispersive terms [6] serve as typical examples of non-Lagrangian equations.

Keeping the above in view we shall work out a Lagrangian representation without taking recourse to the use of Casimir potentials. In Section 2 we introduce a suitable associated equation for an additional field variable $v(x,t)$ and use it to write the action principle for any nonlinear evolution equation. We express the Lagrangian density in terms of $u(x,t)$, $v(x,t)$ and their partial derivatives. We show in Section 3 that the coupled set of equations as introduced by us form a Hamiltonian system. We also cite examples in which the associated fields admit simple physical realization. In Section 4 we summarize our outlook on the present work and make some concluding remarks.

2. Lagrangian representation

The basic philosophy we use here to construct expressions of Lagrangian densities of nonlinear evolution equations has a rather old root in the classical-mechanics literature. For example, as early as 1931, Bateman [7] allowed for an additional degree of freedom to bring the equation of motion for the damped harmonic oscillator within the framework of action principle. The Bateman Lagrangian

$$L = \dot{x}\dot{y} + \frac{\gamma}{2}(x\dot{y} - y\dot{x}) - \omega^2 xy \tag{3}$$

for the one-dimensional linearly damped harmonic oscillator

$$\ddot{x} + \gamma\dot{x} + \omega^2 x = 0\,, \quad x = x(t) \tag{4}$$

has a 'mirror image' equation

$$\ddot{y} - \gamma\dot{y} + \omega^2 y = 0 \quad y = y(t) \tag{5}$$

for the associated coordinate $y(t)$. Here γ is the co-efficient of friction and ω, the natural frequency of the oscillator. The overdots stand for derivatives with respect to time t. Understandably, the complementary equation in (5) represents a physical system which absorbs energy dissipated in the first. Interestingly, Bateman [7] regarded a dissipative system as physically incomplete such that one needs to bring in an additional equation to derive the original one from an action principle. Thus it remains a real curiosity to envisage a similar study in the context of classical fields and look for the Lagrangian representation of nonlinear evolution equations.

Any nonlinear evolution equation that has at least one conserved density $\rho[u]$ can be written in the form

$$u_t + \frac{\partial}{\partial x}\rho[u] = 0\,, \quad u = u(x,\,t)\,. \tag{6}$$

This single evolution equation is non-Lagrangian. When written in terms of the Casimir potential, the equation resulting from (6) may be either Lagrangian or non-Lagrangian. However, we can make use of an elementary lemma to get a Lagrangian representation of (6).

Lemma 1. There exists a prolongation of (6) into another equation

$$v_t + \frac{\delta}{\delta u}(\rho[u]v_x) = 0\,, \qquad v = v(x,\, t) \tag{7}$$

with the variational derivative

$$\frac{\delta}{\delta u} = \sum_{k=0}^{n}(-1)^k \frac{\partial^k}{\partial x^k}\frac{\partial}{\partial u_{kx}}\,, \qquad u_{kx} = \frac{\partial^k u}{\partial x^k} \tag{8}$$

such that the system of equations follows from the action principle

$$\delta \int \mathcal{L}\; dxdt = 0\,. \tag{9}$$

Here $\mathcal{L}$ stands for the Lagrangian density.

Proof. For a direct proof of the lemma let us introduce $\mathcal{L}$ in the form

$$\mathcal{L} = \tfrac{1}{2}(vu_t - uv_t) - \rho[u]v_x\,. \tag{10}$$

From (9) and (10) we obtain the Euler–Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial v_t}\right) - \frac{\delta \mathcal{L}}{\delta v} = 0 \tag{11}$$

and

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial u_t}\right) - \frac{\delta \mathcal{L}}{\delta u} = 0\,. \tag{12}$$

Using (10) in (11) and (12) we obtain (6) and (7), respectively.

We shall now consider two examples of physical interest and apply the rule in (7) to construct the associated equations. We first focus our attention on the Korteweg–de Vries (KdV) equation

$$u_t + 6uu_x + u_{3x} = 0\,. \tag{13}$$

The KdV equation represents the prototypical nonlinear evolution equation that was first solved by the inverse spectral transform method [8]. From (6) and (13) we see that for this equation

$$\rho[u] = 3u^2 + u_{2x}\,. \tag{14}$$

Using (14) in (7) we get the associated equation

$$v_t + 6uv_x + v_{3x} = 0\,. \tag{15}$$

The corresponding Lagrangian density as obtained from (10) reads

$$\mathcal{L} = \tfrac{1}{2}(vu_t - uv_t) - (3u^2 + u_{2x})v_x \,. \tag{16}$$

It is easy to verify that (16), when substituted in (11), reproduces the KdV equation. The second example of our interest is the so-called modified KdV (mKdV) equation given by

$$u_t + 6u^2u_x + u_{3x} = 0 \,. \tag{17}$$

Equation (13) and (17) are connected by Miura transform and as with (13), (17) can also be solved by the inverse spectral method [9]. The mKdV equation appears in a number of applicative contexts including description of Alfvén waves in a collisionless plasma. The associated equation for (17) is obtained in the form

$$v_t + 6u^2v_x + v_{3x} = 0 \tag{18}$$

with the Lagrangian density given again by (10). For the mKdV equation

$$\rho[u] = 2u^3 + u_{2x} \,. \tag{19}$$

Results similar to those for KdV and mKdV equations can also be written for other nonlinear evolution equations which can be expressed in the form (6). We give in Table I, the results for a number of evolution equations which are often believed to be non-Lagrangian.

The first two equations in the table are due to Burgers. These are dissipative and do not support soliton solutions. However, both of them are useful in the study of acoustics and shock waves [10]. In the recent past two of us [11] studied the equations in the Burgers hierarchy and sought a Lagrangian representation in which the appropriate equations were expressed in terms of Casimir potentials. However, the results represented here have the obvious virtue of simplicity and directness. The compound KdV–Burgers equation [12] describes wave propagation in which the effects of nonlinearity, dissipation and dispersion are all present. We believe that the Lagrangian representation of these equations using associated equations is quite interesting. The KdV and mKdV equations are quasi-linear in the sense that the dispersive behaviour of the solution of each equation is governed by a linear term giving the order of the equation. In contrast to this, the third and fifth order equations (FNE3) and (FNE5) [6] in the table are nonlinear partial differential equations with nonlinear dispersive terms. These are, therefore, fully nonlinear evolution (FNE) equations. The solitary wave solutions of these equations have a compact support. To the best of our knowledge no Lagrangian representations of FNE3 and FNE5 have yet been found. There have, however, been attempts [13] to introduce Lagrangian system of FNE equations which support compacton solutions.

TABLE I

Associated equations for a few important nonlinear evolution equations.

Evolution equation	Conserved density	Associated equation
Burgers2: $u_t - u_{2x} - 2uu_x = 0$	$-(u_x + u^2)$	$v_t + v_{2x} - 2uv_x = 0$
Burgers3: $u_t - u_{3x} - 3u^2u_x$ $-3uu_{2x} - 3u_x^2 = 0$	$-(u_{2x} + u^3 + 3uu_x)$	$v_t + 3uv_{2x} - 3u^2v_x$ $-v_{3x} = 0$
KdV–Burgers: $u_t + uu_x - \nu u_{2x}$ $+\mu u_{3x} = 0$	$\frac{1}{2}u^2 - \nu u_x + \mu u_{2x}$	$v_t + uv_x + \nu v_{2x}$ $+\mu v_{3x} = 0$
FNE3: $u_t + 3u^2u_x + 6u_xu_{2x}$ $+2uu_{3x} = 0$	$u^3 + 2u_x^2 + 2uu_{2x}$	$v_t + 3u^2v_x + 2uv_{3x} = 0$
FNE5: $u_t + \beta_1(u^2)_x + \beta_2(u^2)_{3x}$ $+\beta_3(u^2)_{5x} = 0$	$\beta_1u^2 + \beta_2(u^2)_{2x} + \beta_3(u^2)_{4x}$	$v_t + 2\beta_1uv_x + 2\beta_2uv_{3x}$ $+2\beta_3uv_{5x} = 0$

3. Canonical structure

Zakharov and Faddeev [2] developed the Hamiltonian approach to integrability of nonlinear evolution equations in one spatial and one temporal (1+1) dimensions and Gardner [3], in particular, interpreted the KdV equation as a completely integrable Hamiltonian system with ∂_x as the relevant Hamiltonian operator. In this context we introduce the following lemma for the Hamiltonian structure of the coupled equations introduced by us.

Lemma 2. The Hamiltonian density $\mathcal{H}$ constructed from (10), can be used to express (6) and (7) in the Hamiltonian form

$$\eta_t = J\frac{\delta\mathcal{H}}{\delta\eta}\,, \tag{20}$$

with

$$\eta = \begin{pmatrix} u \\ v \end{pmatrix} \tag{21a}$$

and the symplectic matrix

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{21b}$$

Proof. Using the Legendre map the Hamiltonian density for the Lagrangian in (10) is obtained as

$$\mathcal{H} = \rho[u]v_x \,. \tag{22}$$

From (20), (21) and (22), equations in (6) and (7) follow immediately.

A significant development in the Hamiltonian theory is due to Magri [14] who realized that completely integrable Hamiltonian system have an additional structure. They are bi-Hamiltonian systems, *i.e.*, they are Hamiltonian with respect to two different compatible Hamiltonian operators. We have found that both KdV and mKdV can be recast in the bi-Hamiltonian form

$$\eta_t = J_1 \frac{\delta \mathcal{H}_2}{\delta \eta} = J_2 \frac{\delta \mathcal{H}_1}{\delta \eta} \,. \tag{23}$$

The appropriate results for Hamiltonian operators and Hamiltonian densities are given by

$$J_1^{\mathrm{KdV/mKdV}} = J \,, \tag{24}$$

$$J_2^{\mathrm{KdV}} = \begin{pmatrix} 0 & (\partial_x^3 + 2u\partial_x + 2\partial_x u)\partial_x^{-1} \\ -(\partial_x^2 + 4u) & 2v_x\partial_x^{-1} \end{pmatrix}, \tag{25}$$

$$J_2^{\mathrm{mKdV}} = \begin{pmatrix} 0 & (\partial_x^3 + 2u^2\partial_x + \frac{4}{3}\partial_x u^2)\partial_x^{-1} \\ -(\partial_x^2 + 4u^2) & 2uv_x\partial_x^{-1} \end{pmatrix}, \tag{26}$$

$$\mathcal{H}_1^{\mathrm{KdV/mKdV}} = uv_x \,, \tag{27}$$

$$\mathcal{H}_2^{\mathrm{KdV}} = (3u^2 + u_{2x})v_x \tag{28}$$

and

$$\mathcal{H}_2^{\mathrm{mKdV}} = (2u^3 + u_{2x})v_x \,. \tag{29}$$

In solving the inverse variational problem for nonlinear evolution equations we coupled the field variable $u(x,t)$ of some given equation with the field variable $v(x,t)$ of an associated equation such that the system follows from the action principle with a prescribed form of the Lagrangian density as given in (10). Admittedly, one of our tasks in this work will be to study the nature of $v(x,t)$ when the original field variable $u(x,t)$ admits simple physical realization. Keeping this in view, we focus our attention on the KdV and mKdV equations which support soliton solutions. It is well-known that the solutions of (13) and (17) are given in the general form

$$u^{\mathrm{KdV}}(x,t) = A(k)\mathrm{sech}^2(kx - 4k^3t) \,, \tag{30}$$

$$u^{\mathrm{mKdV}}(x,t) = B(k)\mathrm{sech}(kx - k^3t) \,, \tag{31}$$

where k is the wave number for a single bound-state energy of the potential that characterizes the spectral problem in solving the evolution equation [8]. Here $A(k)$ and $B(k)$ represent the amplitudes of the bright solitons in (30) and (31). Understandably, both solutions $u^{\mathrm{KdV}}(x,t)$ and $u^{\mathrm{mKdV}}(x,t)$ are centered at $x=0$. The KdV solution moves to the right with speed $4k^2$ while the mKdV solution moves in the same direction with speed k^2 only. One can verify that the associated fields corresponding to $u^{\mathrm{KdV}}(x,t)$ and $u^{\mathrm{mKdV}}(x,t)$ are given by the dark soliton solutions

$$v^{\mathrm{KdV/mKdV}}(x,t) = \tanh(kx - 4k^3 t)\,. \tag{32}$$

From (30) and (32) it is clear that both soliton solutions of the KdV pair move with equal speed. As opposed to this, comparison of (31) and (32) reveals that the speed of the dark soliton solution for the mKdV pair is four times the speed of the bright soliton solution. This is not immediately clear to us and deserves extensive numerical study for further clarification.

4. Concluding remarks

Nonlinear evolution equations are not directly amenable to Lagrangian representation. We have established that if (6) does not admit a direct analytic or Lagrangian representation, then there exists an auxiliary or associated field which helps us treat the original evolution equation within the framework of the action principle. The method is quite general and works for both integrable and nonintegrable equations. We have dealt with equations in which the effects of nonlinearity, dissipation and dispersion are all present.

It is a well-known belief that there is no exact method for applying variational principle to dissipative systems. In view of this, studies in Lagrangian and Hamiltonian mechanics of nonconservative systems are still regarded as an interesting curiosity. In the context of point mechanics Riewe [15] used the method of fractional calculus to write a Lagrangian for (4) without taking help of the additional equation in (5). Kaup and Malomed [16] sought an application of the variational principle to nonlinear field equations involving dissipative terms. The Ansatz for the Lagrangian density used by these authors is simply related to our expression for $\mathcal{L}$ in Lemma 1. For example, we can add the gauge term $\frac{d}{dt}\left(\frac{1}{2}uv\right) + \frac{d}{dx}\left(3u^2v\right) + \frac{d}{dx}\left(u_{2x}v\right)$ on the right side of (16) and write

$$\mathcal{L} = v\left(u_t + 6uu_x + u_{3x}\right)\,. \tag{33}$$

In this context we note that (33) was originally suggested by Atherton and Homsy [17] and subsequently included as an exercise (5.37, p. 184) in

Ref. [5]. However, it appears that there is no physically founded assumptions in writing $\mathcal{L}$ in this form except that a Lagrangian may involve its own equation of motion provided one introduces a new concept of variational symmetry called the s-equivalence [18]. On the other hand, the treatment presented here is based on a formalism that is specially intended to bring out the reasons why nonlinear evolution equations, as such, do not follow from the action principle.

The KdV-like equations support bright solitons while the associated fields have soliton solutions which are dark. The coupled set of equations for the bright and dark solitons are Lagrangian although, individually, each of them is non-Lagrangian. This observation appears to bring in some similarity with the celebrated work of Bateman [7] on the dissipative system in particle dynamics.

The appearance of dark solitons in the solutions of our coupled equations is not a strange phenomenon. In the past, dynamics of dark solitons induced by stimulated Raman effect in the optical fiber were explained by the use of KdV–Burgers equation [19]. We feel that the inverse variational problem as treated here will be useful to study Noether symmetries and even to construct exact solutions of the nonlinear evolution equations [20].

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