MATTER COLLINEATIONS OF STATIC SPACETIMES WITH MAXIMAL SYMMETRIC TRANSVERSE SPACES

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This paper is devoted to study the symmetries of the energy-momentum tensor for the static spacetimes with maximal symmetric transverse spaces. We solve matter collineation equations for the four main cases by taking one, two, three and four non-zero components of the vector ξ^a. For one component non-zero, we obtain only one matter collineation for the non-degenerate case and for two components non-zero, the non-degenerate case yields maximum three matter collineations. When we take three components non-zero, we obtain three, four and five independent matter collineations for the non-degenerate and for the degenerate cases respectively. This case generalizes the degenerate case of the static spherically symmetric spacetimes. The last case (when all the four components are non-zero) provides the generalization of the non-degenerate case of the static spherically symmetric spacetimes. This gives either four, five, six, seven or ten independent matter collineations in which four are the usual Killing vectors and rest are the proper matter collineations. It is mentioned here that we obtain different constraint equations which, on solving, may provide some new exact solutions of the Einstein field equations.

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1. Introduction

Let M be a spacetime manifold with Lorentz metric g of signature $(+---)$. The manifold M and the metric g are assumed smooth (C^∞). Throughout this article, the usual component notation in local charts will often be used, and a covariant derivative with respect to the symmetric connection Γ associated with the metric g will be denoted by a semicolon and a partial derivative by a comma.

The Einstein's field equations (EFEs) in local coordinates are given by

$$G_{ab} \equiv R_{ab} - \tfrac{1}{2} R g_{ab} = T_{ab}\,, \qquad (a, b = 0, 1, 2, 3)\,, \tag{1}$$

where G_{ab} are the components of the Einstein tensor, R_{ab} those of the Ricci and T_{ab} of the matter (energy-momentum) tensor. Also, $R = g^{ab} R_{ab}$ is the Ricci scalar, and it is assumed that $\kappa = 1$ and $\Lambda = 0$ for simplicity. In General Relativity (GR) theory, the Einstein tensor G_{ab} plays a significant role, since it relates the geometry of spacetime to its source.

The EFEs (1), whose fundamental constituent is the spacetime metric g_{ab}, are highly non-linear partial differential equations, and therefore it is very difficult to obtain their exact solutions. Symmetries of the geometrical/physical relevant quantities of the GR theory are known as *collineations*. In general, these can be represented as $\pounds_\xi \mathcal{A} = \mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are the geometric/physical objects, ξ is the vector field generating the symmetry, and $\pounds_\xi$ signifies the Lie derivative operator along the vector field ξ.

A one-parameter group of conformal motions generated by a *conformal Killing vector* (CKV) ξ is defined as [1]

$$\pounds_\xi g_{ab} = 2\psi g_{ab}\,, \tag{2}$$

where $\psi = \psi(x^a)$ is a conformal factor. If $\psi_{;\,ab} \neq 0$, the CKV is said to be *proper*. Otherwise, ξ reduces to the special *conformal Killing vector* (SCKV) if $\psi_{;\,ab} = 0$, but $\psi_{,a} \neq 0$. Other subcases are *homothetic vector* (HV) if $\psi_{,a} = 0$ and *Killing vector* (KV) if $\psi = 0$.

Using Eq. (2), we find from Eq. (1) that

$$\pounds_\xi T_{ab} = -2\psi_{;ab} + 2 g_{ab} \Box \psi\,, \tag{3}$$

where $\Box$ is the Laplacian operator defined by $\Box \psi \equiv g^{cd} \psi_{;cd}$. Therefore, for a KV, or HV, or SCKV we have

$$\pounds_\xi T_{ab} = 0 \quad \Leftrightarrow \quad \pounds_\xi G_{ab} = 0\,, \tag{4}$$

or in component form

$$T_{ab,c} \xi^c + T_{ac} \xi^c_{,b} + T_{cb} \xi^c_{,a} = 0\,. \tag{5}$$

A vector field ξ satisfying Eq. (4) or (5) on M is called a *matter collineation* (MC). Since the Ricci tensor arises naturally from the Riemann curvature tensor (with components R^a_{bcd} and where $R_{ab} \equiv R^c_{acb}$) and hence from the connection, the study of *Ricci collineation* (RC) defined by $\pounds_\xi R_{ab} = 0$ has a natural geometrical significance [2–12]. Mathematical similarities between the Ricci and energy-momentum tensors mean many techniques for their study should show some similarity. Some papers have recently been published on MCs [13–19]. From the physical viewpoint, a study of MCs, *i.e.* to look into the set of solutions to Eq. (5) seems more relevant. In addition, since energy-momentum tensor T_{ab} is more fundamental in the study of dynamics of fluid spacetimes of GR, the remainder of this paper will be concerned with MCs.

The plan of the paper is as follows. In the next section we shall write down MC Eq. (5) for the maximal symmetric transverse metric. In Section 3, these equations are solved for different cases. Finally, we shall provide a brief summary and discussion of the results obtained.

2. Matter Collineation equations

The general metric for a static spacetime with a maximal symmetric transverse space is given by [20–22].

$$ds^2 = e^{\nu(r)}dt^2 - e^{\mu(r)}dr^2 - r^2(d\theta^2 + f^2(\theta)d\phi^2), \tag{6}$$

where $f(\theta)$ is θ, $\sinh\theta$ or $\sin\theta$ according as $k = -f_{,22}/f = 0, \ -1$ or $+1$ respectively. Notice that $f^2(f_{,2}/f)_{,2} = -1$ if and only if $(ff_{,2})_{,2} = 2f_{,2}^2 - 1$. The non-zero components of the energy-momentum tensor for the above metric are $T_{00},\ T_{11},\ T_{22},\ T_{33}$ given in Appendix A. MC equations in component form can be written as

$$T_0'\xi^1 + 2T_0\xi^0_{,0} = 0\,, \tag{7}$$
$$T_0\xi^0_{,1} + T_1\xi^1_{,0} = 0\,, \tag{8}$$
$$T_0\xi^0_{,2} + T_2\xi^2_{,0} = 0, \tag{9}$$
$$T_0\xi^0_{,3} + f^2T_2\xi^3_{,0} = 0\,, \tag{10}$$
$$T_1'\xi^1 + 2T_1\xi^1_{,1} = 0\,, \tag{11}$$
$$T_1\xi^1_{,2} + T_2\xi^2_{,1} = 0\,, \tag{12}$$
$$T_1\xi^1_{,3} + f^2T_2\xi^3_{,1} = 0\,, \tag{13}$$
$$T_2'\xi^1 + 2T_2\xi^2_{,2} = 0\,, \tag{14}$$
$$T_2(\xi^2_{,3} + f^2\xi^3_{,2}) = 0\,, \tag{15}$$
$$T_2'\xi^1 + 2\frac{f_{,2}}{f}T_2\xi^2 + 2T_2\xi^3_{,3} = 0\,, \tag{16}$$

where prime denotes differentiation with respect to the radial coordinate r. It is to be noted that we have used the notation $T_{aa} = T_a$ for the sake of brevity.

3. Solution of the MC equations

In this section, we solve the MC equations (7)–(16) for the following four cases:

I. One component of $\xi^a(x^b)$ is different from zero;

II. Two components of $\xi^a(x^b)$ are different from zero;

III. Three components of $\xi^a(x^b)$ are different from zero;

IV. All components of $\xi^a(x^b)$ are different from zero.

3.1. One component of $\xi^a(x^b)$ is different from zero

This case has the following four possibilities:

(i) $\xi^a = (\xi^0,\ 0,\ 0,\ 0)$;

(ii) $\xi^a = (0,\ \xi^1,\ 0,\ 0)$;

(iii) $\xi^a = (0,\ 0,\ \xi^2,\ 0)$;

(iv) $\xi^a = (0,\ 0,\ 0,\ \xi^3)$.

In the case I.*(i)*, Eqs. (7)–(9) give $T_0\xi^0_{,a} = 0$. It follows that either $T_0 = 0$ or $T_0 \neq 0$. For $T_0 = 0$, we obtain $\xi^0 = \xi^0(x^a)$ and $T_0 \neq 0$ implies that ξ^0 is an arbitrary constant.

For the case I.*(ii)*, Eqs. (9), (10) and (16) are identically satisfied and the remaining equations become

$$T_0'\xi^1 \ = \ 0 = T_2'\xi^1\,, \tag{17}$$
$$T_1\xi^1_{,a} \ = \ 0, \quad T_1'\xi^1 + 2T_1\xi^1_{,1} = 0\,. \tag{18}$$

Eq. (17) implies that $T_0 = \text{constant} = T_2$. Eq. (18) gives rise to further four possibilities according to

$$\begin{aligned} &\text{(a)} \quad T_1' \ = \ 0\,, \qquad T_1 = 0\,,\\ &\text{(b)} \quad T_1' \ = \ 0\,, \qquad T_1 \neq 0\,,\\ &\text{(c)} \quad T_1' \ \neq \ 0\,, \qquad T_1 \neq 0\,. \end{aligned}$$

For the first possibility, Eq. (18) implies that $\xi^1 = \xi^1(x^a)$. The second case I.*(ii)*(b) gives $T_1 = \text{constant} \neq 0$ and hence $\xi^1 = \text{constant}$. In the third option I.*(ii)*(c) when both $T_1',\ T_1 \neq 0$, it follows from Eq. (18) that $\xi^1 = \frac{c_0}{\sqrt{T_1}}$.

In the third case when $\xi^2 \neq 0$, it follows from MC equations (9), (12), (14) and (15) that $T_2 \xi^2_{,a} = 0$. Also, Eq. (16) yields $T_2 \xi^2 = 0$ which implies that $T_2 = 0$ and hence $\xi^2 = \xi^2(x^a)$.

The last case I.*(iv)*, where $\xi^3 \neq 0$, MC equations (10), (13), (15) and (16) yield $T_2 \xi^3_{,a} = 0$. This implies that either $T_2 = 0$ or $T_2 \neq 0$. When $T_2 = 0$, we obtain $\xi^3 = \xi^3(x^a)$ and for $T_2 \neq 0$, we get $\xi^3 =$ constant.

3.2. Two components of $\xi^a(x^b)$ are different from zero

In this case, we have six different possibilities:

(i) $\xi^a = (\xi^0,\ \xi^1,\ 0,\ 0)\,;$

(ii) $\xi^a = (\xi^0,\ 0,\ \xi^2,\ 0)\,;$

(iii) $\xi^a = (\xi^0,\ 0,\ 0,\ \xi^3)\,;$

(iv) $\xi^a = (0,\ \xi^1,\ \xi^2,\ 0)\,;$

(v) $\xi^a = (0,\ \xi^1,\ 0,\ \xi^3)\,;$

(vi) $\xi^a = (0,\ 0,\ \xi^2,\ \xi^3)\,.$

Case II(*i*): $\xi^a = (\xi^0,\ \xi^1,\ 0,\ 0)$

In this case, Eq. (15) is identically satisfied. From Eq. (14) or (16), it follows that $T_2' \xi^1 = 0$ which implies that $T_2 =$ constant $= c$ (say). From the remaining equations we have four different options according to

$$\begin{array}{llll} \text{(a)} & T_0 = 0\,, \quad T_1 = 0\,, & \text{(b)} & T_0 = 0\,, \quad T_1 \neq 0\,, \\ \text{(c)} & T_0 \neq 0\,, \quad T_1 = 0\,, & \text{(d)} & T_0 \neq 0\,, \quad T_1 \neq 0\,. \end{array}$$

For the first possibility, we obtain $\xi^0 = \xi^0(x^a),\ \xi^1 = \xi^1(x^a)$. In the second case, it follows that $\xi^0 = \xi^0(x^a),\ \xi^1 = \frac{c_0}{\sqrt{T_1}}$. For the third option, we have $\xi^0 = \xi^0(t),\ \xi^1 = \xi^1(x^a)$. Finally, in the last possibility when both $T_0,\ T_1 \neq 0$, we see from Eqs. (9), (10) and (12), (13) that ξ^0 and ξ^1 become functions of $t,\ r$, respectively and we are left with Eqs. (7), (8) and (11). From Eq. (11) we have $\xi^1 = \frac{A(t)}{\sqrt{|T_1|}}$, where $A(t)$ is an integration function. Replacing this value of ξ^0 in Eq. (7), it follows that

$$\xi^0(t,r) = -\frac{T_0'}{2T_0\sqrt{|T_1|}} \int A(t)dr + B(r)\,, \tag{19}$$

where $B(r)$ is an integration function. Substituting this value of ξ^0 together with ξ^1 in Eq. (8), we obtain

$$\frac{\ddot{A}}{A}=\frac{T_0}{\sqrt{|T_1|}}\left(\frac{T_0'}{2T_0\sqrt{|T_1|}}\right)'=-\alpha^2, \tag{20}$$

where α^2 is a separation constant which may be positive, negative or zero. When $\alpha^2>0$, we have

$$\begin{aligned}\xi^0 &= -\frac{T_0'}{2\alpha T_0\sqrt{|T_1|}}(c_0\sin\alpha t-c_1\cos\alpha t)+c_2\,,\\ \xi^1 &= \frac{1}{\sqrt{|T_1|}}(c_0\cos\alpha t+c_1\sin\alpha t)\,,\end{aligned} \tag{21}$$

where $c_0,\ c_1,\ c_2$ are arbitrary constants. It follows that MCs can be written as

$$\begin{aligned}\xi_{(1)} &= \partial_t\,,\\ \xi_{(2)} &= \frac{T_0'}{2\alpha T_0\sqrt{|T_1|}}\sin\alpha t\partial_t-\frac{1}{\sqrt{|T_1|}}\cos\alpha t\partial_r\,,\\ \xi_{(3)} &= \frac{T_0'}{2\alpha T_0\sqrt{|T_1|}}\cos\alpha t\partial_t+\frac{1}{\sqrt{|T_1|}}\sin\alpha t\partial_r\,.\end{aligned} \tag{22}$$

When $\alpha^2<0$, α^2 is replaced by $-\alpha^2$ in Eq. (21) and we obtain the following solution

$$\begin{aligned}\xi_{(1)} &= \partial_t\,,\\ \xi_{(2)} &= \frac{T_0'}{2\alpha T_0\sqrt{|T_1|}}\sinh\alpha t\partial_t-\frac{1}{\sqrt{|T_1|}}\cosh\alpha t\partial_r\,,\\ \xi_{(3)} &= \frac{T_0'}{2\alpha T_0\sqrt{|T_1|}}\cosh\alpha t\partial_t+\frac{1}{\sqrt{|T_1|}}\sinh\alpha t\partial_r\,.\end{aligned} \tag{23}$$

For $\alpha^2=0$, we obtain $\frac{T_0'}{2T_0\sqrt{|T_1|}}=\beta$, where β is an arbitrary constant. This implies that β is either non-zero or zero. For $\beta\neq0$, it follows that

$$\begin{aligned}\xi_{(1)} &= \partial_t\,,\\ \xi_{(2)} &= \left(\beta\frac{t^2}{2}-\int\frac{\sqrt{|T_1|}}{T_0}dr\right)\partial_t-\frac{t}{\sqrt{|T_1|}}\partial_r\,,\\ \xi_{(3)} &= \beta t\partial_t-\frac{1}{\sqrt{|T_1|}}\partial_r\,.\end{aligned} \tag{24}$$

For $\beta = 0$, we have

$$\begin{aligned} \xi_{(1)} &= \partial_t \,, \\ \xi_{(2)} &= \frac{1}{T_0}\int \sqrt{|T_1|}dr\partial_t + \frac{t}{\sqrt{|T_1|}}\partial_r \,, \\ \xi_{(3)} &= \frac{1}{\sqrt{|T_1|}}\partial_r \,. \end{aligned} \tag{25}$$

Thus, in the subcase II.*(i)*(d), MCs turn out to be three in all the possibilities. If Eq. (20) is not satisfied by T_0 and T_1 then $A = 0$ and this reduces to the case I.*(ii)*.

Case II(*ii*): $\xi^a = (\xi^0,\ 0,\ \xi^2,\ 0)$

In this case, Eqs. (11) and (13) are identically satisfied and the remaining equations reduce to

$$T_0\xi^0_{,m} = 0\,, \qquad (m = 0,1,3)\,, \tag{26}$$

$$T_0\xi^0_{,2} + T_2\xi^2_{,0} = 0\,, \tag{27}$$

$$T_2\xi^2_{,i} = 0\,, \qquad (i = 1,2,3)\,, \tag{28}$$

$$T_2\xi^2 = 0\,. \tag{29}$$

Eq. (29) implies that $T_2 = 0$ as $\xi^2 \neq 0$ and consequently, we obtain ξ^2 as an arbitrary function of four-vector. Eqs. (26) and (27) give rise to two possibilities either $T_0 = 0$ or $T_0 \neq 0$. For the first possibility, ξ^0 becomes arbitrary function and for the second case, ξ^0 turns out to be arbitrary constant.

Case II(*iii*): $\xi^a = (\xi^0,\ 0,\ 0,\ \xi^3)$

In this case, Eqs. (11), (12) and (14) are identically satisfied and the remaining equations turn out to be

$$T_0\xi^0_{,n} = 0\,, \quad (n = 0,1,2)\,, \tag{30}$$

$$T_0\xi^0_{,3} + T_2 f\xi^3_{,0} = 0\,, \tag{31}$$

$$T_2\xi^3_{,i} = 0\,, \quad (i = 1,2,3)\,. \tag{32}$$

These equations yield the following four possibilities:

(a) $T_0 = 0$, $T_2 = 0$, (b) $T_0 = 0$, $T_2 \neq 0$,
(c) $T_0 \neq 0$, $T_2 = 0$, (d) $T_0 \neq 0$, $T_2 \neq 0$.

For the first possibility, we obtain $\xi^0=\xi^0(x^a),\ \xi^3=\xi^3(x^a)$. The second option implies that $\xi^0=\xi^0(x^a)$ and ξ^3 to be an arbitrary constant. In the case, when $T_0\neq 0,\ T_2=0$, we have $\xi^0=$ constant, $\xi^3=\xi^3(x^a)$. The last possibility yields both ξ^0 and ξ^3 to be arbitrary constant.

Case II(*iv*): $\xi^a=(0,\ \xi^1,\ \xi^2,\ 0)$

In this case, Eq. (10) is identically satisfied and Eq. (7) implies that $T_0=$ constant. Rest of the equations yield the following constraints

$$\begin{aligned}&(\mathrm{a})\quad T_1=0\,,\quad T_2=0\,,\qquad (\mathrm{b})\quad T_1=0\,,\quad T_2\neq 0\,,\\&(\mathrm{c})\quad T_1\neq 0\,,\quad T_2=0\,,\qquad (\mathrm{d})\quad T_1\neq 0\,,\quad T_2\neq 0\,.\end{aligned}$$

For the first case, MC equations yield $\xi^1=\xi^1(x^a),\ \xi^2=\xi^2(x^a)$. In the second possibility, we obtain $\xi^1=\xi^1(x^a)$ and $\xi^2=cf$. The third option gives the following solution $\xi^1=\frac{c}{\sqrt{T_1}}$ and $\xi^2=\xi^2(x^a)$. In the last possibility, Eqs. (8), (9) and (13) respectively imply that $\xi^1=\xi^1(r,\theta)$ and $\xi^2=\xi^2(r,\theta)$. It follows from Eq. (11) that $\xi^1=\frac{A(\theta)}{\sqrt{T_1}}$. Using this value of ξ^1 in Eqs. (14) and (16) and combining them, we obtain

$$\xi^2_{,2}-\frac{f_{,2}}{f}\xi^2=0 \qquad (33)$$

which gives $\xi^2=B(r)f$. Now we make use of $\xi^1,\ \xi^2$ in Eq. (9), it turns out that

$$\frac{A_{,2}}{f}=-\frac{T_2}{\sqrt{T_1}}B'=\alpha\,, \qquad (34)$$

where α is a constant which can be zero or non-zero. When $\alpha=0$, we obtain

$$\xi_{(1)}=\frac{1}{\sqrt{T_1}}\frac{\partial}{\partial r}\,,\quad \xi_{(2)}=f\partial_\theta\,. \qquad (35)$$

For $\alpha\neq 0$, it follows that

$$\xi^1=\frac{c_0+\alpha\int f d\theta}{\sqrt{T_1}}\,,\qquad \xi^2=\left(c_1-\alpha\int\frac{T_2}{\sqrt{T_1}}dr\right)f\,. \qquad (36)$$

Case II (*v*): $\xi^a=(0,\ \xi^1,\ 0,\ \xi^3)$

It follows from Eqs. (7) and (14) that $T_0=$ constant $=T_2$. From Eqs. (8), (12) and (10), (15) imply that ξ^1 and ξ^3 are functions of r and ϕ respectively.

When we make use of Eqs. (11), (13) and (15), it turns out that $\xi^1 = \frac{c}{\sqrt{T_1}}$ and ξ^3 becomes constant. Thus the two MCs will be

$$\xi_{(1)} = \frac{1}{\sqrt{T_1}}\partial_r\,, \qquad \xi_{(2)} = \partial_\phi\,.$$

Case II (*vi*): $\xi^a = (0,\ 0,\ \xi^2,\ \xi^3)$

In this case, MC Eqs. (7)–(16) reduce to

$$T_2\xi^2_{,m} = 0\,, \quad T_2\xi^3_{,p} = 0\,, \quad (m = 0,1,2)\,, \quad (p = 0,1)\,, \tag{37}$$
$$T_2(\xi^2_{,3} + f^2\xi^3_{,2}) = 0\,, \tag{38}$$
$$T_2(\frac{f^2_{,2}}{f^2}\xi^2 + 2\xi^3_{,3}) = 0\,. \tag{39}$$

These imply that either $T_2 = 0$ or $T_2 \neq 0$. For the first option, Eq. (37)–(39) give $\xi^2 = \xi^2(x^a),\ \xi^3 = \xi^3(x^a)$. In the second possibility, it follows from Eq. (37) that $\xi^2 = \xi^2(\phi)$ and $\xi^3 = \xi^3(\theta,\phi)$. Using Eqs. (38) and (39), it can be shown, after some algebra, that

$$\xi_{(1)} = \frac{2f^2}{f^2_{,2}}\sin\phi\exp\left(-2\int\frac{1}{f^2_{,2}}d\theta\right)\partial_\theta + \cos\phi\exp\left(-2\int\frac{1}{f^2_{,2}}d\theta\right)\partial_\phi\,,$$
$$\xi_{(2)} = \frac{2f^2}{f^2_{,2}}\cos\phi\exp\left(-2\int\frac{1}{f^2_{,2}}d\theta\right)\partial_\theta - \sin\phi\exp\left(-2\int\frac{1}{f^2_{,2}}d\theta\right)\partial\phi\,. \tag{40}$$

3.3. Three components of $\xi^a(x^b)$ are different from zero

It has four different possibilities:

(i) $\xi^a = (\xi^0,\ \xi^1,\ \xi^2,\ 0)$;

(ii) $\xi^a = (\xi^0,\ \xi^1,\ 0,\ \xi^3)$;

(iii) $\xi^a = (\xi^0,\ 0,\ \xi^2,\ \xi^3)$;

(iv) $\xi^a = (0,\ \xi^1,\ \xi^2,\ \xi^3)$;

Case III(*i*): $\xi^a = (\xi^0,\ \xi^1,\ \xi^2,\ 0)$

In this case, using Eqs. (10), (13) and (15) we find that $\xi^0,\ \xi^1$ and ξ^2 are functions of t, r, θ and from Eq. (11), we have

$$\xi^1 = \frac{A(t,\theta)}{\sqrt{|T_1|}}\,,$$

where $A(t,\theta)$ is an integration function. If we make use of this value of ξ^1 in Eq. (16), we obtain

$$\xi^2=-\frac{T_2'}{2T_2\sqrt{|T_1|}}\frac{Af}{f_{,2}}\,.$$

Substituting this value of ξ^2 together with ξ^1 in Eq. (14), we obtain the value of A as follows

$$A(t,\theta)=A_1(t)f_{,2}\,, \tag{41}$$

where $A_1(t)$ is an integration function. From Eq. (7), it follows that

$$\xi^0=-\frac{T_0'}{2T_0\sqrt{|T_1|}}f_{,2}\int A_1dt+B(r,\theta)\,, \tag{42}$$

where $B(r,\theta)$ is an integration function. Using values of ξ^1 and ξ^2 in Eq. (12), we have

$$\frac{T_2}{\sqrt{|T_1|}}\left(\frac{T_2'}{2T_2\sqrt{|T_1|}}\right)'=\frac{f_{,22}}{f}=-k\,. \tag{43}$$

Now plugging the values of ξ^0 and ξ^1 in Eq. (8), we obtain

$$\frac{T_0}{\sqrt{|T_1|}}\left(\frac{T_0'}{2T_0\sqrt{|T_1|}}\right)'=\frac{\ddot{A}_1}{A_1}=-\alpha^2, \tag{44}$$

where α^2 is a separation constant which may be positive, negative or zero. Thus there arise six different possibilities:

(a) $\alpha^2>0\,,\quad k\neq 0\,;$ (b) $\alpha^2<0\,,\quad k\neq 0\,;$ (c) $\alpha^2=0\,,\quad k\neq 0\,;$
(d) $\alpha^2>0\,,\quad k=0\,;$ (e) $\alpha^2<0\,,\quad k=0\,;$ (f) $\alpha^2=0\,,\quad k=0\,.$

For the case III*(i)*(a), after some algebraic manipulation, it is shown that

$$\begin{aligned}
\xi^0&=\frac{1}{\sqrt{|T_1|}}(c_0\cos\alpha t+c_1\sin\alpha t)f_{,2}\,,\\
\xi^1&=-\frac{f_{,2}}{\alpha}\frac{T_0'}{2T_0\sqrt{|T_1|}}(c_0\sin\alpha t-c_1\cos\alpha t)+c_2\,,\\
\xi^2&=-\frac{T_2'}{2T_2\sqrt{|T_1|}}(c_0\cos\alpha t+c_1\sin\alpha t)f\,,\quad T_2=\frac{k}{\alpha^2}T_0+c\,,
\end{aligned} \tag{45}$$

where c is an arbitrary constants and k can take values ± 1. It follows that MCs are three which can be written as

$$\begin{aligned}
\xi_{(1)} &= \frac{f_{,2}}{\sqrt{|T_1|}} \cos \alpha t \partial_t - \frac{f_{,2}}{\alpha} \frac{T_0'}{2T_0\sqrt{|T_1|}} \sin \alpha t \partial_r - \frac{fT_2'}{2T_2\sqrt{|T_0|}} \cos \alpha t \partial_\theta \,, \\
\xi_{(2)} &= \frac{f_{,2}}{\sqrt{|T_1|}} \sin \alpha t \partial_t + \frac{f_{,2}}{\alpha} \frac{T_0'}{2T_0\sqrt{|T_1|}} \cos \alpha t \partial_r - \frac{fT_2'}{2T_2\sqrt{|T_1|}} \sin \alpha t \partial_\theta \,, \\
\xi_{(3)} &= \partial_r \,.
\end{aligned} \tag{46}$$

In the case III*(i)*(b), α^2 is replaced by $-\alpha^2$ in Eq. (40) and it follows that

$$\begin{aligned}
\xi_{(1)} &= \frac{f_{,2}}{\sqrt{|T_1|}} \cosh \alpha t \partial_t - \frac{f_{,2}}{\alpha} \frac{T_0'}{2T_0\sqrt{|T_1|}} \sinh \alpha t \partial_r - \frac{fT_2'}{2T_2\sqrt{|T_0|}} \cosh \alpha t \partial_\theta \,, \\
\xi_{(2)} &= \frac{f_{,2}}{\sqrt{|T_1|}} \sinh \alpha t \partial_t - \frac{f_{,2}}{\alpha} \frac{T_0'}{2T_0\sqrt{|T_1|}} \cosh \alpha t \partial_r - \frac{fT_2'}{2T_2\sqrt{|T_1|}} \sinh \alpha t \partial_\theta \,, \\
\xi_{(3)} &= \partial_r \,.
\end{aligned} \tag{47}$$

For the third subcase, we have $A_1 = c_0 t + c_1$ and $\frac{T_0'}{2T_0\sqrt{|T_1|}} = \beta$, where β is an arbitrary constant. This implies that either β is non-zero or zero. If $\beta \neq 0$, it reduces to the case I*(ii)*. For $\beta = 0$, we have $T_0 = \text{constant}$. Using these values in Eq. (15) and (18) we obtain

$$\begin{aligned}
\xi_{(1)} &= f_{,2} \frac{1}{\sqrt{|T_1|}} t \partial_t - \frac{f_{,2} T_2'}{2kT_0\sqrt{|T_1|}} \partial_r - \frac{T_2' f}{2T_2\sqrt{|T_1|}} t \partial_\theta \,, \\
\xi_{(2)} &= \frac{1}{\sqrt{|T_1|}} f_{,2} \partial_t - \frac{T_2' f}{2T_2\sqrt{|T_1|}} \partial_\theta \,, \\
\xi_{(3)} &= \partial_r \,, \\
T_0 = \text{constant}, \quad T_2' &= 2k\sqrt{|T_1|} \int \sqrt{|T_1|} dt \,.
\end{aligned} \tag{48}$$

The fourth and fifth subcases reduce to I*(ii)*.

In the case I*(i)*(f), we obtain $A_1 = c_0 t + c_1$, $\frac{T_0'}{2T_0\sqrt{|T_1|}} = \beta$ and $\frac{T_2'}{2T_2\sqrt{|T_1|}} = \gamma$, where β and γ are arbitrary constants. We see from Eq. (18) that $\gamma T_2 = 0$ which implies that either $\gamma = 0$ or $T_2 = 0$. This gives rise to the following three possibilities

$$\begin{array}{lll}
(1) & \gamma \neq 0 \neq \beta \,, & T_2 = 0 \,, \\
(2) & \gamma = 0 \,, & \beta \neq 0 \neq T_2 \,, \\
(3) & \gamma = 0 = \beta \,, & T_2 \neq 0 \,.
\end{array}$$

In the subcase III*(i)*(f)(1), for $k = 0$, we have

$$\begin{aligned}
\xi_{(1)} &= \frac{1}{\sqrt{|T_1|}}t\partial_t + (\beta\frac{t^2}{2} + \int\frac{\sqrt{|T_1|}}{T_0}dr)\partial_r - \gamma\theta t\partial_\theta\,,\\
\xi_{(2)} &= \frac{1}{\sqrt{|T_1|}}\partial_t - \beta t\partial_\theta\,,\\
\xi_{(3)} &= \partial_r\,. \qquad (49)
\end{aligned}$$

For $k=\pm1$, $c_0=0$. The subcases III*(i)*(f)(2) and III*(i)*(f)(3) reduce to the case II*(i)*. In the former subcase T_0 becomes constant while in the latter subcase both T_0 and T_2 become constants.

Case III(*ii*): $\xi^a=(\xi^0,\ \xi^1,\ 0,\ \xi^3)$

In this case, when we replace $\xi^2=0$ in the MC Eqs. (7)–(16), it follows eight different possibilities according to the values of $T_0,\ T_1,\ T_2$. If we take at least one of these three components of the energy-momentum tensor zero, we obtain infinite dimensional MCs. The only case which gives finite dimensional MCs is that where all $T_0,\ T_1,\ T_2$ are non-zero. Here we give the solution of this case only. Solving MC equations (7)–(16) under the constraints of this case, we obtain

$$\xi^0 = \dot{A}\int\frac{\sqrt{T_1}}{T_0}dr + c_0\,,\qquad \xi^1=\frac{A}{\sqrt{T_1}}\,,\qquad \xi^3=c_1 \qquad (50)$$

with

$$\frac{\ddot{A}}{A} = \frac{T_0'\sqrt{T_1}}{2T_0}\int\frac{\sqrt{T_1}}{T_0}dr = \alpha\,,$$

where α is a separation constant which gives three different possibilities according to the value of α whether positive, zero or negative. For $\alpha>0$, we have the following MCs

$$\begin{aligned}
\xi_{(1)} &= \partial_t\,,\\
\xi_{(2)} &= \sqrt{\alpha}\exp(-\sqrt{\alpha}t)\partial_t + \frac{\exp(-\sqrt{\alpha}t)}{\sqrt{T_1}}\partial_r\,,\\
\xi_{(3)} &= \exp(\sqrt{\alpha}t)\int\frac{\sqrt{T_1}}{T_0}dr\partial_t - \frac{\exp(\sqrt{\alpha}t)}{\sqrt{T_1}}\partial_r\,,\\
\xi_{(4)} &= \partial_\phi\,. \qquad (51)
\end{aligned}$$

When $\alpha = 0$, MCs turn out to be

$$\begin{aligned}
\xi_{(1)} &= \partial_t\,,\\
\xi_{(2)} &= \int \frac{\sqrt{T_1}}{T_0} dr\partial_t + \frac{t}{\sqrt{T_1}}\partial_r,\\
\xi_{(3)} &= \frac{1}{\sqrt{T_1}}\partial_r\,,\\
\xi_{(4)} &= \partial_\phi\,.
\end{aligned} \tag{52}$$

For $\alpha < 0$, we obtain the following MCs

$$\begin{aligned}
\xi_{(1)} &= \partial_t\,,\\
\xi_{(2)} &= \sqrt{\alpha}\sin(\sqrt{\alpha}t)\partial_t + \frac{\cos(\sqrt{\alpha}t)}{\sqrt{T_1}}\partial_r\,,\\
\xi_{(3)} &= \cos(\sqrt{\alpha}t)\int \frac{\sqrt{T_1}}{T_0} dr\partial_t - \frac{\sin(\sqrt{\alpha}t)}{\sqrt{T_1}}\partial_r\,,\\
\xi_{(4)} &= \partial_\phi\,.
\end{aligned} \tag{53}$$

We see that in each case MCs turn out to be four.

Case III(*iii*): $\xi^a = (\xi^0,\ 0,\ \xi^2,\ \xi^3)$

When we substitute $\xi^1 = 0$ in MC equations, we obtain the following three cases.

(a) $T_0 = 0\,,\quad T_2 \neq 0\,,$ (b) $T_0 \neq 0\,,\quad T_2 = 0\,,$ (c) $T_0 \neq 0,\quad T_2 \neq 0\,.$

The first two cases yield infinite dimensional MCs and the third case is the interesting one which gives finite MCs. When we solve MC equations simultaneously for this case, after some algebra, we obtain the following MCs

$$\begin{aligned}
\xi_{(1)} &= \partial_t\,,\\
\xi_{(2)} &= \frac{T_2}{T_0} f^2\cos\phi\exp\left(-2\int\frac{d\theta}{f_{,2}^2}\right)\partial_t - t\cos\phi\partial_\theta + t\sin\phi\exp\left(-2\int\frac{d\theta}{f_{,2}^2}\right)\partial_\phi\,,\\
\xi_{(3)} &= \frac{T_2}{T_0} f^2\sin\phi\exp\left(-2\int\frac{d\theta}{f_{,2}^2}\right)\partial_t - t\sin\phi\partial_\theta - t\cos\phi\partial_\phi\,,\\
\xi_{(4)} &= \cos\phi\partial_\theta - \sin\phi\exp\left(-2\int\frac{d\theta}{f_{,2}^2}\right)\partial_\phi\,,\\
\xi_{(5)} &= \sin\phi\partial_\theta + \cos\phi\partial_\phi\,.
\end{aligned} \tag{54}$$

This gives five independent MCs.

Case III(*iv*): $\xi^a = (0,\ \xi^1,\ \xi^2,\ \xi^3)$

In this case, MC equations give the following four possibilities:

$$\begin{array}{lllll} \text{(a)} & T_1 = 0\,, & T_2 = 0\,, & \text{(b)} & T_1 \neq 0\,, \quad T_2 = 0\,, \\ \text{(c)} & T_1 = 0\,, & T_2 \neq 0\,, & \text{(d)} & T_1 \neq 0\,, \quad T_2 \neq 0\,. \end{array}$$

The first two cases reduce to the earlier one. For the case III*(iv)*(c), we are left with the MC equations (14)–(16) along with $\xi^1 = \xi^1(r,\theta,\phi)$, $\xi^2 = \xi^2(\theta,\phi)$, $\xi^3 = \xi^3(\theta,\phi)$. For $k = -1$, when we solve MC equations (14)–(16) we obtain the following MCs

$$\begin{aligned} \xi_{(1)} &= \frac{2T_2}{T_2'}\cos\phi\sinh\theta\partial_r - \cos\phi\cosh\theta\partial_\theta + \sin\phi\,\text{cosech}\,\theta\partial_\phi\,, \\ \xi_{(2)} &= \frac{2T_2}{T_2'}\sin\phi\sinh\theta\partial_r - \sin\phi\cosh\theta\partial_\theta - \cos\phi\,\text{cosech}\,\theta\partial_\phi\,, \\ \xi_{(3)} &= \cos\phi\partial_\theta - \sin\phi\coth\theta\partial_\phi\,, \\ \xi_{(4)} &= \sin\phi\partial_\theta + \cos\phi\coth\theta\partial_\phi\,, \\ \xi_{(5)} &= \partial_\phi\,. \end{aligned} \tag{55}$$

When $k = 0$, MCs are

$$\begin{aligned} \xi_{(1)} &= \frac{2T_2}{T_2'}\cos\phi\theta\partial_r - \cos\phi\frac{\theta^2}{2}\partial_\theta - \sin\phi\frac{\theta}{2}\partial_\phi\,, \\ \xi_{(2)} &= \frac{2T_2}{T_2'}\sin\phi\theta\partial_r - \sin\phi\frac{\theta^2}{2}\partial_\theta + \cos\phi\frac{\theta}{2}\partial_\phi\,, \\ \xi_{(3)} &= \cos\phi\partial_\theta - \sin\phi\frac{1}{\theta}\partial_\phi\,, \\ \xi_{(4)} &= \sin\phi\partial_\theta - \cos\phi\frac{1}{\theta}\partial_\phi\,, \\ \xi_{(5)} &= \partial_\phi\,. \end{aligned} \tag{56}$$

If we take $k = 1$, MCs turn out to be

$$\begin{aligned} \xi_{(1)} &= \frac{2T_2}{T_2'}\cos\phi\sin\theta\partial_r - \cos\phi\cos\theta\partial_\theta + \sin\phi\,\text{cosec}\,\theta\partial_\phi\,, \\ \xi_{(2)} &= \frac{2T_2}{T_2'}\sin\phi\sin\theta\partial_r - \sin\phi\cos\theta\partial_\theta - \cos\phi\,\text{cosec}\,\theta\partial_\phi\,, \\ \xi_{(3)} &= \cos\phi\partial_\theta - \sin\phi\cot\theta\partial_\phi\,, \\ \xi_{(4)} &= \sin\phi\partial_\theta + \cos\phi\cot\theta\partial_\phi\,, \\ \xi_{(5)} &= \partial_\phi\,. \end{aligned} \tag{57}$$

The case III*(iv)*(d) yields $\xi^1,\ \xi^2,\ \xi^3$ as functions of r,θ,ϕ with MC equations (14)–(16). For $k=-1$, when we solve MC equations (14)–(16) we obtain the following MCs

$$\begin{aligned}
\xi_{(1)} &= a\Bigg[\frac{2T_2}{T_2'}(\cos\phi+\sin\phi)\sinh\theta\partial_r-(\cos\phi+\sin\phi)\cosh\theta\partial_\theta \\
&\quad -(\sin\phi-\cos\phi)\,\mathrm{cosech}\,\theta\partial_\phi\Bigg]\,, \\
\xi_{(2)} &= \cos\phi\partial_\theta-\sin\phi\coth\theta\partial_\phi\,, \\
\xi_{(3)} &= \sin\phi\partial_\theta+\cos\phi\coth\theta\partial_\phi\,, \\
\xi_{(4)} &= \partial_\phi\,.
\end{aligned} \tag{58}$$

When $k=0$, MCs take the following form

$$\begin{aligned}
\xi_{(1)} &= a\Bigg[\frac{2T_2}{T_2'}(\cos\phi+\sin\phi)\theta\partial_r-(\cos\phi+\sin\phi)\frac{\theta^2}{2}\partial_\theta \\
&\quad -(\sin\phi-\cos\phi)\frac{\theta}{2}\partial_\phi\Bigg], \\
\xi_{(2)} &= \cos\phi\partial_\theta-\sin\phi\frac{1}{\theta}\partial_\phi\,, \\
\xi_{(3)} &= \sin\phi\partial_\theta+\cos\phi\frac{1}{\theta}\partial_\phi\,, \\
\xi_{(4)} &= \partial_\phi\,.
\end{aligned} \tag{59}$$

If we take $k=1$, MCs turn out to be

$$\begin{aligned}
\xi_{(1)} &= a\Bigg[\frac{2T_2}{T_2'}(\cos\phi+\sin\phi)\sin\theta\partial_r-(\cos\phi+\sin\phi)\cos\theta\partial_\theta \\
&\quad -(\sin\phi-\cos\phi)\,\mathrm{cosec}\,\theta\partial_\phi\Bigg], \\
\xi_{(2)} &= \cos\phi\partial_\theta-\sin\phi\cot\theta\partial_\phi\,, \\
\xi_{(3)} &= \sin\phi\partial_\theta+\cos\phi\cot\theta\partial_\phi\,, \\
\xi_{(4)} &= \partial_\phi\,.
\end{aligned} \tag{60}$$

This gives four independent MCs.

3.4. Four components of $\xi^a(x^b)$ are different from zero

This is an interesting and a bit difficult case. In this section, we shall evaluate MCs when all the four components of $\xi^a(x^b)$ are non-zero. In other words, we find MCs only for those cases which have non-degenerate energy-momentum tensor, *i.e.*, $\det(T_{ab}) \neq 0$. To this end, we set up the general conditions for the solution of MC equations for the non-degenerate case.

When we solve Eqs. (7)–(16) simultaneously, after some tedious algebra, we get the following solution

$$\begin{aligned}\xi^0 &= \frac{T_2}{T_0}f^2\left[(\dot{A}_1(t,r)\sin\phi - \dot{A}_2(t,r)\cos\phi)\int\frac{1}{f^2}\left(\int f d\theta\right)d\theta\right]\\ &\quad -\frac{T_2}{T_1}\dot{A}_3(t,r)\int f d\theta + A_4(t,r)\,,\end{aligned} \tag{61}$$

$$\begin{aligned}\xi^1 &= \frac{T_2}{T_0}f^2\left[(A_1'(t,r)\sin\phi - A_2'(t,r)\cos\phi)\int\frac{1}{f^2}\left(\int f d\theta\right)d\theta\right]\\ &\quad -\frac{T_2}{T_1}A_3'(t,r)\int f d\theta + A_5(t,r)\,,\end{aligned} \tag{62}$$

$$\begin{aligned}\xi^2 &= [A_1(t,r)\sin\phi - A_2(t,r)\cos\phi]\int f d\theta + c_1\sin\phi - c_2\cos\phi\\ &\quad +A_3(t,r)f\,,\end{aligned} \tag{63}$$

$$\begin{aligned}\xi^3 &= -[(A_1(t,r)\cos\phi + A_2(t,r)\sin\phi)]\int\frac{1}{f^2}\left(\int f d\theta\right)d\theta\\ &\quad -(c_1\sin\phi - c_2\cos\phi)\int\frac{1}{f^2}d\theta + c_3\,,\end{aligned} \tag{64}$$

where c_1, c_2, c_3 are arbitrary constants and $A_\nu = A_\nu(t,r),\ \nu = 1,2,3,4,5.$ Here dot and prime indicate the differentiation with respect to time and r coordinate respectively. When we replace these values of ξ^a in MC Eqs. (7)–(16), we obtain the following differential constraints on A_ν with $c_4 = 0$

$$2T_1\ddot{A}_i + T_{0,1}A_i' = 0\,, \quad i = 1,2,3\,, \tag{65}$$

$$2T_0\dot{A}_4 + T_{0,1}A_5 = 0\,, \tag{66}$$

$$2T_2\dot{A}_i' + T_0\left(\frac{T_2}{T_0}\right)'\dot{A}_i = 0\,, \tag{67}$$

$$T_0A_4' + T_1\dot{A}_5 = 0\,, \tag{68}$$

$$\left\{T_{1,1}\frac{T_2}{T_1}+2T_1\left(\frac{T_2}{T_1}\right)'\right\}A_i'+2T_2A_i'' = 0\,, \tag{69}$$

$$T_{1,1}A_5+2T_1A_5' = 0\,, \tag{70}$$

$$T_{2,1}A_i'+2T_1A_i = 0\,, \quad c_0=0\,, \tag{71}$$

$$T_{2,1}A_5=0\,. \tag{72}$$

Thus the problem of working out MCs for all possibilities of $A_i,\ A_4,\ A_5$ is reduced to solving the set of Eqs. (61)–(64) subject to the above constraints. We start the classification of MCs by considering the constraint Eq. (72). This can be satisfied for three different possible cases

$$(i)\ T_2'=0\,,\quad A_5\neq 0\,,\quad (ii)\ T_2'\neq 0\,,\quad A_5=0\,,\quad (iii)\ T_2'=0\,,\quad A_5=0\,.$$

Case (*i*): In this case, all the constraints remain unchanged except (65), (69) and (71). Thus we have

$$\dot{A}_i'-\frac{1}{2}\frac{T_0'}{T_0}\dot{A}_i=0\,, \tag{73}$$

$$A_i''-\frac{T_1'}{2T_1}A_i'=0\,, \tag{74}$$

$$T_1A_i=0\,. \tag{75}$$

The last equation is satisfied only if $A_i=0$. As a result, all the differential constraints involving A_i and its derivatives disappear identically and we are left with Eqs. (66), (68) and (70) only. Now integrating constraint Eq. (70) w.r.t. r and replacing the value of A_5 in constraint Eq. (66), we have

$$T_0'\frac{A(t)}{\sqrt{T_1}}+2T_0\dot{A}_4=0\,,$$

where $A(t)$ is an integration function. This gives rise to the following two possibilities:

$$\text{(a)}\quad T_0'=0\,,\quad \dot{A}_4=0\,,\qquad \text{(b)}\quad T_0'\neq 0\,,\quad \dot{A}_4\neq 0\,.$$

For the case IV*(i)*(a), after some algebra, we arrive at the following MCs

$$\begin{aligned}\xi_{(1)} &= \partial_t\,,\\ \xi_{(2)} &= \frac{1}{\sqrt{T_1}}\partial_r\,,\\ \xi_{(3)} &= \cos\phi\partial_\theta+\sin\phi\partial_\phi\int\frac{d\theta}{f^2}\,,\end{aligned}$$

$$\begin{aligned}
\xi_{(4)} &= \sin\phi\partial_\theta - \cos\phi\partial_\phi\int\frac{d\theta}{f^2}\,,\\
\xi_{(5)} &= \frac{1}{a}\int\sqrt{T_1}dr\partial_t - \frac{t}{\sqrt{T_1}}\partial_r\,,\\
\xi_{(6)} &= \partial_\phi\,. \qquad (76)
\end{aligned}$$

This shows that we have six MCs. In the case IV*(i)*(b), we have $\dot{A}_4 \neq 0$ and $T_0' \neq 0$. Solving Eqs. (66) and (68) and re-arranging terms, we get

$$\frac{\ddot{A}}{A} = \frac{T_0}{2\sqrt{T_1}}\left(\frac{T_0'}{T_0\sqrt{T_1}}\right)' = \alpha\,, \qquad (77)$$

where α is a separation constant and this gives the following three possible cases:

$$(1)\quad \alpha<0\,,\qquad (2)\quad \alpha=0\,,\qquad (3)\quad \alpha>0\,.$$

The first case $\alpha < 0$ reduces to the case III(*ii)*(a) of the previous section. The subcase IVi(b)(2) gives

$$A(t) = c_3t + c_4$$

and

$$\frac{T_0'}{T_0\sqrt{T_1}} = \beta\,, \qquad (78)$$

where β is an integration constant which yields the following two possibilities

$$(*)\quad \beta\neq 0\,,\quad (**)\quad \beta=0\,.$$

The first possibility implies that

$$T_0 = \beta_0 e^{\beta\int\sqrt{T_1}dr}\,,$$

where β_0 is an integration constant. Now we solve Eqs. (70) and (72) by using this constraint, we can get the following MCs

$$\begin{aligned}
\xi_{(1)} &= \partial_t\,,\\
\xi_{(2)} &= \sin\phi\partial_\theta - \cos\phi\int\frac{d\theta}{f^2}\partial_\phi\,,\\
\xi_{(3)} &= \cos\phi\partial_\theta + \sin\phi\int\frac{d\theta}{f^2}\partial_\phi\,,\\
\xi_{(4)} &= \frac{1}{\beta_0}\int\frac{\sqrt{T_1}}{e^{\beta\int\sqrt{T_1}dr}}dr\partial_t - \frac{t}{\sqrt{T_1}}\partial_r\,,\\
\xi_{(5)} &= \frac{1}{\sqrt{T_1}}\partial_r\,. \qquad (79)
\end{aligned}$$

This gives five independent MCs. For the case IV*(i)*(b)(2)(**), $T_0 = \text{const.}$ Using this fact Eq. (70) yields $A_4 = g(r)$. Thus we have the solution

$$\begin{aligned}
\xi_{(1)} &= \partial_t \,, \\
\xi_{(2)} &= \sin\phi\partial_\theta - \cos\phi\int\frac{d\theta}{f^2}\partial_\phi \,, \\
\xi_{(3)} &= \cos\phi\partial_\theta + \sin\phi\int\frac{d\theta}{f^2}\partial_\phi \,, \\
\xi_{(4)} &= \frac{1}{b}\int\sqrt{T_1}dr\partial_t - \frac{t}{\sqrt{T_1}}\partial_r \,, \\
\xi_{(5)} &= \frac{1}{\sqrt{T_1}}\partial_r \,.
\end{aligned} \tag{80}$$

We again have five MCs. The case IV*(i)*(b)(3) gives the same results as the case IV*(i)*(b)(1).

Case *(ii)*: In the case IV*(ii)*, when $T_2' \neq 0$, it follows from Eqs. (69)–(72) that for $\frac{T_2}{\sqrt{T_1}}(\frac{T_2'}{2T_2\sqrt{T_1}})' + 1 \neq 0$, we obtain the following four MCs

$$\begin{aligned}
\xi_{(1)} &= \partial_t \,, \\
\xi_{(2)} &= \sin\phi\partial_\theta - \cos\phi\int\frac{d\theta}{f^2}\partial_\phi \,, \\
\xi_{(3)} &= \cos\phi\partial_\theta + \sin\phi\int\frac{d\theta}{f^2}\partial_\phi \,, \\
\xi_{(4)} &= \partial_\phi \,.
\end{aligned} \tag{81}$$

If $\frac{T_2}{\sqrt{T_1}}\left(\frac{T_2'}{2T_2\sqrt{T_1}}\right)' + 1 = 0$ and $\left(\frac{T_2'}{\sqrt{T_0T_1T_2}}\right)' \neq 0$, we have seven MCs given by

$$\begin{aligned}
\xi_{(1)} &= \partial_t \,, \\
\xi_{(2)} &= \sin\phi\partial_\theta - \cos\phi\int\frac{d\theta}{f^2}\partial_\phi \,, \\
\xi_{(3)} &= \cos\phi\partial_\theta + \sin\phi\int\frac{d\theta}{f^2}\partial_\phi \,, \\
\xi_{(4)} &= \partial_\phi \,, \\
\xi_{(5)} &= \frac{1}{\sqrt{T_1}}\sin\phi\partial_r f^2\int\left(\frac{1}{f^2}\int f d\theta\right)d\theta + X\sin\phi\partial_\theta\int f d\theta \\
&+ X\cos\phi\partial_\phi\int\left(\frac{1}{f^2}\int f d\theta\right)d\theta,
\end{aligned}$$

$$
\begin{aligned}
\xi_{(6)} &= \frac{1}{\sqrt{T_1}}\cos\phi\partial_r f^2\int\left(\frac{1}{f^2}\int fd\theta\right)d\theta+X\cos\phi\partial_\theta\int fd\theta\\
&+ X\sin\phi\partial_\phi\int\left(\frac{1}{f^2}\int fd\theta\right)d\theta,\\
\xi_{(7)} &= \frac{1}{\sqrt{T_1}}\partial_r\int fd\theta+Xf\sin\theta\partial_\theta\,. \qquad (82)
\end{aligned}
$$

where $X=\frac{T_2'}{2T_2\sqrt{T_1}}$. If we have $\frac{T_2}{\sqrt{T_1}}\left(\frac{T_2'}{2T_2\sqrt{T_1}}\right)'+1=0$, $\left(\frac{T_2'}{\sqrt{T_0T_1T_2}}\right)'=0$ and $\left(\frac{T_0'}{T_2'}\right)'\neq 0$, then we get the same MCs as given by Eq. (81).

When $\frac{T_2}{\sqrt{T_1}}\left(\frac{T_2'}{2T_2\sqrt{T_1}}\right)'+1=0$, $\left(\frac{T_2'}{\sqrt{T_0T_1T_2}}\right)'=0$ and $\frac{T_0'}{T_2'}=\delta$, an arbitrary constant. For $\delta>0$, we obtain

$$
\begin{aligned}
\xi_{(1)} &= \partial_t\,,\\
\xi_{(2)} &= \sin\phi\partial_\theta-\cos\phi\int\frac{d\theta}{f^2}\partial_\phi\,,\\
\xi_{(3)} &= \cos\phi\partial_\theta+\sin\phi\int\frac{d\theta}{f^2}\partial_\phi\,,\\
\xi_{(4)} &= \partial_\phi\,,\\
\xi_{(5)} &= \left(\frac{T_2}{T_0}X\sqrt{\delta}\sinh\sqrt{\delta}t\partial_t-\frac{1}{\sqrt{T_1}}\cosh\sqrt{\delta}t\partial_r\right)\sin\phi f^2\int\left(\frac{1}{f^2}\int fd\theta\right)d\theta\\
&\quad -\left(\sin\phi\partial_\theta\int fd\theta+\cos\phi\partial_\phi\int\left(\frac{1}{f^2}\int fd\theta\right)d\theta\right)X\cosh\sqrt{\delta}t\,,\\
\xi_{(6)} &= \left(\frac{T_2}{T_0}X\sqrt{\delta}\sinh\sqrt{\delta}t\partial_t-\frac{1}{\sqrt{T_1}}\cosh\sqrt{\delta}t\partial_r\right)\cos\phi f^2\int\left(\frac{1}{f^2}\int fd\theta\right)d\theta\\
&\quad -\left(\cos\phi\partial_\theta\int fd\theta-\sin\phi\partial_\phi\int\left(\frac{1}{f^2}\int fd\theta\right)d\theta\right)X\cosh\sqrt{\delta}t\,,\\
\xi_{(7)} &= \left(\frac{T_2}{T_0}X\sqrt{\delta}\sinh\sqrt{\delta}t\partial_t-\frac{1}{\sqrt{T_1}}\cosh\sqrt{\delta}t\partial_r\right)\int fd\theta\\
&+ X\cosh\sqrt{\delta}t\partial_\theta f^2\int\left(\frac{1}{f^2}\int fd\theta\right)d\theta\,,\\
\xi_{(8)} &= \left(\frac{T_2}{T_0}X\sqrt{\delta}\cosh\sqrt{\delta}t\partial_t-\frac{1}{\sqrt{T_1}}\sinh\sqrt{\delta}t\partial_r\right)\sin\phi f^2\int\left(\frac{1}{f^2}\int fd\theta\right)d\theta\\
&- \left(\sin\phi\partial_\theta\int fd\theta+\cos\phi\partial_\phi\int\left(\frac{1}{f^2}\int fd\theta\right)d\theta\right)X\sinh\sqrt{\delta}t\,,
\end{aligned}
$$

$$
\begin{aligned}
\xi_{(9)} &= \left(\frac{T_2}{T_0}X\sqrt{\delta}\cosh\sqrt{\delta}t\partial_t - \frac{1}{\sqrt{T_1}}\sinh\sqrt{\delta}t\partial_r\right)\cos\phi f^2\int\left(\frac{1}{f^2}\int f d\theta\right)d\theta \\
&\quad - \left(\cos\phi\partial_\theta\int f d\theta - \sin\phi\partial_\phi\int\left(\frac{1}{f^2}\int f d\theta\right)d\theta\right)X\sinh\sqrt{\delta}t\,, \\
\xi_{(10)} &= \left(\frac{T_2}{T_0}X\sqrt{\delta}\cosh\sqrt{\delta}t\partial_t - \frac{1}{\sqrt{T_1}}X\sinh\sqrt{\delta}t\partial_r\right)\int f d\theta \\
&\quad + X\sinh\sqrt{\delta}t\partial_\theta f^2\int\left(\frac{1}{f^2}\int f d\theta\right)d\theta\,. \qquad (83)
\end{aligned}
$$

If $\delta = 0$, we have

$$
\begin{aligned}
\xi_{(1)} &= \partial_t, \\
\xi_{(2)} &= \sin\phi\partial_\theta - \cos\phi\int\frac{d\theta}{f^2}\partial_\phi\,, \\
\xi_{(3)} &= \cos\phi\partial_\theta + \sin\phi\int\frac{d\theta}{f^2}\partial_\phi\,, \\
\xi_{(4)} &= \partial_\phi, \\
\xi_{(5)} &= \left(\frac{T_2}{T_0}X\partial_t - \frac{1}{\sqrt{T_1}}t\partial_r\right)\sin\phi f^2\int\left(\frac{1}{f^2}\int f d\theta\right)d\theta \\
&\quad - \left(\sin\phi\partial_\theta\int f d\theta + \cos\phi\partial_\phi\int\left(\frac{1}{f^2}\int f d\theta\right)d\theta\right)Xt\,, \\
\xi_{(6)} &= \left(-\frac{T_2}{T_0}X\partial_t + \frac{1}{\sqrt{T_1}}t\partial_r\right)\cos\phi f^2\int\left(\frac{1}{f^2}\int f d\theta\right)d\theta \\
&\quad + \left(\cos\phi\partial_\theta\int f d\theta - \sin\phi\partial_\phi\int\left(\frac{1}{f^2}\int f d\theta\right)d\theta\right)Xt\,, \\
\xi_{(7)} &= \left(\frac{T_2}{T_0}X\partial_t - \frac{1}{\sqrt{T_1}}t\partial_r\right)\int f d\theta + Xt\partial_\theta f^2\int\left(\frac{1}{f^2}\int f d\theta\right)d\theta\,, \\
\xi_{(8)} &= \left(\frac{1}{\sqrt{T_1}}\partial_r f^2\int\left(\frac{1}{f^2}\int f d\theta\right)d\theta + X\partial_\theta\int f d\theta\right)\sin\phi \\
&\quad + X\cos\phi\partial_\phi\int\left(\frac{1}{f^2}\int f d\theta\right)d\theta\,, \\
\xi_{(9)} &= \left(\frac{1}{\sqrt{T_1}}\partial_r f^2\int\left(\frac{1}{f^2}\int f d\theta\right)d\theta + X\partial_\theta\int f d\theta\right)\cos\phi \\
&\quad - X\sin\phi\partial_\phi\int\left(\frac{1}{f^2}\int f d\theta\right)d\theta\,, \\
\xi_{(10)} &= \frac{1}{\sqrt{T_1}}\partial_r\int f d\theta - X\partial_\theta f^2\int\left(\frac{1}{f^2}\int f d\theta\right)d\theta\,. \qquad (84)
\end{aligned}
$$

For $\delta<0$, MCs are given by

$$
\begin{aligned}
\xi_{(1)} &= \partial_t\,,\\
\xi_{(2)} &= \sin\phi\partial_\theta-\cos\phi\int\frac{d\theta}{f^2}\partial_\phi\,,\\
\xi_{(3)} &= \cos\phi\partial_\theta+\sin\phi\int\frac{d\theta}{f^2}\partial_\phi\,,\\
\xi_{(4)} &= \partial_\phi\,,\\
\xi_{(5)} &= \left(-\frac{T_2}{T_0}X\sqrt{-\delta}\sin\sqrt{-\delta}t\partial_t-\frac{1}{\sqrt{T_1}}\cos\sqrt{-\delta}t\partial_r\right)\sin\phi f^2\int\left(\frac{1}{f^2}\int fd\theta\right)d\theta\\
&\quad-\left(\sin\phi\partial_\theta\int fd\theta+\cos\phi\partial_\phi\int\left(\frac{1}{f^2}\int fd\theta\right)d\theta\right)X\cos\sqrt{-\delta}t\,,\\
\xi_{(6)} &= \left(\frac{T_2}{T_0}X\sqrt{-\delta}\sin\sqrt{-\delta}t\partial_t+\frac{1}{\sqrt{T_1}}\cos\sqrt{-\delta}t\partial_r\right)\cos\phi f^2\int\left(\frac{1}{f^2}\int fd\theta\right)d\theta\\
&\quad+\left(\cos\phi\partial_\theta\int fd\theta-\sin\phi\partial_\phi f^2\int\left(\frac{1}{f^2}\int fd\theta\right)d\theta\right)X\cos\sqrt{-\delta}t\,,\\
\xi_{(7)} &= \left(\frac{T_2}{T_0}X\sqrt{-\delta}\sin\sqrt{-\delta}t\partial_t+\frac{1}{\sqrt{T_1}}\cos\sqrt{-\delta}t\partial_r\right)\int fd\theta\\
&\quad- X\cos\sqrt{-\delta}t\partial_\theta f^2\int\left(\frac{1}{f^2}\int fd\theta\right)d\theta\,,\\
\xi_{(8)} &= \left(\frac{T_2}{T_0}X\sqrt{-\delta}\cos\sqrt{-\delta}t\partial_t-\frac{1}{\sqrt{T_1}}\sin\sqrt{-\delta}t\partial_r\right)\sin\phi f^2\int\left(\frac{1}{f^2}\int fd\theta\right)d\theta)\\
&\quad-\left(\sin\phi\partial_\theta\int fd\theta+\cos\phi\partial_\phi\int\left(\frac{1}{f^2}\int fd\theta\right)d\theta\right)X\sin\sqrt{-\delta}t\,,\\
\xi_{(9)} &= \left(\frac{T_2}{T_0}X\sqrt{\delta}\cos\sqrt{-\delta}t\partial_t-\frac{1}{\sqrt{T_1}}\sin\sqrt{-\delta}t\partial_r\right)\cos\phi f^2\int\left(\frac{1}{f^2}\int fd\theta\right)d\theta\\
&\quad-\left(\cos\phi\partial_\theta\int fd\theta-\sin\phi\partial_\phi f^2\int\left(\frac{1}{f^2}\int fd\theta\right)d\theta\right)X\sin\sqrt{-\delta}t\,,\\
\xi_{(10)} &= \left(\frac{T_2}{T_0}X\sqrt{-\delta}\cos\sqrt{-\delta}t\partial_t-\frac{1}{\sqrt{T_1}}X\sin\sqrt{-\delta}t\partial_r\right)\int fd\theta\\
&\quad+X\sin\sqrt{-\delta}t\partial_\theta f^2\int\left(\frac{1}{f^2}\int fd\theta\right)d\theta\,. \qquad (85)
\end{aligned}
$$

From Eqs. (83)–(85), it follows that for each value of δ, we obtain ten independent MCs.

Case *(iii)*: In this case, we have $T_2=$ constant and $A_5=0$. This can be solved trivially and gives similar results as in the case IV*(i)*(b)(1).

4. Conclusion

A large part of GR research is the consequence of classifying solutions of the EFEs. There are various approaches to classify spacetimes, including the Segre classification of the energy-momentum tensor or the Petrov classification of the Weyl tensor and have been studied extensively by many researchers [23]. They also classify spacetimes using symmetry vector fields, in particular KVs and HVs. KVs may be used to classify spacetimes as there is a limit to the number of global, smooth Killing vector fields that a spacetime may possess (the maximum being 10 for 4-dimensional spacetimes). Generally speaking, the higher the dimension of the algebra of symmetry vector fields on a spacetime, the more symmetry the spacetime admits. For example, the Schwarzschild solution has a Killing algebra of dimension 4 (3 spatial rotational vector fields and a time translation), whereas the Friedmann–Robertson–Walker metric (excluding the Einstein static subcase) has a Killing algebra of dimension 6 (3 translations and 3 rotations). The Einstein static metric has a Killing algebra of dimension 7 (the previous 6 plus a time translation). The assumption of a spacetime admitting a certain symmetry vector field can place severe restrictions on the spacetime.

The relation between geometry and physics can be highlighted if the vector field ξ is regarded as preserving certain physical quantities along the flow lines of ξ. Since every KV is a MC hence for a given solution of the EFE, a vector field that preserves the metric necessarily preserves the corresponding energy-momentum tensor. When the energy-momentum tensor represents a perfect fluid, every KV preserves the energy density, pressure and the fluid flow vector field. When the energy-momentum tensor represents an electromagnetic field, a KV does not necessarily preserve the electric and magnetic fields. The main application of these symmetries occur in GR, where solutions of EFEs may be classified by imposing some certain symmetries on the spacetime.

This paper continues the study of the symmetries of the energy-momentum tensor for the static spacetimes with maximal symmetric transverse spaces. We have classified static spacetimes with maximal symmetric transverse spaces according to their MCs. The MC equations are solved by taking one non-zero component, two non-zero components, three non-zero components and finally all non-zero components. The case with all the non-zero components gives the generalization of the static spherically symmetric spacetimes and we recover all the results from the spacetime under consideration. The results can be summarized in the form of tables given below:

When we take one component non-zero (Case I), we obtain infinite dimensional MCs for the degenerate case and one MC for the non-degenerate case.

TABLE I

MCs when one component of ξ^a is non-zero.

Cases	**MCs**	**Constraints**
I*(i)*(a)	Infinite dimensional MCs	$T_0 = 0,\ T_1 \neq 0,\ T_2 \neq 0$
I*(i)*(b)	1	$T_0 \neq 0,\ T_1 \neq 0,\ T_2 \neq 0$
I*(ii)*(a)	Infinite dimensional MCs	$T_0 = \text{const.} = T_2,\ T_1 = 0,\ T_1' = 0$
I*(ii)*(b)	1	$T_0 = \text{const.} = T_2, T_1 = \text{const.} \neq 0$
I*(ii)*(c)	1	$T_0 = \text{const.} = T_2, T_1' \neq 0,\ T_1 \neq 0$
I*(iii)*	Infinite dimensional MCs	$T_0 \neq 0,\ T_1 \neq 0,\ T_2 = 0$
I*(iv)*(a)	Infinite dimensional MCs	$T_0 \neq 0,\ T_1 \neq 0,\ T_2 = 0$
I*(iv)*(b)	1	$T_0 \neq 0,\ T_1 \neq 0,\ T_2 \neq 0$

TABLE II

MCs when two components of ξ^a are non-zero.

Cases	**MCs**	**Constraints**
II*(i)*(a)	Infinite dimensional MCs	$T_0 = 0 = T_1,\ T_2 = \text{const.}$
II*(i)*(b)	Infinite dimensional MCs	$T_0 = 0,\ T_1 \neq 0,\ T_2 = \text{const.}$
II*(i)*(c)	Infinite dimensional MCs	$T_0 \neq 0,\ T_1 = 0,\ T_2 = \text{const.}$
II*(i)*(d)(1)	3	$T_0 \neq 0,\ T_1 \neq 0,\ T_2 = \text{const.},$ $\frac{T_0}{\sqrt{\lvert T_1\rvert}}\left(\frac{T_0'}{2T_0\sqrt{\lvert T_1\rvert}}\right)' = -\alpha^2,\ \alpha^2 > 0$
II*(i)*(d)(2)	3	$T_0 \neq 0,\ T_1 \neq 0,\ T_2 = \text{const.},$ $\alpha^2 < 0$
II*(i)*(d)(3)(*)	3	$T_0 \neq 0,\ T_1 \neq 0,\ T_2 = \text{const.},$ $\alpha^2 = 0,$ $\frac{T_0'}{2T_0\sqrt{\lvert T_1\rvert}} = \beta \neq 0$
II*(i)*(d)(3)(**)	3	$T_0 \neq 0,\ T_1 \neq 0,\ T_2 \neq 0,\ \alpha^2 = 0,$ $\beta = 0$
II*(ii)*(a)	Infinite dimensional MCs	$T_0 = 0,\ T_1 \neq 0,\ T_2 \neq 0$
II*(ii)*(b)	1	$T_0 \neq 0,\ T_1 \neq 0,\ T_2 \neq 0$
II*(iii)*(a)	Infinite dimensional MCs	$T_0 = 0,\ T_1 \neq 0,\ T_2 = 0$
II*(iii)*(b)	Infinite dimensional MCs	$T_0 = 0,\ T_1 \neq 0,\ T_2 \neq 0$
Cases	**RCs**	**Constraints**
II*(iii)*(c)	Infinite dimensional MCs	$T_0 \neq 0,\ T_1 \neq 0,\ T_2 = 0$
II*(iii)*(d)	2	$T_0 \neq 0,\ T_1 \neq 0,\ T_2 \neq 0$
II*(iv)*(a)	Infinite dimensional MCs	$T_0 = \text{const.},\ T_1 = 0 = T_2$
II*(iv)*(b)	Infinite dimensional MCs	$T_0 = \text{const.},\ T_1 = 0,\ T_2 \neq 0$
II*(iv)*(c)	Infinite dimensional MCs	$T_0 = \text{const.},\ T_1 \neq 0,\ T_2 = 0$
II*(iv)*(d)(1)	2	$T_0 \neq 0,\ T_1 \neq 0,\ T_2 \neq 0,$ $-\frac{T_2}{\sqrt{T_1}}B' = \alpha,\ \alpha = 0$
II*(iv)*(d)(2)	2	$T_0 \neq 0,\ T_1 \neq 0,\ T_2 \neq 0,\ \alpha \neq 0$
II*(v)*	2	$T_0 = \text{const.},\ T_1 \neq 0, T_2 = \text{const.}$
II*(vi)*(a)	Infinite dimensional MCs	$T_0 \neq 0,\ T_1 \neq 0,\ T_2 = 0$
II*(vi)*(b)	2	$T_0 \neq 0,\ T_1 \neq 0,\ T_2 \neq 0$

TABLE III

MCs when three components of ξ^a are non-zero.

Cases	**MCs**	**Constraints**
III*(i)*(a)	3	$T_0 \neq 0,\ T_1 \neq 0,\ k \neq 0$ $\frac{T_0}{\sqrt{\lvert T_1\rvert}}\left(\frac{T_0'}{2T_0\sqrt{\lvert T_1\rvert}}\right)' = -\alpha^2,\ \alpha^2 > 0,$ $T_2 = \frac{k}{\alpha^2}T_0 + c$
III*(i)*(b)	3	$T_0 \neq 0,\ T_1 \neq 0,\ k \neq 0,\ \alpha^2 < 0,$ $T_2 = \frac{k}{\alpha^2}T_0 + c$
III*(i)*(c)(1)	I*(ii)*	$T_0 = \text{const.},\ T_1 \neq 0,\ k \neq 0,$ $\alpha^2 = 0,\ T_2' = 2k\sqrt{T_1}\int\sqrt{T_1}dt,$ $\frac{T_0'}{2T_0\sqrt{T_1}} = \beta,\ \beta \neq 0$
III*(i)*(c)(2)	3	$T_0 = \text{const.},\ T_1 \neq 0,\ k \neq 0,$ $\alpha^2 = 0,\ T_2' = 2k\sqrt{T_1}\int\sqrt{T_1}dt,$ $\beta = 0$
III*(i)*(d)	II*(i)*	$T_0 \neq 0,\ T_1 \neq 0,\ T_2 \neq 0,\ k = 0,$ $\alpha^2 > 0$
III*(i)*(e)	II*(i)*	$T_0 \neq 0,\ T_1 \neq 0,\ T_2 \neq 0,\ k = 0,$ $\alpha^2 < 0$
Cases	**RCs**	**Constraints**
III*(i)*(f)(1)	3	$T_0 \neq 0,\ T_1 \neq 0,\ T_2 = 0,\ k = 0,$ $\alpha^2 = 0,\ \frac{T_0'}{2T_0\sqrt{T_1}} = \beta,\ \frac{T_2'}{2T_2\sqrt{T_1}} = \gamma,$ $\beta \neq 0,\ \gamma \neq 0$
III*(i)*(f)(2)	I*(ii)*	$T_0 = \text{const.},\ T_1 \neq 0,\ T_2 \neq 0,$ $k = 0,\ \beta \neq 0,\ \gamma = 0$
III*(i)*(f)(3)	I*(ii)*	$T_0 = \text{const.},\ T_1 \neq 0,$ $T_2 = \text{const.},\ k = 0,\ \beta = 0 = \gamma$
III*(ii)*(a)	Infinite dimensional MCs	$T_0 = 0,\ T_1 = 0,\ T_2 = 0$
III*(ii)*(b)	Infinite dimensional MCs	$T_0 = 0,\ T_1 = 0,\ T_2 = \text{const.} \neq 0$
III*(ii)*(c)	Infinite dimensional MCs	$T_0 = 0,\ T_1 \neq 0,\ T_2 = 0$
III*(ii)*(d)	Infinite dimensional MCs	$T_0 \neq 0,\ T_1 = 0,\ T_2 = 0$
III*(ii)*(e)	Infinite dimensional MCs	$T_0 = 0,\ T_1 \neq 0,\ T_2 = \text{const.} \neq 0$
III*(ii)*(f)	Infinite dimensional MCs	$T_0 \neq 0,\ T_1 \neq 0,\ T_2 = 0$
III*(ii)*(g)	Infinite dimensional MCs	$T_0 \neq 0,\ T_1 = 0,\ T_2 = \text{const.} \neq 0$
III*(ii)*(h)(1)	4	$T_0 \neq 0,\ T_1 \neq 0,\ T_2 = \text{const.} \neq 0,$ $\frac{T_0'\sqrt{T_1}}{2T_0}\int\frac{\sqrt{T_1}}{T_0}dr = \alpha, \alpha > 0$
III*(ii)*(h)(2)	4	$T_0 \neq 0,\ T_1 \neq 0,\ T_2 = \text{const.} \neq 0, \alpha = 0$
III*(ii)*(h)(3)	4	$T_0 \neq 0,\ T_1 \neq 0,\ T_2 = \text{const.} \neq 0, \alpha < 0$
III*(iii)*(a)	Infinite dimensional MCs	$T_0 = 0,\ T_1 \neq 0,\ T_2 \neq 0$
III*(iii)*(b)	Infinite dimensional MCs	$T_0 \neq 0,\ T_1 \neq 0,\ T_2 = 0$
III*(iii)*(c)	5	$T_0 \neq 0,\ T_1 \neq 0,\ T_2 \neq 0$
III*(iv)*(a)	Infinite dimensional MCs	$T_0 \neq 0,\ T_1 = 0,\ T_2 = 0$
III*(iv)*(b)	Infinite dimensional MCs	$T_0 \neq 0,\ T_1 \neq 0,\ T_2 = 0$
III*(iv)*(c)	5	$T_0 \neq 0,\ T_1 = 0,\ T_2 \neq 0$
III*(iv)*(d)	4	$T_0 \neq 0,\ T_1 \neq 0,\ T_2 \neq 0$

TABLE IV
MCs when all components of ξ^a are non-zero.

Cases	**MCs**	**Constraints**
IV*(i)*(a)	6	$T_2'=0,\ A_5(t,r)\neq 0,\ T_0'=0,$ $\dot{A}_4(t,r)=0,$
IV*(i)*(b)(1)	III*(ii)*(a)	$T_2'\neq 0,\ A_5(t,r)\neq 0,\ T_0'=0,$ $\dot{A}_4(t,r)\neq 0,\ \frac{T_0}{\sqrt{T_1}}\left(\frac{T_0'}{2T_0\sqrt{T_1}}\right)'=\alpha,$ $\alpha<0$
IV*(i)*(b)(2)(*)	5	$T_2'\neq 0,\ A_5(t,r)\neq 0,\ T_0'=0,$ $\alpha=0,\ \frac{T_0'}{T_0\sqrt{T_1}}=\beta,\ \beta\neq 0$
IV*(i)*(b)(2)(**)	5	$T_2'\neq 0,\ A_5(t,r)\neq 0,\ T_0'=0,$ $\alpha=0,\ \beta=0$
IV*(i)*(b)(3)	IV*(i)*(b)(1)	$T_2'\neq 0,\ A_5(t,r)\neq 0,\ T_0'=0,$ $\alpha>0$
IV*(ii)*(a)	4	$T_2'\neq 0,\ A_5(t,r)=0,\ \frac{T_2}{\sqrt{T_1}}\left(\frac{T_2'}{2T_2\sqrt{T_1}}\right)'+1\neq 0$
IV*(ii)*(b)(1)	7	$T_2'\neq 0,\ A_5(t,r)=0,\ \frac{T_2}{\sqrt{T_1}}\left(\frac{T_2'}{2T_2\sqrt{T_1}}\right)'+1=0,$ $(\frac{T_2'}{2T_2\sqrt{T_1}})'\neq 0$
IV*(ii)*(b)(2)(*)	4	$T_2'\neq 0,\ A_5(t,r)=0,\ \frac{T_2}{\sqrt{T_1}}\left(\frac{T_2'}{2T_2\sqrt{T_1}}\right)'+1=0,$ $(\frac{T_2'}{2T_2\sqrt{T_1}})'=0,\ (\frac{T_0'}{T_2'})'\neq 0$
IV*(ii)*(b)(2)(**)(+)	10	$T_2'\neq 0,\ A_5(t,r)=0,\ \frac{T_2}{\sqrt{T_1}}\left(\frac{T_2'}{2T_2\sqrt{T_1}}\right)'+1=0,$ $(\frac{T_2'}{2T_2\sqrt{T_1}})'=0,\ \frac{T_0'}{T_2'}=\delta,\ \delta>0$
IV*(ii)*(b)(2)(**)(++)	10	$T_2'\neq 0,\ A_5(t,r)=0,\ \frac{T_2}{\sqrt{T_1}}\left(\frac{T_2'}{2T_2\sqrt{T_1}}\right)'+1=0,$ $(\frac{T_2'}{2T_2\sqrt{T_1}})'=0,\ \frac{T_0'}{T_2'}=\delta,\ \delta=0$
IV*(ii)*(b)(2)(**)(+++)	10	$T_2'\neq 0,\ A_5(t,r)=0,\ \frac{T_2}{\sqrt{T_1}}\left(\frac{T_2'}{2T_2\sqrt{T_1}}\right)'+1=0,$ $(\frac{T_2'}{2T_2\sqrt{T_1}})'=0,\ \frac{T_0'}{T_2'}=\delta,\ \delta<0$
IV*(iii)*	IV*(i)*9b)(1)	$T_2=\text{const.},\ A_5=0$

For two components non-zero (Case II), the degenerate case gives infinite dimensional Lie algebra while the non-degenerate case yields maximum three MCs. The case III has four different possibilities. For the first possibility, we obtain three MCs and the second possibility gives four independent MCs. It is mentioned here that finite results have been obtained for the non-degenerate cases. The cases III*(iii)* and III*(iv)* are interesting one where we obtain finite MCs even for the degenerate case. These are five independent MCs. It is worth noticing that this (Case III) generalizes the degenerate case of the static spherically symmetric spcetimes [15]. We know that every KV is an MC, but the converse is not always true. The last case (Case IV) generalizes the non-degenerate case of the static spherically symmetric spacetimes [15] for the case $k=1$. Here we take all the components of the vector

ξ^a non-zero and solve the MC equations generally. We obtain either four, five, six, seven or ten independent MCs in which four are the usual KVs and rest are the proper MCs.

Finally, it is remarked that RCs obtained by Akbar [24] are similar to MCs. However, the constraint equations are different. If we solve these constraint equations, we may have a family of spacetimes. It would be interesting to look for solutions from these constraint equations.

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Appendix

The surviving components of the Ricci tensor are

$$R_{00} = \left(\frac{\nu''}{2} - \frac{\nu'\mu'}{4} + \frac{\nu'^2}{4} + \frac{\nu'}{r}\right) e^{\nu-\mu}\,, \tag{A.1}$$

$$R_{11} = -\frac{\nu''}{2} + \frac{\nu'\mu'}{4} + \frac{\nu'^2}{4} + \frac{\mu'}{r}\,, \tag{A.2}$$

$$R_{22} = \left(\frac{r\mu'}{2} - \frac{r\nu'}{2} - 1\right) e^{-\mu} + k\,, \tag{A.3}$$

$$R_{33} = f^2 T_{22}\,. \tag{A.4}$$

The Ricci scalar is

$$R = (\nu'' - \frac{\nu'\mu'}{2} + \frac{\nu'^2}{2} - \frac{2\mu'}{r} + \frac{2\nu'}{r} + \frac{2}{r^2})e^{-\mu} - \frac{2k}{r^2}\,. \tag{A.5}$$

The non-vanishing components of the energy-momentum tensor turn out to be

$$T_{00} = \frac{1}{r}\left[\left(\mu' - \frac{1}{r}\right) e^{-\mu} + \frac{k}{r}\right] e^{\nu}\,, \tag{A.6}$$

$$T_{11} = \frac{1}{r}(\nu' - \frac{k}{r} + \frac{e^{\mu}}{r})\,, \tag{A.7}$$

$$T_{22} = \frac{r}{2}\left(\nu' - \mu' + r\nu'' - \frac{r\nu'\mu'}{2} + \frac{r\nu'^2}{2}\right) e^{-\mu}\,, \tag{A.8}$$

$$T_{33} = f^2 T_{22}\,. \tag{A.9}$$

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