SPACE, PHASE SPACE AND QUANTUM NUMBERS OF ELEMENTARY PARTICLES

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We recall the arguments that there should be a close connection between the properties of elementary particles and the arena used for the description of macroscopic processes, and argue that a natural choice for this arena is provided by nonrelativistic phase space with momentum and position being independent variables. Accepting standard commutation relations for these variables, and adopting $\boldsymbol{x}^2+\boldsymbol{p}^2$ as an invariant, we linearise the latter *á la* Dirac. Phase space $\mathrm{U}(1)\otimes\mathrm{SU}(3)$ symmetry is then represented in the relevant Clifford algebra. Within this algebra, the eigenvalues of the U(1) generator are $\pm(+1/3,+1/3,+1/3,-1)$, characteristic of weak hypercharge Y for three coloured quarks and one lepton. The total U(1) generator contains contributions from the phase space and the Clifford algebra, and leads to a relation, which we propose to identify with the Gell-Mann–Nishijima–Glashow formula $Q=I_3+Y/2$.

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1. Introduction

In standard approaches the origin of internal symmetries such as colour, flavour and various other experimentally unconfirmed "exotic" quantum numbers is placed "outside" the familiar macroscopic space. In recent years a lot of work has been devoted to approaches in which this "outside" is augmented by endowing the configuration space with additional dimensions. These additional dimensions are to be "hidden" from our macroscopic point of view (*e.g.* by being discernible at very small distances only), but could be identified through the detection of new particles carrying new exotic types of quantum numbers [1]. In summary, this approach adopts the widely held view that particles move in configuration space and then enlarges the latter.

The above view is driven by our every-day experience. However, it does not have to be a good starting point for all questions. Physics provides descriptions of Nature only, and the choice of the language of description may be crucial. In fact, the choice of the language may hinder our ability to describe the microscopic world and to identify possible connections between the properties of this world and those of the surrounding macroscopic world. There are, therefore, physicists who advocate for an altogether different description of classical macroscopic world.

Many arguments may be used to support the assumption that what we really have to do is to find the unknown hidden links between the known properties of the classical macroscopic world and the known properties of the micro-world. Some of them are given in [2–5]. The approach of [5], as argued therein at length, is based on the fundamental conjecture that the properties of a continuous arena used for the description of classical macroscopic processes must be strictly and closely related to the discrete quantum properties of elementary particles, including the very appearance of *all* quantum numbers characterising these particles, *i.e.* not only spin and parity, but also colour, flavour, charge, *etc.* The relevance of this conjecture for colour, flavour, *etc.* clearly depends on what is meant by the "arena". In the present paper I attempt to develop further the ideas originally proposed in [4, 5].

As observed in [5], *all* of those quantum numbers of elementary particles for which a connection with the properties of macroscopic space is known, are related to the description of physical phenomena at a nonrelativistic level. This concerns not only spin and parity, whose connection to the symmetry properties of the three-dimensional space is well known, but also the very existence of particles and antiparticles. Indeed, despite the widespread view, based on the properties of the Dirac equation and holding that relativity is at play here, the existence of the particle–antiparticle dichotomic variable has a strictly nonrelativistic origin. The claim that the origin of the existence of particles and antiparticles is in its essence nonrelativistic can be understood intuitively by thinking of a nonrelativistic particle moving, as Feynman put it, "backwards in time". From this picture, one can see that the existence of antiparticles should be closely connected with symmetry under time reflection, which has nothing to do with truly relativistic transformations. Thus, it should be possible to contemplate the emergence of antiparticles within a purely nonrelativistic description. Our version of relevant arguments will be given in Section 2.

In view of the nonrelativistic nature of all quantum numbers for which their connection with the macroscopic classical arena has been established, *the simplest expectation is that other observed quantum numbers of elementary particles may also be inferred through a nonrelativistic reasoning.* This makes our guiding principle more specific. Consequently, as in [5] we limit ourselves to a nonrelativistic description, hoping that proper extension to include relativity can be done at a later stage.

In Ref. [5] it was argued that instead of identifying the arena of nonrelativistic classical physics with the observable three-dimensional space, as it is customarily done, one should adopt the description given by the language of the nonrelativistic Hamiltonian formalism, in which momentum and position coordinates are treated as independent variables. In this description, the arena in question appears to be that of phase space, and the issue of a possible symmetry between the momentum and position coordinates can be formulated in a more natural way. In the present paper, the idea of connecting properties of phase space with quantum numbers of elementary particles is pursued further.

The paper is organised as follows. In Section 2 we present our argument that the existence of the particle–antiparticle degree of freedom may be deduced via a purely nonrelativistic reasoning. In Section 3 we introduce language appropriate for the description of nonrelativistic phase space with noncommuting momentum and position coordinates (technically, just the three-dimensional harmonic oscillator), which leads us to view the well-known $\mathrm{U}(1) \otimes \mathrm{SU}(3)$ symmetry of the harmonic oscillator as the basic symmetry of Nature. In Section 4 we linearise the relevant phase space invariants *á la* Dirac and discuss the extension of the $\mathrm{U}(1) \otimes \mathrm{SU}(3)$ symmetry of phase space into the space of the relevant Clifford algebra. This leads to the appearance of threefold-degenerate fractional eigenvalues for the U(1) generator, and the appearance of a Gell-Mann–Nishijima–Glashow-like relation. The connection between the fractional eigenvalues of the U(1) generator and the rishon model of Harari is also established.

2. Particles and antiparticles without relativity

The purpose of this section is to show that the existence of the two-valued particle–antiparticle degree of freedom can be ascertained in a purely nonrelativistic framework, as mentioned in [5].

We will follow the approach of Dirac closely. Our aim is to linearise the bilinear form $\boldsymbol{p}^2 - 2mE$ which vanishes when (kinetic) energy E is given by the relevant nonrelativistic expression (see also [6]). Let us write

$$\Theta \equiv \boldsymbol{p}^2 - 2mE = L'L\,, \tag{1}$$

and attempt to linearise this form *á la* Dirac by writing (summation over repeated indices implied; we use k, l, m, n running from 1 to 3)

$$L \ = \ \alpha_k p_k + \mu m + \nu E\,, \tag{2}$$

$$L' \ = \ \alpha'_k p_k + \mu' m + \nu' E\,, \tag{3}$$

and seeking α_k, μ, ν, (α'_k, μ', ν') that would satisfy condition (1). As in the case of the original derivation of Dirac, we must search for a solution among the 4×4 matrices, since within the space of Hermitean 2×2 matrices there is no matrix available that would anticommute with all three Pauli matrices. It is straightforward to see that the following expressions constitute a solution meeting our condition (with α_k, β being standard Dirac matrices):

$$L(\beta) \;=\; \boldsymbol{\alpha}\cdot\boldsymbol{p} + (1+\beta)\frac{m}{m_0} + (1-\beta)\frac{m_0}{2}E\,, \tag{4}$$

$$L'(\beta) \;=\; \boldsymbol{\alpha}\cdot\boldsymbol{p} - (1-\beta)\frac{m}{m_0} - (1+\beta)\frac{m_0}{2}E\,, \tag{5}$$

where m_0 is an arbitrary positive factor. Since the relative sign with which β enters into the second and third term vanishes after evaluating $L'(\beta)L(\beta) = L(\beta)L'(\beta) = \boldsymbol{p}^2 - 2mE$, the following expressions also constitute a solution:

$$L_a(\beta) \;=\; \boldsymbol{\alpha}\cdot\boldsymbol{p} + (1-\beta)\frac{m}{m_0} + (1+\beta)\frac{m_0}{2}E\,, \tag{6}$$

$$L'_a(\beta) \;=\; \boldsymbol{\alpha}\cdot\boldsymbol{p} - (1+\beta)\frac{m}{m_0} - (1-\beta)\frac{m_0}{2}E\,, \tag{7}$$

as we clearly have $L_a(\beta)L'_a(\beta) = L'_a(\beta)L_a(\beta) = \boldsymbol{p}^2 - 2mE$. Note that under operation $m_0 \to -m_0$ expressions $L(\beta)$, $L'(\beta)$) turn into $L'_a(\beta)$, $L_a(\beta)$. Thus, if one admits negative m_0, Eqs. (6), (7) are already included in Eqs. (4), (5).

Let us assume that solution $L(\beta)$ corresponds to particles. In order to see that solution $L_a(\beta)$ corresponds to antiparticles, we shall now try to implement the operation of charge conjugation. We shall follow the traditional procedure with one necessary change, as explained below. This procedure consists of the following steps:

1. Consider operator $L(\beta)$ supplemented with minimal electromagnetic coupling:

$$L(\beta,e) = \boldsymbol{\alpha}\cdot(\boldsymbol{p} - e\,\boldsymbol{\mathcal{A}}) + (1+\beta)\frac{m}{m_0} + (1-\beta)\frac{m_0}{2}(E - e\,\boldsymbol{\mathcal{A}}_0)\,. \tag{8}$$

2. Take "complex conjugation" of $L(\beta,e)$. This complex conjugation replaces operators $\boldsymbol{p}$ and E with $-\boldsymbol{p}$ and $-E$, leaving e, $\boldsymbol{\mathcal{A}}$, $\mathcal{A}_0$ unchanged. In addition, in order to keep $\boldsymbol{p}^2 - 2mE$ invariant, one must replace m with $-m$. This is the additional requirement that *must* be added in order to keep the invariance of the nonrelativistic formula relating energy to mass and momentum. It is also the condition necessary for the form-invariance of operator $\boldsymbol{p}^2 - 2mE$ under time reversal, *i.e.* for viewing antiparticles as particles "moving backwards in time". It suggests that m is actually an operator. Thus, one gets

$$L^*(\beta,e) = \boldsymbol{\alpha}^*(-\boldsymbol{p} - e\,\boldsymbol{\mathcal{A}}) - (1+\beta)\frac{m}{m_0} + (1-\beta)\frac{m_0}{2}(-E - e\,\boldsymbol{\mathcal{A}}_0)\,. \tag{9}$$

3. Now take matrix $C = -i\sigma_2 \otimes \sigma_2 = -C^{-1}$ and transform $-L^*(\beta, e)$ with its help:

$$C(-L^*(\beta,e)C^{-1} = \boldsymbol{\alpha}\cdot(\boldsymbol{p}+e\,\boldsymbol{\mathcal{A}})+(1-\beta)\frac{m}{m_0}+(1+\beta)\frac{m_0}{2}(E+e\,\boldsymbol{\mathcal{A}}_0)\,. \quad (10)$$

The last equation constitutes a solution meeting our original requirement (it corresponds directly to operator $L_a(\beta)$ given in Eq. (6) or $L'(\beta)$ for negative m_0), and describes objects with opposite charges, *i.e.* antiparticles. The transition from particles to antiparticles may be obtained simply by reversing the signs of β and of charge e, *i.e.* $L(\beta, e) \to L(-\beta, -e)$. In this way, from Eqs. (4), (8) one obtains Eqs. (6), (10).

In conclusion, the existence of the particle–antiparticle degree of freedom may be established using purely nonrelativistic reasoning. Irrelevance of Lorentzian relativity was stressed also in Ref. [7], where the existence of antiparticles was inferred under more fully fledged Galilean invariance. Thus, we repeat the statement made in the introduction: *all* of those quantum numbers of elementary particles for which a connection with the properties of macroscopic space is known are related to the description of phenomena at a nonrelativistic level. Consequently, it is natural to assume that also other observed quantum numbers of elementary particles may be deduced via a nonrelativistic reasoning. We are thus entitled to study a purely nonrelativistic approach.

3. Phase space and its symmetries

As stressed in [4, 5], the symmetry of the observed three-dimensional macroscopic world (*i.e.* group O(3) of transformations leaving $\boldsymbol{r}^2$ (or $\boldsymbol{p}^2$) form-invariant, which includes proper rotations and reflections) is closely connected with quantum numbers characterising elementary particles such as spin and parity. Ref. [5] argues then at length that instead of the three-dimensional coordinate–space-based description, one should adopt a description in which momenta and positions are treated on a more equal footing. Similar ideas were advocated in the past *e.g.* by Born, whose point of view concerning the properties of particles and the emergence of mass is well described by the following quote from [8] (in this quote, Born talks about the values of $P \equiv E^2 - \boldsymbol{p}^2$, $R \equiv t^2 - \boldsymbol{x}^2$ for individual particles): "It looks, therefore, as if the distance P in momentum space is capable of an infinite number of discrete values which can be roughly determined while the distance R in coordinate space is not an observable at all. This lack of symmetry seems to me very strange and rather improbable." Basing on such ideas, Born suggested that the "reciprocity" transformations, *i.e.* $\boldsymbol{p}' = \boldsymbol{x}$ and $\boldsymbol{x}' = -\boldsymbol{p}$, under which various basic equations of classical physics remain invariant, should play an important role. Invariance under reciprocity transformations suggests the introduction of a new physical constant, κ, of dimension $[g\ s^{-1}]$ (or [GeV/c fm^{-1}] when units used in particle physics are chosen). The actual value of κ is irrelevant for the purposes of the present paper. (However, when κ is specified, its value and that of the Planck's constant $\hbar$ together define "natural units" in position and momentum spaces.) The physical position $\boldsymbol{x}_{[x]}$, where the subscript $_{[x]}$ indicates that position is measured in such standard units of distance as meters or their derivatives (*eg.* fm), is then related to $\boldsymbol{x}$ through $\boldsymbol{x} \equiv \boldsymbol{x}_{[p]} = \kappa \boldsymbol{x}_{[x]}$, with subscript $_{[p]}$ denoting that position is expressed in units of momentum.

A natural framework in which momentum and positions are treated in a fully symmetric way as mutually independent variables, is the nonrelativistic Hamiltonian formalism with phase space as the arena of classical events. When the phase–space-based description is admitted, it is further natural to combine the two invariants of momentum space and position space into a single invariant, in which momentum and position coordinates are treated in a completely symmetrical way. This singles out the form $\boldsymbol{p}^2 + \boldsymbol{x}^2$ and admits other transformations besides the proper rotations and reflections. These other transformations include transformations in which momentum is transformed into position and *vice versa*, but in a more general fashion than that allowed for by Born's reciprocity transformations.

When of all transformations which keep $\boldsymbol{p}^2 + \boldsymbol{x}^2$ form-invariant one admits only those that keep the Poisson brackets unchanged, the relevant group of transformations turns out to be $\mathrm{U}(1) \otimes \mathrm{SU}(3)$ [5]. By moving from the 3-dimensional description in the configuration space (x_1, x_2, x_3) to the 6-dimensional description in phase–space $(z_1, z_2, \ldots, z_6) \equiv (x_1, x_2, x_3, p_1, p_2, p_3) = (\boldsymbol{x}, \boldsymbol{p})$, we "expand" O(3) to $\mathrm{U}(1) \otimes \mathrm{SU}(3)$. Thus, $\mathrm{U}(1) \otimes \mathrm{SU}(3)$ defines an "ideal figure" in phase space. The U(1) factor takes care of reflections ($\boldsymbol{p}' = -\boldsymbol{p}$, $\boldsymbol{x}' = -\boldsymbol{x}$) and Born's reciprocity transformations ($\boldsymbol{p}' = \boldsymbol{x}$, $\boldsymbol{x}' = -\boldsymbol{p}$), the former being the squares of the latter; while the SU(3) group constitutes an extension of the group of proper rotations, where the latter are understood as simultaneous rotations of $\boldsymbol{p}$ and $\boldsymbol{x}$.

Present-day Standard Model of elementary particles is a gauge theory based on group $\mathrm{U}(1) \otimes \mathrm{SU}(3) \otimes \mathrm{SU}(2)_{\mathrm{L}}$. There are two ingredients here: the group structure and the gauge principle. The latter leads to dynamics. In this paper we will be interested in the symmetry aspects only. The question how to introduce dynamics in the phase–space language will be put aside. However, some hints exist. For example, in the phase–space formulation of Ref. [9] canonical reciprocity transformations are used to describe a minimally coupled particle in a "magnetic" field in one-dimensional space. This seems to suggests that our $\mathrm{U}(1) \otimes \mathrm{SU}(3)$ could be related to the $\mathrm{U}(1)_{\mathrm{Y}} \otimes \mathrm{SU}(3)_{\mathrm{C}}$ symmetry of the Standard Model.

In Ref. [5] we started with the classical description, in which momenta and positions were c-numbers. Below, we will treat them as operators which satisfy the standard commutation rules of quantum mechanics. Although from the technical point of view the following part of this section deals with the well-known U(3) symmetry properties of the three-dimensional harmonic oscillator, it is needed here as it provides the background for the subsequent section. Consider therefore, the operator

$$\boldsymbol{p}^2 + \boldsymbol{x}^2 \,, \tag{11}$$

where p_k and x_k are Hermitean operators of momentum and position. Standard commutation relations between position and momentum coordinates $[x_{[x],m}, p_{[p],n}] = i\hbar\ \delta_{mn}$ or, equivalently, $[x_{[p],m}, p_{[p],n}] = i\hbar\ \kappa\ \delta_{mn}$ may be brought to a dimensionless form $[x_m, p_n] = i\delta_{mn}$ by an appropriate choice of $\boldsymbol{x}$ and $\boldsymbol{p}$:

$$\begin{aligned} \boldsymbol{x} &= \frac{1}{\sqrt{\hbar\kappa}}\, \boldsymbol{x}_{[p]} = \sqrt{\frac{\kappa}{\hbar}}\, \boldsymbol{x}_{[x]} \,, \\ \boldsymbol{p} &= \frac{1}{\sqrt{\hbar\kappa}}\, \boldsymbol{p}_{[p]} \,. \end{aligned} \tag{12}$$

Consequently, position $\boldsymbol{x}$ and momentum $\boldsymbol{p}$ operators used in the whole remaining part of this paper (and the operator of Eq. (11)) are dimensionless. Their eigenvalues correspond to numerical values of position and momentum coordinates when the latter are expressed in terms of "natural" units in position ($\sqrt{\hbar/\kappa}$) and momentum ($\sqrt{\hbar\kappa}$) spaces. The corresponding physical quantities in standard units *i.e.* $\boldsymbol{x}_{[x]}$ and $\boldsymbol{p}_{[p]}$ may be recovered from Eq. (12). Let us denote:

$$\begin{aligned} a_k &= \frac{1}{\sqrt{2}}(x_k + ip_k) \,, \\ a_k^\dagger &= \frac{1}{\sqrt{2}}(x_k - ip_k) \,, \end{aligned} \tag{13}$$

with a_m, $a_n^\dagger$ and x_n, p_m satisfying standard commutation rules:

$$\begin{aligned} &\left[a_m, a_n^\dagger\right] = \delta_{mn} \,, && \left[a_m, a_n\right] = 0 \,, && \left[a_m^\dagger, a_n^\dagger\right] = 0 \,, \\ &[x_m, p_n] = i\delta_{mn} \,, && [x_m, x_n] = 0 \,, && [p_m, p_n] = 0 \,. \end{aligned} \tag{14}$$

We have

$$\begin{aligned} \boldsymbol{x}^2 + \boldsymbol{p}^2 &= (x_k - ip_k)\,(x_k + ip_k) - i\,[x_k, p_k] \\ &= (x_k + ip_k)\,(x_k - ip_k) + i\,[x_k, p_k] \,. \end{aligned} \tag{15}$$

There are two Hermitean invariants of U(1) ⊗ SU(3):

$$a_k^\dagger a_k \;=\; \frac{1}{2}\left(\boldsymbol{x}^2+\boldsymbol{p}^2+i\left(\boldsymbol{x}\cdot\boldsymbol{p}-\boldsymbol{p}\cdot\boldsymbol{x}\right)\right)\,, \tag{16}$$

$$a_k a_k^\dagger \;=\; \frac{1}{2}\left(\boldsymbol{x}^2+\boldsymbol{p}^2-i\left(\boldsymbol{x}\cdot\boldsymbol{p}-\boldsymbol{p}\cdot\boldsymbol{x}\right)\right)\,, \tag{17}$$

or, alternatively:

$$\begin{aligned}\{a_k,a_k^\dagger\} &= \boldsymbol{x}^2+\boldsymbol{p}^2\,,\\ [a_k,a_k^\dagger] &= -i\,[\boldsymbol{x},\boldsymbol{p}]=3\,.\end{aligned} \tag{18}$$

They are quantised, with the eigenvalues of $\boldsymbol{x}^2+\boldsymbol{p}^2$ being $2N+3$ $(N=0,1,2,3,\ldots)$.

3.1. Generalised reciprocity — the U(1) factor

The condition that $\boldsymbol{p}^2+\boldsymbol{x}^2$ and $-i\,[\boldsymbol{x},\boldsymbol{p}]=3$ are to be invariant under the transformations of the symmetry group means that $\boldsymbol{p}^2+\boldsymbol{x}^2$ has to commute with all the generators of this group. Consequently, one of these generators must be proportional to $\boldsymbol{p}^2+\boldsymbol{x}^2$ itself. Up to a positive factor, there are two possible choices of a Hermitean generator:

$$R^z\equiv\left(\boldsymbol{p}^2+\boldsymbol{x}^2\right)=\{a_k,a_k^\dagger\}=\sum_{k=1}^{3}R_k^z\,, \tag{19}$$

with (no sum)

$$R_k^z\equiv\{a_k,a_k^\dagger\}\,, \tag{20}$$

or $R^{z\prime}=-R^z$. We shall see later how these two possibilities may be combined. The superscript z means that the corresponding operator is expressed in terms of phase–space variables $z_{1,\ldots,6}$, or $z\equiv(\boldsymbol{x},\boldsymbol{p})$, *i.e.* through x_k and p_k, or a_k and $a_k^\dagger$.

For R^z we have

$$[R^z,x_k]=-2ip_k\,,\qquad\qquad [R^z,p_k]=+2ix_k\,, \tag{21}$$

$$[R^z,a_k]=-2a_k\,,\qquad\qquad \left[R^z,a_k^\dagger\right]=+2a_k^\dagger\,, \tag{22}$$

and for general transformations:

$$\begin{aligned}a_k' &= \exp(i\tfrac{\phi}{2}R^z)a_k\exp(-i\tfrac{\phi}{2}R^z) &&= e^{-i\phi}a_k\,,\\ a_k'^{\,\dagger} &= \exp(i\tfrac{\phi}{2}R^z)a_k^\dagger\exp(-i\tfrac{\phi}{2}R^z) &&= e^{+i\phi}a_k^\dagger\,,\end{aligned} \tag{23}$$

with ϕ defined as the common angle of three identical simultaneous rotations in each of the (x_k, p_k) planes. The above transformations of operators a_k and $a_k^\dagger$ are overall U(1) phase transformations.

For $\phi = \pm\pi$ we have

$$a'_k = -a_k\,, \qquad\qquad {a'_k}^\dagger = -a_k^\dagger\,, \tag{24}$$

i.e. we obtain reflection $\boldsymbol{x} \to \boldsymbol{x}' = -\boldsymbol{x}$, $\boldsymbol{p} \to \boldsymbol{p}' = -\boldsymbol{p}$.

For $\phi = -\pi/2$ (and similarly for $\phi = +\pi/2$) we have

$$a'_k = i a_k\,, \qquad\qquad {a'_k}^\dagger = -i a_k^\dagger\,, \tag{25}$$

i.e. we get Born's reciprocity transformation: $\boldsymbol{x} \to \boldsymbol{x}' = -\boldsymbol{p}$, $\boldsymbol{p} \to \boldsymbol{p}' = \boldsymbol{x}$, which constitutes a "square root" of the standard reflection, as its two consecutive applications yield $\boldsymbol{x} \to \boldsymbol{x}'' = -\boldsymbol{p}' = -\boldsymbol{x}$, $\boldsymbol{p} \to \boldsymbol{p}'' = \boldsymbol{x}' = -\boldsymbol{p}$. In summary, R^z is the generator of generalised reciprocity transformations.

3.2. Generalised rotation — the SU(3) factor

In the standard discussion of the SU(3) properties of the three-dimensional harmonic oscillator, one introduces nine shift operators

$$H^z_{kl} = \tfrac{1}{2}\,\{a_k, a_l^\dagger\}\,, \tag{26}$$

(*i.e.* $R^z_k = 2H^z_{kk}$, no sum), satisfying:

$$\begin{aligned} [H^z_{kl}, a_n^\dagger] &= \delta_{kn} a_l^\dagger\,, \\ [H^z_{kl}, a_n] &= -\delta_{ln} a_k\,. \end{aligned} \tag{27}$$

The eight Hermitian operators of SU(3) are then:

$$\begin{aligned}
F^z_1 &= H^z_{12} + H^z_{21} &&= a_2^\dagger a_1 + a_1^\dagger a_2\,, \\
F^z_2 &= i(H^z_{12} - H^z_{21}) &&= i(a_2^\dagger a_1 - a_1^\dagger a_2)\,, \\
F^z_3 &= H^z_{11} - H^z_{22} &&= a_1^\dagger a_1 - a_2^\dagger a_2\,, \\
F^z_4 &= H^z_{31} + H^z_{13} &&= a_1^\dagger a_3 + a_3^\dagger a_1\,, \\
F^z_5 &= i(H^z_{13} - H^z_{31}) &&= -i(a_1^\dagger a_3 - a_3^\dagger a_1)\,, \\
F^z_6 &= H^z_{23} + H^z_{32} &&= a_3^\dagger a_2 + a_2^\dagger a_3\,, \\
F^z_7 &= i(H^z_{23} - H^z_{32}) &&= i(a_3^\dagger a_2 - a_2^\dagger a_3)\,, \\
F^z_8 &= \tfrac{1}{\sqrt{3}}(H^z_{11} + H^z_{22} - 2H^z_{33}) &&= \tfrac{1}{\sqrt{3}}(a_1^\dagger a_1 + a_2^\dagger a_2 - 2a_3^\dagger a_3)\,.
\end{aligned} \tag{28}$$

When expressed in terms of operators p_k and x_k, the above equations read:

$$\begin{aligned}
F^z_1 &= & p_1p_2 + x_1x_2\,,\\
F^z_2 &= +L_3 = & x_1p_2 - x_2p_1\,,\\
F^z_3 &= & \frac{1}{2}({x_1}^2 + {p_1}^2 - {x_2}^2 - {p_2}^2)\,,\\
F^z_4 &= & p_3p_1 + x_3x_1\,,\\
F^z_5 &= -L_2 = & -(x_3p_1 - x_1p_3)\,,\\
F^z_6 &= & p_2p_3 + x_2x_3\,,\\
F^z_7 &= +L_1 = & x_2p_3 - x_3p_2\,,\\
F^z_8 &= & \tfrac{1}{2\sqrt{3}}({x_1}^2 + {p_1}^2 + {x_2}^2 + {p_2}^2 - 2{x_3}^2 - 2{p_3}^2)\,.
\end{aligned} \tag{29}$$

As before, the superscript z is used to denote $(\boldsymbol{x},\boldsymbol{p})$ jointly: $F^z_b \equiv F_b^{(\boldsymbol{x},\boldsymbol{p})}$, and it reminds us that we are dealing with the representation of SU(3) in terms of momentum and position (or a_k and $a^\dagger_k$) operators. Thanks to the commutation relations of Eq. (14), the order of operators x_k and p_m (a_k and $a^\dagger_m$) on the r.h.s of Eq. (29) is irrelevant.

The nine shift operators may be decomposed in a spherical basis as:

$$H^z_{kl} = \frac{1}{3}H^z_{mm}\delta_{kl} + \frac{1}{2}(H^z_{kl} - H^z_{lk}) + \left(\frac{1}{2}(H^z_{kl} + H^z_{lk}) - \frac{1}{3}H^z_{mm}\delta_{kl}\right)\,, \tag{30}$$

with the first term constituting the trace-only part and equal to

$$\frac{1}{2}\frac{R^z}{3}\delta_{kl}\,, \tag{31}$$

the second term being antisymmetric and equal to

$$-\frac{i}{2}e_{klm}L_m\,, \tag{32}$$

and the third term forming the symmetric traceless part.

It is straightforward to check that

$$\sum_{b=1}^{8}(F^z_b)^2 = \frac{1}{3}(R^z)^2 - 3\,. \tag{33}$$

Thus, the eigenvalues of R^z determine the eigenvalues of the SU(3) invariant $\sum_{b=1}^{8}(F^z_b)^2$ to be $4N(1+N/3)$ $(N = 0,1,2,3,\ldots)$.

The above generators F^z_a (we use a, b, c running from 1 to 8) satisfy standard SU(3) commutation rules

$$[F_a, F_b] = 2if_{abc}F_c\,, \tag{34}$$

with the antisymmetric structure constants f_{abc} equal to: 1 for $abc = (123)$ (where (123) denotes cyclic permutations of 123); $\frac{1}{2}$ for $abc = (147)$, (165), (246), (257), (345), (376); $\frac{\sqrt{3}}{2}$ for $abc = (458)$, (678); and zero otherwise. This corresponds to the defining standard matrix representation of SU(3) generators given for example in [10] (our F_b's are equal to λ_b's of [10]).

When the SO(3) transformation properties of a_k and $a^\dagger_k$ are extended to SU(3), the two sets of operators (a_k and $a^\dagger_k$) transform under SU(3) in two inequivalent ways, *i.e.* as a triplet and an antitriplet. Denoting

$$\boldsymbol{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \qquad (\boldsymbol{a}^\dagger)^T = \begin{bmatrix} a^\dagger_1 \\ a^\dagger_2 \\ a^\dagger_3 \end{bmatrix} \tag{35}$$

the relevant commutation relations take the form

$$[F^z_b, \boldsymbol{a}] \;=\; -F_b\boldsymbol{a}\,, \tag{36}$$

$$\left[F^z_b, (\boldsymbol{a}^\dagger)^T\right] \;=\; +F^*_b(\boldsymbol{a}^\dagger)^T\,. \tag{37}$$

Since $F_b = -F^*_b$ only for $b = 2, 5, 7$, it follows that only under the transformations generated by F^z_2, F^z_5, F^z_7 the two sets of operators (a_k and $a^\dagger_k$) transform in the same way. With $F^z_2 \equiv L_3$, $F^z_5 \equiv -L_2$, $F^z_7 \equiv L_1$, we obtain identical transformations of operators a_k and $a^\dagger_k$ (alternatively: momentum and position) under SO(3):

$$\begin{aligned} [L_k, a_l] &= i\; e_{klm}\; a_m\,, & \left[L_k, a^\dagger_l\right] &= i\; e_{klm}\; a^\dagger_m\,, \\ [L_k, x_l] &= i\; e_{klm}\; x_m\,, & [L_k, p_l] &= i\; e_{klm}\; p_m\,, \end{aligned} \tag{38}$$

with the generators of the SO(3) rotation subgroup of SU(3) satisfying standard commutation rules:

$$[L_k, L_l] = i\; e_{klm}\; L_m\,. \tag{39}$$

On the other hand, while for $b = 1, 3, 4, 6, 8$ the fifteen commutators of F^z_b with x_k are:

$$\begin{aligned} [F^z_1, x_2] = [F^z_3, x_1] = [F^z_4, x_3] = \sqrt{3}\,[F^z_8, x_1] &\;=\; -ip_1\,, \\ [F^z_1, x_1] = -\,[F^z_3, x_2] = [F^z_6, x_3] = \sqrt{3}\,[F^z_8, x_2] &\;=\; -ip_2\,, \\ [F^z_4, x_1] = [F^z_6, x_2] = -\frac{\sqrt{3}}{2}\,[F^z_8, x_3] &\;=\; -ip_3\,, \\ [F^z_1, x_3] = [F^z_3, x_3] = [F^z_4, x_2] = [F^z_6, x_1] &\;=\; 0\,, \end{aligned} \tag{40}$$

the corresponding fifteen commutators of F_b^z with p_k are obtained from the above equations by interchanging p_k with x_k and replacing $-i$ by i, *i.e.*:

$$\begin{aligned}
[F_1^z, p_2] = [F_3^z, p_1] = [F_4^z, p_3] = \sqrt{3}\,[F_8^z, p_1] \ &= \ ix_1\,, \\
[F_1^z, p_1] = -\,[F_3^z, p_2] = [F_6^z, p_3] = \sqrt{3}\,[F_8^z, p_2] \ &= \ ix_2\,, \\
[F_4^z, p_1] = [F_6^z, p_2] = -\frac{\sqrt{3}}{2}\,[F_8^z, p_3] \ &= \ ix_3\,, \\
[F_1^z, p_3] = [F_3^z, p_3] = [F_4^z, p_2] = [F_6^z, p_1] \ &= \ 0\,.
\end{aligned} \tag{41}$$

Born's reciprocity transformations (*i.e.* $\boldsymbol{x} \to \boldsymbol{x}' = -\boldsymbol{p}$, $\boldsymbol{p} \to \boldsymbol{p}' = \boldsymbol{x}$) interchange the above two sets of the commutators of $F^z_{1,3,4,6,8}$ with x_k and p_k, *e.g.* $[F_3^z, p_1] = +ix_1 \to [F_3'^z, x_1'] = [F_3^z, x_1'] = -ip_1'$. (One has $F_b'^z = F_b^z$ since reciprocity transformations commute with the SU(3) generators.)

When we restrict our considerations to the SU(3) transformations only, we observe that they distinguish between the momentum and the position coordinates (*i.e.* the two sets of commutators above, Eqs. (40), (41), differ in the sign in front of i). The SU(3) transformations of Eqs. (40), (41) are more general than the SO(3) ones, and they do permit some $p_k \leftrightarrow x_l$ transformations. However, the SU(3) transformations never permit the full exchange of $\boldsymbol{p}$ into $\boldsymbol{x}$ and *vice versa*. This is embodied in the structure of the commutation relations of Eqs. (40), (41) above. Full interchange of the role of $\boldsymbol{x}$ and $\boldsymbol{p}$ requires consideration of the reciprocity transformations.

This feature of SU(3) transformations was exploited in [5] when proposing a generalisation of the concept of mass. Namely, it was pointed out there that energy of free particles, whether given by a relativistic or non-relativistic formula, is always given in terms of mass and momentum, never in terms of mass and position. In fact, this was essentially Born's remark. In other words, it was observed in [5] that *the standard concept of mass may be said to be directly associated with the concept of momentum* $\boldsymbol{p}$, *not position* $\boldsymbol{x}$. Then, Ref. [5] uses SU(3) to propose a generalisation of this association of mass with momentum.

Let us recall briefly the spirit of the argument of [5]. According to [5], before the concept of mass is introduced there is no difference between the momentum and position coordinates. Thus, the unknown mechanism generating particle masses must somehow divide the six-dimensional object $\boldsymbol{x} \oplus \boldsymbol{p}$ into a pair of canonically conjugated 3-dimensional variables, of which only one is associated with the concept of mass. However, such a division may proceed in four typical ways, namely:

$$\begin{array}{cc} \text{“canonical position“} & \text{“canonical momentum“} \\ (x_1, x_2, x_3) & (p_1, p_2, p_3) \end{array} \tag{42}$$

and

$$\begin{array}{cc} (x_1, p_2, p_3) & (p_1, x_2, x_3) \\ (p_1, x_2, p_3) & (x_1, p_2, x_3) \\ (p_1, p_2, x_3) & (x_1, x_2, p_3)\,. \end{array} \tag{43}$$

Thus, according to the proposal of [5], the concept of mass may be associated not only with the division of Eq. (42) and the canonical momentum being the standard momentum (p_1, p_2, p_3), but also with the remaining three possibilities of Eqs. (43) with canonical momenta involving one component of “standard momentum” and two components of “standard position”. Each of the three additional choices clearly violates ordinary rotational invariance (translational invariance might be satisfied by admitting position differences only). Thus, if there are objects for which mass is/would be linked to generalised momenta as in Eq. (43), they cannot belong — as individual objects — to our classical macro-world, since the latter is rotationally invariant. However, these objects could belong to the macro-world as unseparable components of composite objects, provided the latter are constructed in such a way as to satisfy all the necessary invariance conditions.

4. Linearisation

As hinted above and discussed at length in [5], when the choice of what is considered to be a “momentum” is generalised according to the requirements of the SU(3) symmetry, it follows that the basic inputs of present theories, which are based on the direct association of the concept of mass with standard momentum, should be appropriately generalised. In particular, this should concern the Dirac Hamiltonian, in which no SU(3) “$\boldsymbol{x} \oplus \boldsymbol{p}$” symmetry is seen: when Dirac Hamiltonian is written in the momentum representation, it is completely oblivious to space and time.

In the following, we shall linearise the $\boldsymbol{p}^2 + \boldsymbol{x}^2$ form *á la* Dirac, as proposed in [5], and discuss the structure obtained and its symmetries at some length. In order to achieve this linearisation, we have to enlarge the Dirac matrices by doubling their size and introducing

$$\begin{aligned} A_k &= \sigma_k \otimes \sigma_0 \otimes \sigma_1\,, \\ B_k &= \sigma_0 \otimes \sigma_k \otimes \sigma_2\,, \\ B &= \sigma_0 \otimes \sigma_0 \otimes \sigma_3\,. \end{aligned} \tag{44}$$

The above matrices satisfy the conditions:

$$\begin{aligned} A_kA_l + A_lA_k &= 2\delta_{kl}\,, \\ A_kB_l + B_lA_k &= 0\,, \\ B_kB_l + B_lB_k &= 2\delta_{kl}\,, \\ A_kB + BA_k &= 0\,, \\ B_kB + BB_k &= 0\,, \\ BB &= 1\,. \end{aligned} \tag{45}$$

Matrices A_k,B (or B_k,B) satisfy standard anticommutation relations of Dirac matrices α_k, β. Matrix $B = iA_1A_2A_3B_1B_2B_3$ is the seventh anticommuting matrix of the relevant Clifford algebra. Perhaps it might be somehow related to mass terms, as the nonrelativistic version of the Dirac equation suggests. However, as stressed in [5], we do not know how to introduce the generalised concept of mass into the present approach in any other way than through symmetry arguments based on an analogy with the way in which the concept of mass enters in standard approaches. Rather, it is the other way round, *i.e.* it is hoped that after the present approach is developed sufficiently far, it will provide us with some ideas on the mechanism of mass generation. Consequently, for the purposes of the present paper, we will restrict ourselves to the algebra/geometry of the $\boldsymbol{x} \oplus \boldsymbol{p}$ phase space alone, without writing in an explicit way how mass enters into the game.

Linearisation of form $\boldsymbol{p}^2 + \boldsymbol{x}^2$ suggests that we square

$$\boldsymbol{A} \cdot \boldsymbol{p} + \boldsymbol{B} \cdot \boldsymbol{x}\,, \tag{46}$$

as is certainly appropriate when $\boldsymbol{x}$ and $\boldsymbol{p}$ commute. Assuming that both $\boldsymbol{A}$ and $\boldsymbol{B}$ commute with both $\boldsymbol{p}$ and $\boldsymbol{x}$, we then have

$$(\boldsymbol{A} \cdot \boldsymbol{p}+\boldsymbol{B} \cdot \boldsymbol{x})(\boldsymbol{A} \cdot \boldsymbol{p}+\boldsymbol{B} \cdot \boldsymbol{x}) = (\boldsymbol{p}^2+\boldsymbol{x}^2)\,\sigma_0 \otimes \sigma_0 \otimes \sigma_0 + \sigma_k \otimes \sigma_k \otimes \sigma_3\,. \tag{47}$$

The first term on the right-hand side above is clearly $\mathrm{U}(1) \otimes \mathrm{SU}(3)$ invariant. The second term, whose appearance is due to the nonvanishing of commutators $[x_k, p_k]$ ($k = 1, 2, 3$), is also invariant under $\mathrm{U}(1) \otimes \mathrm{SU}(3)$ transformations, as we shall explicitly see in the following.

The operation of charge conjugation consists in particular of the following substitutions [5]:

$$\begin{aligned} i &\rightarrow -i\,, \\ \boldsymbol{p} &\rightarrow \boldsymbol{p}' = -\boldsymbol{p}\,, \\ \boldsymbol{x} &\rightarrow \boldsymbol{x}' = +\boldsymbol{x}\,, \end{aligned}$$

$$\begin{aligned} \boldsymbol{A} &\rightarrow \boldsymbol{A}' = C\boldsymbol{A}^*C^{-1} = \boldsymbol{A}\,, \\ \boldsymbol{B} &\rightarrow \boldsymbol{B}' = C\boldsymbol{B}^*C^{-1} = \boldsymbol{B}\,, \\ B &\rightarrow B' = CB^*C^{-1} = -B\,, \end{aligned} \tag{48}$$

with $C = -i\sigma_2 \otimes \sigma_2 \otimes \sigma_2 = -C^{-1}$. After the transformation, Eq. (47) reads:

$$\begin{aligned} (-\boldsymbol{A}\cdot\boldsymbol{p} + \boldsymbol{B}\cdot\boldsymbol{x})(-\boldsymbol{A}\cdot\boldsymbol{p} + \boldsymbol{B}\cdot\boldsymbol{x}) \;=\; & (\boldsymbol{p}^2 + \boldsymbol{x}^2)\;\sigma_0 \otimes \sigma_0 \otimes \sigma_0 \\ & - \sigma_k \otimes \sigma_k \otimes \sigma_3\,. \end{aligned} \tag{49}$$

For antiparticles, therefore, the sign of the last term above is reversed.

Since in the discussions of phase–space properties, in addition to x_k and p_k one uses the concept of operators a_k and $a_k^\dagger$, it is natural to introduce their analogs in matrix space, namely:

$$\begin{aligned} C_k &= \frac{1}{\sqrt{2}}(B_k + iA_k)\,, \\ C_k^\dagger &= \frac{1}{\sqrt{2}}(B_k - iA_k)\,. \end{aligned} \tag{50}$$

In terms of C_k and $C_k^\dagger$, the relevant anticommutation relations of Eqs. (45) read:

$$\begin{aligned} \{C_k, C_l\} = \{C_k^\dagger, C_l^\dagger\} &= 0\,, \\ \{C_k, C_l^\dagger\} = \{C_k^\dagger, C_l\} &= 2\delta_{kl}\,, \\ \{C_k, B\} = \{C_k^\dagger, B\} &= 0\,. \end{aligned} \tag{51}$$

4.1. U(1) transformations

The generic linearised expression (46) is form-invariant under the reciprocity transformations, understood here as the following simultaneous transformations of momenta $\boldsymbol{p}$, positions $\boldsymbol{x}$, and matrices $\boldsymbol{A}$, $\boldsymbol{B}$:

$$\begin{aligned} x_k \rightarrow x_k' &= -p_k\,, & p_k \rightarrow p_k' &= x_k\,, \\ B_k \rightarrow B_k' &= -A_k\,, & A_k \rightarrow A_k' &= B_k\,. \end{aligned} \tag{52}$$

It is also invariant under U(1) (generalised reciprocity) transformations acting simultaneously on $\boldsymbol{x}, \boldsymbol{p}$ and $\boldsymbol{B}, \boldsymbol{A}$. In the space of matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, the counterpart of Eq. (19) is:

$$\begin{aligned} R^\sigma = \sum_{k=1}^{3} R_k^\sigma \;&=\; -\frac{i}{2}\,[A_k, B_k] = -\frac{1}{2}\left[C_k, C_k^\dagger\right] \\ &=\; (\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_3) \otimes \sigma_3\,, \end{aligned} \tag{53}$$

with (no sum)

$$R^{\sigma}_k \equiv \sigma_k \otimes \sigma_k \otimes \sigma_3 \,, \tag{54}$$

and we have the following analogs of Eqs. (21), (22):

$$[R^{\sigma}, B_k] \;=\; -2iA_k \,, \qquad [R^{\sigma}, B] = 0 \,, \qquad [R^{\sigma}, A_k] = +2iB_k \,. \tag{55}$$

$$[R^{\sigma}, C_k] \;=\; -2C_k \,, \qquad \left[R^{\sigma}, C^{\dagger}_k\right] = +2C^{\dagger}_k \,. \tag{56}$$

The superscript σ is used to denote that the corresponding operator is represented in terms of tensor products of matrices σ_k. The r.h.s of Eq. (47) is just total R:

$$R = R^z \hat{1} + R^{\sigma} \,, \tag{57}$$

(with $\hat{1} = \sigma_0 \otimes \sigma_0 \otimes \sigma_0$, which shall be omitted from now on), *i.e.* it is the sum of U(1) generators in respective spaces. Both terms on the r.h.s. of Eq. (47) are thus clearly invariant under U(1) transformations.

In addition to R^{σ}, we shall also consider

$$Y \equiv Y^{\sigma} = \frac{1}{3} R^{\sigma} B \,, \tag{58}$$

and

$$Y^z = \frac{1}{3} R^z B \,. \tag{59}$$

The operator $Y^z = \frac{1}{3} R^z B$ is twice the trace-only part of the spherical decomposition of Eq. (30) multiplied by matrix B. Since the eigenvalues of B are ± 1, the two options for the generator discussed earlier (*i.e.* R^z and $-R^z$) are in this way combined.

We have

$$Y = \sum_{k=1}^{3} Y_k = \sum_{k=1}^{3} y_k \otimes \sigma_0 = y \otimes \sigma_0 \,, \tag{60}$$

with (no sum)

$$Y_k \;=\; y_k \otimes \sigma_0 = \frac{1}{3}\, \sigma_k \otimes \sigma_k \otimes \sigma_0 \,, \tag{61}$$

$$y \;=\; \frac{1}{3} \left(\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_3 \right) , \tag{62}$$

satisfying

$$[Y, R^{\sigma}] = [Y, B] = [Y_k, R^{\sigma}] = [Y_k, B] = 0 \,. \tag{63}$$

One finds that y satisfies the following equation:

$$3y^2 + 2y - \sigma_0 \otimes \sigma_0 = 0 \,. \tag{64}$$

Therefore, the eigenvalues of y are $+1/3$ and -1.

Since

$$[y, y_k] = [y_k, y_l] = [Y, Y_k] = [Y_k, Y_l] = 0\,, \tag{65}$$

for any k, l, it follows that y, y_1, y_2, y_3 (Y, Y_1, Y_2, Y_3) can be simultaneously diagonalized. For any k, the eigenvalues of $\sigma_k \otimes \sigma_k$ (no sum) are ± 1, whence the eigenvalues of y_k (Y_k) are $\pm 1/3$. The eigenvalue of y equal to $+1/3$ is obtained three times, and the eigenvalue of -1 once, as shown in Table I. For Y, Y_k the relevant pattern is doubled.

TABLE I

Structure of eigenvalues of y and y_k.

y_1	y_2	y_3	y
$-\frac{1}{3}$	$+\frac{1}{3}$	$+\frac{1}{3}$	$+\frac{1}{3}$
$+\frac{1}{3}$	$-\frac{1}{3}$	$+\frac{1}{3}$	$+\frac{1}{3}$
$+\frac{1}{3}$	$+\frac{1}{3}$	$-\frac{1}{3}$	$+\frac{1}{3}$
$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	-1

Since $R_k^\sigma = 3y_k \otimes \sigma_3$ and $R^\sigma = 3y \otimes \sigma_3$, the eigenvalues of R_k^σ are ± 1, while those of R^σ are ± 1, ± 3. Thus, there is an essential difference between the eigenvalues of R^z (which is constructed from $\boldsymbol{x}$ and $\boldsymbol{p}$), and the eigenvalues of its matrix counterpart R^σ, which together add up to the operator of total generalised reciprocity $R = R^z + R^\sigma$. Namely, we observe that among the eigenvalues of R^σ we have not only ± 3, which are actually the lowest eigenvalues for $\pm R^z$, but also ± 1. This resembles somewhat the situation for the total angular momentum $\boldsymbol{J}$, which is composed of the orbital angular momentum and spin: $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$. Indeed, for the orbital angular momentum (constructed from $\boldsymbol{x}$ and $\boldsymbol{p}$) we have $L = 0, 1, 2, 3, \ldots$, which does not admit half-integer spin values $S = 1/2, 3/2, \ldots$ obtained from matrix representations only.

4.2. SU(3) transformations

In analogy to the shift operators H_{kl}^z of the previous section, we introduce

$$H_{kl}^\sigma = -\frac{1}{4}[C_k, C_l^\dagger]\,, \tag{66}$$

satisfying the analogues of Eqs. (27):

$$[H_{kl}^\sigma, C_n^\dagger] = \delta_{kn} C_l^\dagger\,, \qquad [H_{kl}^\sigma, C_n] = -\delta_{ln} C_k\,. \tag{67}$$

Then, SU(3) generators are represented by:

$$
\begin{aligned}
F_1^\sigma &= \quad H_{12}^\sigma + H_{21}^\sigma = & -\frac{i}{4}\left([A_1,B_2]+[A_2,B_1]\right)\,,\\
F_2^\sigma &= i(H_{12}^\sigma - H_{21}^\sigma) = & -\frac{i}{4}\left([A_1,A_2]+[B_1,B_2]\right)\,,\\
F_3^\sigma &= \quad H_{11}^\sigma - H_{22}^\sigma = & -\frac{i}{4}\left([A_1,B_1]-[A_2,B_2]\right)\,,\\
F_4^\sigma &= \quad H_{13}^\sigma + H_{31}^\sigma = & -\frac{i}{4}\left([A_1,B_3]+[A_3,B_1]\right)\,,\\
F_5^\sigma &= i(H_{13}^\sigma - H_{31}^\sigma) = & -\frac{i}{4}\left([A_1,A_3]+[B_1,B_3]\right)\,,\\
F_6^\sigma &= \quad H_{23}^\sigma + H_{32}^\sigma = & -\frac{i}{4}\left([A_2,B_3]+[A_3,B_2]\right)\,,\\
F_7^\sigma &= i(H_{23}^\sigma - H_{32}^\sigma) = & -\frac{i}{4}\left([A_2,A_3]+[B_2,B_3]\right)\,,\\
F_8^\sigma &= \tfrac{1}{\sqrt{3}}(H_{11}^\sigma + H_{22}^\sigma - 2H_{33}^\sigma) = & -\tfrac{i}{4\sqrt{3}}\left([A_1,B_1]+[A_2,B_2]-2\,[A_3,B_3]\right).
\end{aligned}
\tag{68}
$$

As before, the nine matrices H_{kl}^σ may be decomposed as follows:

$$
H_{kl}^\sigma = \tfrac{1}{3}H_{mm}^\sigma\delta_{kl} + \tfrac{1}{2}(H_{kl}^\sigma - H_{lk}^\sigma) + \left(\tfrac{1}{2}(H_{kl}^\sigma + H_{lk}^\sigma) - \tfrac{1}{3}H_{mm}^\sigma\delta_{kl}\right)\,, \tag{69}
$$

with

$$
\tfrac{1}{3}H_{mm}^\sigma\delta_{kl} = -\tfrac{1}{12}\left[C_m, C_m^\dagger\right]\delta_{kl} = -\tfrac{i}{12}[A_m,B_m]\delta_{kl} = \tfrac{1}{2}\tfrac{R^\sigma}{3}\delta_{kl} \tag{70}
$$

being the trace-only part,

$$
\tfrac{1}{2}(H_{kl}^\sigma - H_{lk}^\sigma) = -\tfrac{1}{8}\left(\left[C_k, C_l^\dagger\right] - \left[C_l, C_k^\dagger\right]\right) = -\tfrac{i}{2}\,e_{klm}S_m \tag{71}
$$

constituting the antisymmetric part with S_m being spin generators

$$
S_k = -\tfrac{i}{8}e_{kmn}\left([A_m,A_n]+[B_m,B_n]\right) = \tfrac{1}{2}\left(\sigma_k\otimes\sigma_0 + \sigma_0\otimes\sigma_k\right)\otimes\sigma_0 \tag{72}
$$

satisfying standard relations

$$
[S_k, S_m] = i\ e_{kmn}S_n\,, \tag{73}
$$

and the rest, *i.e.*

$$
-\tfrac{1}{8}\left(\left[C_k, C_l^\dagger\right] + \left[C_l, C_k^\dagger\right]\right) + \tfrac{1}{12}\left[C_m, C_m^\dagger\right]\,\delta_{kl}\,, \tag{74}
$$

being the symmetric traceless part.

Below we list explicit expressions for all SU(3) generators:

$$
\begin{aligned}
F_1^\sigma &= && \tfrac{1}{2}\left(\sigma_1 \otimes \sigma_2 + \sigma_2 \otimes \sigma_1\right) \otimes \sigma_3 \,, \\
F_2^\sigma &= +S_3 = && \tfrac{1}{2}\left(\sigma_3 \otimes \sigma_0 + \sigma_0 \otimes \sigma_3\right) \otimes \sigma_0 \,, \\
F_3^\sigma &= && \tfrac{1}{2}\left(\sigma_1 \otimes \sigma_1 - \sigma_2 \otimes \sigma_2\right) \otimes \sigma_3 \,, \\
F_4^\sigma &= && \tfrac{1}{2}\left(\sigma_3 \otimes \sigma_1 + \sigma_1 \otimes \sigma_3\right) \otimes \sigma_3 \,, \\
F_5^\sigma &= -S_2 = && -\tfrac{1}{2}\left(\sigma_2 \otimes \sigma_0 + \sigma_0 \otimes \sigma_2\right) \otimes \sigma_0 \,, \\
F_6^\sigma &= && \tfrac{1}{2}\left(\sigma_2 \otimes \sigma_3 + \sigma_3 \otimes \sigma_2\right) \otimes \sigma_3 \,, \\
F_7^\sigma &= +S_1 = && \tfrac{1}{2}\left(\sigma_1 \otimes \sigma_0 + \sigma_0 \otimes \sigma_1\right) \otimes \sigma_0 \,, \\
F_8^\sigma &= && \tfrac{1}{2\sqrt{3}}\left(\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2 - 2\sigma_3 \otimes \sigma_3\right) \otimes \sigma_3 \,.
\end{aligned} \tag{75}
$$

Linear expression (46) is invariant both under U(1) transformations Eqs. (21), (22), (55), (56):

$$[R^z + R^\sigma, (B_k + iA_k)(x_k - ip_k)] = [R^z + R^\sigma, (B_k - iA_k)(x_k + ip_k)] = 0 \tag{76}$$

and under SU(3) transformations. Its SU(3) invariance is particularly transparent when Eq. (46) is rewritten as

$$\tfrac{1}{2}\left((B_k + iA_k)(x_k - ip_k) + (x_k + ip_k)(B_k - iA_k)\right) = C_k a_k^\dagger + a_k C_k^\dagger \,. \tag{77}$$

Under SU(3) transformations, matrices B_k and A_k transform like x_k and p_k of Eqs. (38), (40), (41), $C_k = \frac{1}{\sqrt{2}}(B_k + iA_k)$ behaves like a_k, while $C_k^\dagger = \frac{1}{\sqrt{2}}(B_k - iA_k)$ like $a_k^\dagger$, *i.e.* they transform like a triplet and an antitriplet of SU(3) (see Eqs. (36), (37)). Under standard rotations, A_k and B_k transform like momentum and position (*c.f.* Eq. (38)), while B is a scalar:

$$[S_k, A_m] = i\, e_{kml} A_l \,, \qquad [S_k, B_m] = i\, e_{kml} B_l \,, \qquad [S_k, B] = 0 \,. \tag{78}$$

With $(\boldsymbol{A} \cdot \boldsymbol{p} + \boldsymbol{B} \cdot \boldsymbol{x})$ being SU(3) invariant, its square is also SU(3) invariant and, consequently, both the $R^z = \boldsymbol{p}^2 + \boldsymbol{x}^2$ and the $R^\sigma = \sigma_k \otimes \sigma_k \otimes \sigma_3$ terms on the r.h.s. of Eq. (47) are SU(3) invariant. The latter is thus clearly a singlet of SU(3). Explicitly, in the matrix space, the representations of the U(1) and SU(3) generators commute as they should:

$$[F_b^\sigma, R^\sigma] = 0 \,. \tag{79}$$

Using the expressions of Eq. (75) for the F_b^σ, one calculates that

$$\sum_{b=1}^{8} (F_b^\sigma)^2 = 4(1 + y) \otimes \sigma_0 = 4(1 + Y) \equiv 6(1 - Y)(1 + Y) \,, \tag{80}$$

(where the last equality uses $3Y^2+2Y-1=0$ of Eq. (64)), to be compared with

$$\sum_{b=1}^{8}(F_b^z)^2 = 3(Y^z+1)(Y^z-1)\,. \tag{81}$$

The two eigenvalues of Y label the representations of SU(3) within the Clifford algebra. For the eigenvalue of $Y=-1$ we get $\boldsymbol{F}^2=0$, *i.e.* an SU(3) singlet, while for the eigenvalue of $Y=+1/3$ we get $\boldsymbol{F}^2=16/3$, *i.e.* an SU(3) triplet.

Apart from F_b^σ, we may consider also the following eight matrices $\tilde{F}_b$:

$$\tilde{F}_b^\sigma \;=\; F_b^\sigma \qquad\qquad \text{for } b=2,5,7\,, \tag{82}$$

$$\tilde{F}_b^\sigma \;=\; F_b^\sigma B = BF_b^\sigma \qquad \text{for } b=1,3,4,6,8\,. \tag{83}$$

As the SU(3) structure constants f_{abc} are nonzero only when none or two of indices a,b,c take their values from the second group above, and moreover $B^2=1$, it follows that $\tilde{F}_b^\sigma$ satisfy the standard SU(3) commutation relations as well.

4.3. Gell-Mann–Nishijima–Glashow relation

With $Y=-1$ corresponding to SU(3) singlet and $Y=+1/3$ corresponding to SU(3) triplet, it is tempting to identify Y with the weak hypercharge quantum number $\mathcal{Y}$ appearing in the classification of leptons ($\mathcal{Y}=-1$) and three-coloured quarks ($\mathcal{Y}=+1/3$).

Let us now introduce operator Q defined as

$$Q=\tfrac{1}{6}(R^z+R^\sigma)B=\tfrac{1}{2}\,(Y^z+Y)\,. \tag{84}$$

The above operator constitutes the trace-only part of total shift operators $H_{kl}^z+H_{kl}^\sigma$ multiplied by B. It clearly commutes with the original form $\boldsymbol{p}^2+\boldsymbol{x}^2$. For the lowest eigenvalue of R^z (equal to $+3$) we have $Y^z=B$, *i.e.*

$$Q=\tfrac{1}{2}(B+Y)\,, \tag{85}$$

which, upon introducing

$$I_3\equiv\tfrac{1}{2}B=\tfrac{1}{2}\sigma_0\otimes\sigma_0\otimes\sigma_3 \tag{86}$$

may be rewritten as

$$Q=I_3+\frac{Y}{2}. \tag{87}$$

Thus, for $Y=-1$ we obtain integer values $Q=0,-1$ (each value once), while for $Y=+1/3$ we get fractional values $Q=+2/3,-1/3$ (each value three times), exactly as needed for the description of electric charges of one generation of leptons and three-coloured quarks.

The application of charge conjugation yields

$$\begin{aligned} Y \to Y' &= CY^*C^{-1} = Y\,, \\ B \to B' &= CB^*C^{-1} = -B\,. \end{aligned} \tag{88}$$

Now, the sign of charge is defined relative to the $\boldsymbol{A}\cdot\boldsymbol{p}$ term. Consequently, as under charge conjugation we have $\boldsymbol{A}\cdot\boldsymbol{p}\to\boldsymbol{A}'\cdot\boldsymbol{p}' = -\boldsymbol{A}\cdot\boldsymbol{p}$, we obtain opposite values of $Y_a = -Y$ for antiparticles as needed for the description of antileptons ($\mathcal{Y} = +1$) and antiquarks ($\mathcal{Y} = -1/3$), while $I_3 = B_a/2 = B/2$ is left unchanged. This reversal of the sign of the effective values of Y under particle–antiparticle conjugation can be seen also directly from Eq. (49), which reads:

$$R' = R^z - R^\sigma\,. \tag{89}$$

Consequently, for antiparticles we obtain:

$$Q_a = I_3 + \frac{Y_a}{2}\,, \tag{90}$$

with $Y_a = +1$ and $-1/3$. Hence, proper values of antilepton and antiquark charges follow. I believe that the emergence of relation (87), (90) which mimics the Gell-Mann–Nishijima–Glashow relation [11], with Y being a weak hypercharge and I_3 — the third component of weak isospin, is not accidental.

The SU(2) counterparts of generator I_3, *i.e.* $I_k = \frac{1}{2}\sigma_0\otimes\sigma_0\otimes\sigma_k$ (k=1,2) commute with Y. However, they do not commute with the generators of the original SU(3). In fact, of all 64 elements of the relevant Clifford algebra only the unit element and the three elements: Y, $B = 2I_3$, and $R^\sigma = 3YB = 3BY$ commute with all nine generators R^σ, F^σ_b. Instead, the I_k ($k = 1,2,3$) commute with the modified $\widetilde{\mathrm{SU}(3)}$ generated by $\tilde{F}^\sigma_b$. Now, in the scheme considered, I_3 is proportional to the reflection operator. Indeed: generalised reciprocity transformations in matrix space follow from the analog of Eq. (23) in which R^z is replaced with R^σ, *etc.* Born's reciprocity transformation $\mathcal{R}^\sigma$ is then obtained by setting $\phi = -\pi/2$:

$$\mathcal{R}^\sigma \equiv \exp\left(-i\frac{\pi}{4}R^\sigma\right) = \frac{1}{2\sqrt{2}}(1-iB)(1+3Y)\,, \tag{91}$$

and the full reflection $\mathcal{P}^\sigma$ is

$$\mathcal{P}^\sigma = (\mathcal{R}^\sigma)^2 = -iB\,. \tag{92}$$

Thus, while the generators of U(1) and SU(3) (or $\widetilde{\mathrm{SU}(3)}$) commute with the reflection operator, this is not true for the SU(2) generators $I_{1,2}$. Bearing in mind that in our world parity is violated, I consider this an attractive feature of the approach. Clearly, it would also be interesting to see what is the precise connection (if any) between the above derivation of the Gell-Mann–Nishijima–Glashow relation and the general ideas of [5] and Eqs. (42), (43).

The way in which total hypercharge Y is built out of Y_k exhibits a strict correspondence to the rishon model of leptons and quarks [12]. In that model, neutrino ν_e is constructed from three electrically neutral rishons V as VVV, while up quarks u_{R}, u_{G}, u_{B} are built from one rishon V and two rishons T of charge $Q(T) = +1/3$ each, as VTT, TVT, TTV. Furthermore, electron e^- is built as $\bar{T}\bar{T}\bar{T}$, while down quarks d_R, d_G, d_B are constructed as $\bar{T}\bar{V}\bar{V}$, $\bar{V}\bar{T}\bar{V}$, $\bar{V}\bar{V}\bar{T}$. If one assigns hypercharge values to rishons as $Y(V) = -1/3$, $Y(T) = +1/3$ (corresponding to rishon charges $Q(V) = 1/6 + Y(V)/2 = 0$ and $Q(T) = 1/6 + Y(T)/2 = +1/3$ as in [12]) one reproduces Table I.

4.4. Second invariant

Eq. (77) exhibits two U(1) $\otimes$ SU(3) invariant terms:

$$\begin{aligned}&(B_k - iA_k)(x_k + ip_k)\,,\\&(B_k + iA_k)(x_k - ip_k)\,,\end{aligned} \tag{93}$$

combined to form a Hermitean expression $\boldsymbol{A}\cdot\boldsymbol{p} + \boldsymbol{B}\cdot\boldsymbol{x}$. The two terms of Eq. (93) may be combined to form a second U(1) $\otimes$ SU(3) invariant Hermitean expression:

$$\frac{i}{2}\left((B_k+iA_k)(x_k-ip_k)-(x_k+ip_k)(B_k-iA_k)\right)=-\boldsymbol{A}\cdot\boldsymbol{x}+\boldsymbol{B}\cdot\boldsymbol{p}\,. \tag{94}$$

The existence of the two invariants of Eq. (93) is directly related to the existence of two U(1) $\otimes$ SU(3) invariant Hermitean expressions: $a_k^\dagger a_k$ and $a_k a_k^\dagger$ of Eqs. (16), (17). These two Hermitean expressions lead to Eq. (93) upon the substitution of the first factor in Eq. (16) (*i.e.* $a_k^\dagger$) with $B_k - iA_k$, and the first factor in Eq. (17) (*i.e.* a_k) with $B_k + iA_k$. Although $\boldsymbol{p}\cdot\boldsymbol{p}+\boldsymbol{x}\cdot\boldsymbol{x}$ and $-i\,(\boldsymbol{x}\cdot\boldsymbol{p} - \boldsymbol{p}\cdot\boldsymbol{x})$ do not look akin to each other, their linearised forms, obtained by replacing left $\boldsymbol{p}$ with $\boldsymbol{A}$ and left $\boldsymbol{x}$ with $\boldsymbol{B}$, are just Eqs. (77), (94) and look much more similar.

Squaring "$-\boldsymbol{A}\cdot\boldsymbol{x} + \boldsymbol{B}\cdot\boldsymbol{p}$" leads to an analog of Eq. (47)

$$(-\boldsymbol{A}\cdot\boldsymbol{x}+\boldsymbol{B}\cdot\boldsymbol{p})(-\boldsymbol{A}\cdot\boldsymbol{x}+\boldsymbol{B}\cdot\boldsymbol{p}) = (\boldsymbol{x}^2+\boldsymbol{p}^2)\,\sigma_0\otimes\sigma_0\otimes\sigma_0+\sigma_k\otimes\sigma_k\otimes\sigma_3\,, \tag{95}$$

in which the r.h.s. is completely identical with the r.h.s. in Eq. (47). Changing the sign of $\boldsymbol{x}$ above will again lead, in accordance with the prescription for charge conjugation, to a different sign of the second term on the right-hand side.

Transformations between the two U(1) ⊗ SU(3) invariant structures of Eqs. (77), (94) may be obtained by a formal application of reciprocity transformation in only one of the two spaces involved, *i.e.* either in the phase–space or in the matrix space. In addition to Eqs. (47), (95), we have (see Table II for partial results):

$$\{\boldsymbol{A}\cdot\boldsymbol{p}+\boldsymbol{B}\cdot\boldsymbol{x}\,,-\boldsymbol{A}\cdot\boldsymbol{x}+\boldsymbol{B}\cdot\boldsymbol{p}\}=0\,. \tag{96}$$

Thus, the anticommutators of $\boldsymbol{A}\cdot\boldsymbol{p}+\boldsymbol{B}\cdot\boldsymbol{x}$, and $-\boldsymbol{A}\cdot\boldsymbol{x}+\boldsymbol{B}\cdot\boldsymbol{p}$ lead either to an extension of invariant $\boldsymbol{p}^2+\boldsymbol{x}^2$ into matrix space or to zero. Likewise, the commutators of $\boldsymbol{A}\cdot\boldsymbol{p}+\boldsymbol{B}\cdot\boldsymbol{x}$, and $-\boldsymbol{A}\cdot\boldsymbol{x}+\boldsymbol{B}\cdot\boldsymbol{p}$ yield either zero or the extension of the invariant $\boldsymbol{x}\cdot\boldsymbol{p}-\boldsymbol{p}\cdot\boldsymbol{x}$, *i.e.*:

$$[\boldsymbol{A}\cdot\boldsymbol{p}+\boldsymbol{B}\cdot\boldsymbol{x},-\boldsymbol{A}\cdot\boldsymbol{x}+\boldsymbol{B}\cdot\boldsymbol{p}]=2(\boldsymbol{x}\cdot\boldsymbol{p}-\boldsymbol{p}\cdot\boldsymbol{x})+2i(3Y^zY^\sigma+2F_b^zF_b^\sigma)\,. \tag{97}$$

TABLE II

Commutators $[X,Y]$ and anticommutators $\{X,Y\}$ for $X,Y=\boldsymbol{A}\cdot\boldsymbol{p}$, $\boldsymbol{B}\cdot\boldsymbol{x}$, $\boldsymbol{B}\cdot\boldsymbol{p}$, and $\boldsymbol{A}\cdot\boldsymbol{x}$. Entries for anticommutators are given along the diagonal and above it; entries for commutators — below the diagonal.

$X\backslash Y$	$\boldsymbol{A}\cdot\boldsymbol{p}$	$\boldsymbol{B}\cdot\boldsymbol{x}$	$\boldsymbol{B}\cdot\boldsymbol{p}$	$\boldsymbol{A}\cdot\boldsymbol{x}$
$\boldsymbol{A}\cdot\boldsymbol{p}$	$2\boldsymbol{p}^2$	$-\frac{i}{2}[A_n,B_n]$	0	$\{x_k,p_k\}$
$\boldsymbol{B}\cdot\boldsymbol{x}$	$-\frac{1}{2}[A_k,B_n]\{x_n,p_k\}$	$2\boldsymbol{x}^2$	$\{x_k,p_k\}$	0
$\boldsymbol{B}\cdot\boldsymbol{p}$	$-[A_k,B_n]p_kp_n$	$-3i-\frac{1}{2}[B_n,B_k]\{x_n,p_k\}$	$2\boldsymbol{p}^2$	$+\frac{i}{2}[A_n,B_n]$
$\boldsymbol{A}\cdot\boldsymbol{x}$	$+3i+\frac{1}{2}[A_n,A_k]\{x_n,p_k\}$	$+[A_k,B_n]x_kx_n$	$+\frac{1}{2}[A_n,B_k]\{x_n,p_k\}$	$2\boldsymbol{x}^2$

5. Summary

We argued that nonrelativistic phase space constitutes a natural choice in the search for a space-related origin of quantum numbers of elementary particles and we studied the consequences of this choice somewhat further. Thus, our results follow from geometry of nonrelativistic phase space. We exploited the fact that invariance of the form $\boldsymbol{x}^2+\boldsymbol{p}^2$ (assumed to be fundamental), and of the standard commutation relations, selects U(1) ⊗ SU(3) as the symmetry group. We linearised the fundamental form *á la* Dirac and represented the U(1) ⊗ SU(3) transformations in the relevant Clifford algebra. The eigenvalues of the U(1) generator Y in this algebra were shown to be $\pm(+1/3, +1/3, +1/3, -1)$. We proposed to identify this generator with weak hypercharge $\mathcal{Y}$ in the Standard Model of elementary particles. We showed that the generator of total U(1) transformations contains additive contributions from the phase space and the Clifford algebra and leads to a relation, which we proposed to identify with the Gell-Mann–Nishijima–Glashow formula $Q = I_3 + Y/2$. Connections between the fractional Y eigenvalues and the rishon model of Harari were established.

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