

INFRARED EVOLUTION EQUATIONS: METHOD AND APPLICATIONS*

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It is a brief review on composing and solving Infrared Evolution Equations. They can be used in order to calculate amplitudes of high-energy reactions in different kinematic regions in the double-logarithmic approximation.

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1. Introduction

Double-logarithmic (DL) contributions are of a special interest among radiative corrections. They are interesting in two aspects: first, in every fixed order of the perturbation theories they are the largest terms among the radiative corrections depending on the total energy, and second, they are easiest kind of the corrections to sum up. DL corrections were discovered by Sudakov in Ref. [1] in the QED context. He showed that DL terms appear from integrations over soft, infrared (IR) divergent momenta of virtual photons. All-order resummation of such contributions led to their exponentiations.

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Next important step was done in Refs. [2], where calculation and summation of DL contributions was considered in a systematic way. They found a complementary source of DL terms: soft virtual fermions. This situation appears in the Regge kinematics. The all-order resummations of DL contributions in the Regge kinematic are quite involved and yield more complicated expressions than the Sudakov exponentials. Nonetheless important was the proof of the factorization of bremsstrahlung photons with small k_{\perp} in the high-energy hadronic reactions found in Ref. [3] and often addressed as the Gribov's bremsstrahlung theorem. This statement, suggested originally in the framework of the phenomenological QED of hadrons was extended to QCD in Refs. [4].

Calculation in the double-logarithmic approximation (DLA) amplitudes of the fermion–antifermion annihilation in the Regge forward and backward kinematics involves accounting for DL contributions from soft quarks and soft gluons. These reactions in QED and QCD have many common features. The e^+e^- annihilation was studied in Refs. [2]. The quark–antiquark annihilation DLA was investigated in Ref. [5]. The method of calculation here was based on factorization of virtual quarks and gluons with minimal k_{\perp} . Generally speaking, the results obtained in Ref. [5] could be obtained with the method of Ref. [2], however the technique of calculations suggested in Ref. [5] was much more elegant and efficient. Although Ref. [5] is about quark scattering only, it contains almost all technical ingredients necessary to compose Infrared Evolution Equations for any of elastic scattering amplitudes. Nevertheless, it could not directly be applied to inelastic processes involving emission of soft particles. Such a generalization was obtained in Refs. [4, 6]. The basic idea of the above mentioned method was suggested by L.N. Lipatov: to investigate evolution with respect to the infrared cut-off. The present, sounding naturally term “Infrared Evolution Equations” (IREE) for this method was suggested by M. Krawczyk in Ref. [7], where amplitudes for the backward Compton scattering were calculated in DLA.

The aim of the present brief review is to show how to compose and solve IREE for scattering amplitudes in different field theories and kinematic regions. The paper is organized as follows: in Section 2 we consider composing IREE in the technically simplest hard kinematics. In Section 3 we consider composing IREE in the forward kinematics and apply it to studying the structure function g_1 of the polarized Deep Inelastic scattering (DIS) at small- x . The point is that the commonly used theoretical instrument to study g_1 is DGLAP [11]. It collects logarithms of Q^2 to all orders in α_s but does not include the total resummation of logarithms of $1/x$, though it is important at small- x . Accounting for such a resummation leads to the steep rise of g_1 at the small- x region. As is shown in Section 4, DGLAP lacks the resummation but mimics it in explicitly, through the special choice of

fits for the initial parton densities. Invoking such peculiar fits together with DGLAP to describe g_1 at $x \ll 1$ led to various misconceptions in the literature. They are enlisted and corrected in Section 5. The total resummation of the leading logarithms is essential in the region of small- x . In the opposite region of large x , DGLAP is quite efficient. It is attractive to combine the resummation with DGLAP. The manual for doing it is given in Section 6. Finally, Section 7 is for concluding remarks.

2. IREE for scattering amplitudes in the hard kinematics

From the technical point of view, the hard kinematics, where all invariants are of the same order, is the easiest for analysis. For the simplest, $2 \rightarrow 2$ processes, the hard kinematics means that the Mandelstamm variables s, t, u obey

$$s \sim -t \sim -u. \tag{1}$$

In other words, the cmf scattering angles $\theta \sim 1$ in the hard kinematics. This kinematics is the easiest because the ladder Feynman graphs do not yield DL contributions here and usually the total resummation of DL contributions leads to multiplying the Born amplitude by exponentials decreasing with the total energy. Let us begin with composing and solving an IREE for the well-known object: electromagnetic vertex Γ_μ of an elementary fermion (lepton or quark). As is known,

$$\Gamma_\mu = \bar{u}(p_2)[\gamma_\mu f(q^2) - \frac{\sigma_{\mu\nu} q_\nu}{2m} g(q^2)]u(p_1), \tag{2}$$

where $p_{1,2}$ are the initial and final momenta of the fermion, m stands for the fermion mass and the transfer momentum $q = p_2 - p_1$. Scalar functions f and g in Eq. (2) are called form factors. Historically, DL contributions were discovered by V. Sudakov when he studied the QED radiative corrections to the form factor f at $|q^2| \gg |p_{1,2}^2|$. Following him, let us consider vertex V_μ at

$$|q^2| \gg p_1^2 = p_2^2 = m^2, \tag{3}$$

i.e. we assume the fermion to be on-shell and account for DL electromagnetic contributions. We will drop m for the sake of simplicity.

2.1. IREE for the form factor $f(q^2)$ in QED

Step 1: is to introduce the infrared cut-off μ in the transverse (with respect to the plane formed by momenta $p_{1,2}$) momentum space for all virtual momenta k_i :

$$k_{i \perp} > \mu, \tag{4}$$

where $i = 1, 2, \dots$

Step 2: is to look for the softest virtual particle among soft external and virtual particles. The only option we have is the softest virtual photon. Let denote its transverse momenta $\equiv k_{\perp}$. By definition,

$$k_{\perp} = \min k_{i \perp}. \tag{5}$$

Step 3: according to the Gribov theorem, the propagator of the softest photon can be factorized (*i.e.* it is attached to the external lines in all possible ways) whereas k_{\perp} acts as a new cut-off for other integrations. Adding the Born contribution $f^{\text{Born}} = 1$ we arrive at the IREE for f in the diagrammatic form. It is depicted in Fig. 1.

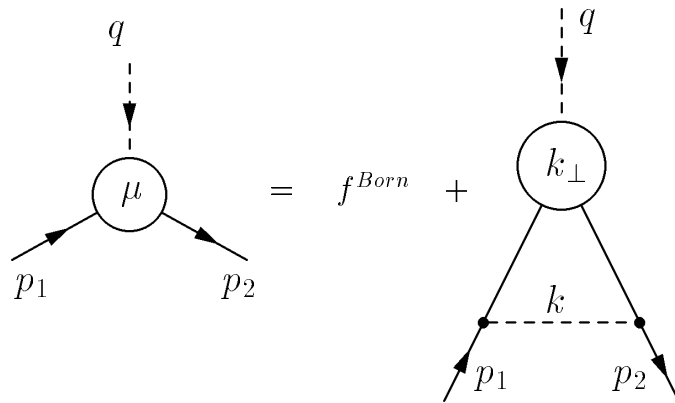


Fig. 1. The IREE for the Sudakov form factor. The letters in the blobs stand for IR cut-off.

IREE in the analytic form are written in the gauge-invariant way, but their diagrammatical writing depends on the gauge. In the present paper we use the Feynman gauge.

Applying to it the standard Feynman rules, we write it in the analytic form:

$$f(q^2, \mu^2) = f^{\text{Born}} - \frac{e^2}{8\pi^2} \times \int \frac{d\alpha d\beta dk_{\perp}^2 \Theta(k_{\perp}^2 - \mu^2) f(q^2, k_{\perp}^2)}{(s\alpha\beta - k_{\perp}^2 + i\varepsilon)(-s\alpha + s\alpha\beta - k_{\perp}^2 + i\varepsilon)(s\beta + s\alpha\beta - k_{\perp}^2 + i\varepsilon)}, \tag{6}$$

where we have used the Sudakov parametrization $k = \alpha p_2 + \beta p_1 + k_{\perp}$ and denoted $s = -q^2 \approx 2p_1 p_2$. As $f(q^2, k_{\perp}^2)$ does not depend on α and β , the DL integration over them can be done with the standard way, so we are left with a simple integral equation to solve:

$$f(q^2, \mu^2) = f^{\text{Born}} - \frac{e^2}{8\pi^2} \int_{\mu^2}^s \frac{dk_{\perp}^2}{k_{\perp}^2} \ln(s/k_{\perp}^2) f(q^2, k_{\perp}^2) . \quad (7)$$

Differentiation of Eq. (7) over μ^2 (more exactly, applying $-\mu^2 \partial/\partial\mu^2$) reduces it to a differential equation

$$\partial f/\partial \ln(s/\mu^2) = -(e^2/8\pi^2) \ln(s/\mu^2) f , \quad (8)$$

with the obvious solution

$$f = f^{\text{Born}} \exp[-(\alpha/4\pi) \ln^2(q^2/m^2)] , \quad (9)$$

where we have replaced μ by m and used $\alpha = e^2/4\pi$. Eq. (9) is the famous Sudakov exponential obtained in Ref. [1].

2.2. IREE for the form factor $g(q^2)$ in QED

Repeating the same steps (see Ref. [8] for detail) leads to a similar IREE for the form factor g :

$$g(q^2, m^2, \mu^2) = g^{\text{Born}}(s, m^2) - \frac{e^2}{8\pi^2} \int_{\mu^2}^s \frac{dk_{\perp}^2}{k_{\perp}^2} \ln(s/k_{\perp}^2) g(q^2, m^2, k_{\perp}^2) , \quad (10)$$

where $g^{\text{Born}}(s, m^2) = -(m^2/s) (\alpha/\pi) \ln(s/m^2)$. Solving this equation and putting $\mu = m$ in the answer leads to the following relation between form factors f and g :

$$g(s) = -2 \frac{\partial f}{\partial \rho} , \quad (11)$$

with $\rho = s/m^2$. Combining Eqs. (9), (11) allows to write a simple expression for the DL asymptotics of the vertex Γ_{μ} :

$$\Gamma_{\mu} = \bar{u}(p_2) \left[\gamma_{\mu} + \frac{\sigma_{\mu\nu} q_{\nu}}{m} \frac{\partial}{\partial \rho} \right] u(p_1) \exp [-(\alpha/4\pi) \ln^2 \rho] . \quad (12)$$

2.3. e^+e^- annihilation into a quark-antiquark pair

Let us consider the e^+e^- annihilation into a quark $q(p_1)$ and $\bar{q}(p_2)$ at high energy when $2p_1 p_2 \gg p_{1,2}^2$. We consider the channel where the e^+e^- pair annihilates into one heavy photon which decays into the $q(p_1) \bar{q}(p_2)$ -pair:

$$e^+e^- \rightarrow \gamma^* \rightarrow q(p_1) \bar{q}(p_2) . \quad (13)$$

We call this process elastic. In this case the most sizable radiative corrections arise from the graphs where the quark and antiquark exchange with gluons and these graphs look absolutely similar to the graphs for the electromagnetic vertex Γ_μ considered in the previous subsection. As a result, accounting for the QCD radiative corrections in DLA to the elastic form factors f_q, g_q of quarks can be obtained directly from Eqs. (9) and (11) by replacement

$$\alpha \rightarrow \alpha_s C_F, \quad (14)$$

with $C_F = (N^2 - 1)/2N = 4/3$.

2.4. e^+e^- annihilation into a quark–antiquark pair and gluons

In addition to the elastic annihilation (13), the final state can include gluons:

$$e^+e^- \rightarrow \gamma^* \rightarrow q(p_1) \bar{q}(p_2) + g(k_1), \dots, g(k_n). \quad (15)$$

We call this process the inelastic annihilation. The QED radiative corrections to the inelastic annihilation (15) in DLA are absolutely the same as the corrections to the elastic annihilation. On the contrary, the QCD corrections account for gluon exchanges between all final particles. This makes composing the IREE for the inelastic annihilation be more involved (see Ref. [4]). The difference to the considered elastic case appears at **Step 2**: look for the softest virtual particle among soft external and virtual particles. Indeed, now the softest particle can be both a virtual gluon and an emitted gluon. For the sake of simplicity let us discuss the 3-particle final state, *i.e.* the process

$$e^+e^- \rightarrow \gamma^* \rightarrow q(p_1) \bar{q}(p_2) + g(k_1). \quad (16)$$

The main ingredient of the scattering amplitude of this process is the new electromagnetic vertex $\Gamma_\mu^{(1)}$ of the quark. In DLA, it is parameterized by new form factors $F^{(1)}$ and $G^{(1)}$

$$\Gamma_\mu = B_1(k_1) \bar{u}(p_2) \left[\gamma_\mu F^{(1)}(q, k_1) - \frac{\sigma_{\mu\nu} q_\nu}{2m} G^{(1)}(q, k_1) \right] u(p_1), \quad (17)$$

where (1) corresponds to the number of emitted gluons, $q = p_1 + p_2$ and l is the polarization vector of the emitted gluon. The bremsstrahlung factor B_1 in Eq. (17) at high energies is expressed through $k_{1\perp}$:

$$B_1 = \left(\frac{p_2 l}{p_2 k_1} - \frac{p_1 l}{p_1 k_1} \right) \approx \frac{2}{k_{1\perp}}. \quad (18)$$

We call $F^{(n)}, G^{(n)}$ inelastic form factors. Let us start composing the IREE for $F^{(1)}$. **Step 1** is the same like in the previous case. **Step 2** opens

more options. Let us first choose the softest gluon among virtual gluons and denote its transverse momentum k_{\perp} . The integration over k_{\perp} runs from μ to s . As $\mu < k_{1\perp} < s$, we have two regions to consider: Region D_1 were

$$\mu < k_{1\perp} < k_{\perp} < \sqrt{s}, \tag{19}$$

and Region D_2 were

$$\mu < k_{\perp} < k_{1\perp} < \sqrt{s}. \tag{20}$$

Obviously, the softest particle in Region D_1 is the emitted gluon, so it can be factorized as depicted in graphs (b), (b') of Fig. 2.

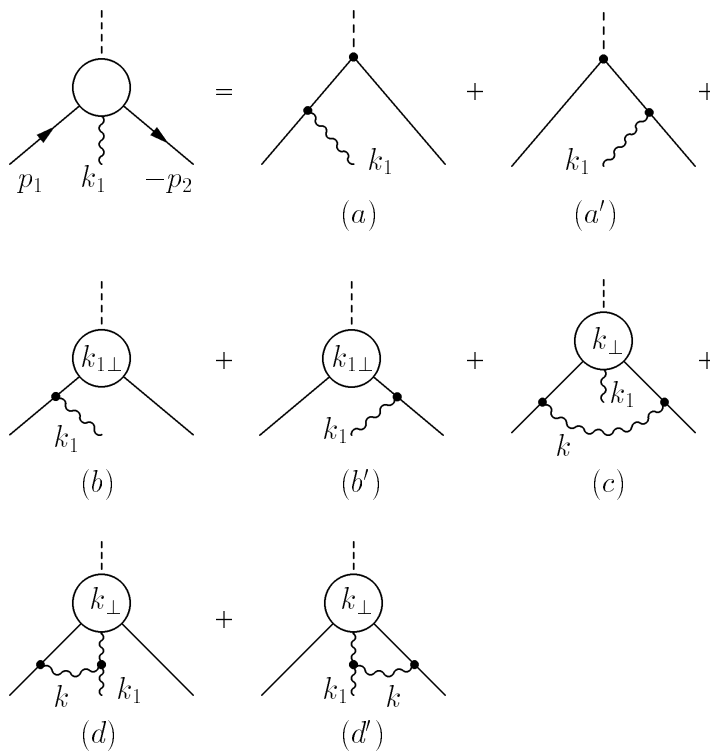


Fig. 2. The IREE for the inelastic quark form factor.

On the contrary, the virtual gluon is the softest in Region D_2 were its propagator is factorized as shown in graphs (c), (d), (d') of Fig. 2. Adding the Born contribution (graphs (a), (a') in Fig. 2) completes the IREE for $F^{(1)}$ depicted in Fig. 2. Graphs (a)–(b') do not depend on μ and vanish when differentiated with respect to μ . Blobs in graphs (c)–(d') do not depend on the longitudinal Sudakov variables, so integrations over α, β can be done like

in the first loop. After that the differential IREE for $F^{(1)}$ is

$$-\mu^2 \frac{\partial F^{(1)}}{\partial \mu^2} = -\frac{\alpha_s}{2\pi} \left[C_F \ln\left(\frac{s}{\mu^2}\right) + \frac{N}{2} \ln\left(\frac{2p_2 k_1}{\mu^2}\right) + \frac{N}{2} \ln\left(\frac{2p_1 k_1}{\mu^2}\right) \right] F^{(1)}. \quad (21)$$

Solving Eq. (21) and using that $(2p_1 k_1)(2p_2 k_1) = s k_{1\perp}^2$ leads to the expression

$$F^{(1)} = \exp\left(-\frac{\alpha_s}{4\pi} \left[C_F \ln^2\left(\frac{s}{\mu^2}\right) + \frac{N}{2} \ln^2\left(\frac{k_{1\perp}^2}{\mu^2}\right) \right]\right), \quad (22)$$

suggested in Ref. [9] and proved in Ref. [4] for any n . The IREE for the form factor $G^{(n)}$ was obtained and solved in Ref. [8]. It was shown that

$$G^{(n)} = -2\partial F^{(n)}/\partial\rho. \quad (23)$$

2.5. Exponentiation of Sudakov electroweak double-logarithmic contributions

The IREE method was applied in Ref. [10] to prove exponentiation of DL correction to the electroweak (EW) reactions in the hard kinematics. There is an essential technical difference between the theories with the exact gauge symmetry (QED and QCD) and the EW interactions theory with the broken $SU(2) \otimes U(1)$ gauge symmetry: only DL contributions from virtual photons yield IR singularities needed to be regulated with the cut-off μ whereas DL contributions involving W and Z bosons are IR stable because the boson masses M_W and M_Z act as IR regulators. In Ref. [10] the difference between M_W and M_Z was neglected and the parameter

$$M \gtrsim M_W \approx M_Z \quad (24)$$

was introduced, in addition to μ , as the second IR cut-off. It allowed to drop masses $M_{W,Z}$. The IREE with two IR cut-offs was composed quite similarly to Eq. (6), with factorizing one by one the softest virtual photon, Z -boson and W -boson. As a result the EW Sudakov form factor F_{EW} is

$$F_{EW} = \exp\left(-\frac{\alpha(Q_1^2 + Q_2^2)}{8\pi} \ln^2(s/\mu^2) - \left[\frac{g^2 C_F^{SU(2)}}{16\pi^2} + \frac{g'^2}{16\pi^2} \frac{(Y_1^2 + Y_2^2)}{4} - \frac{\alpha(Q_1^2 + Q_2^2)}{8\pi}\right] \ln^2(s/M^2)\right), \quad (25)$$

where $Q_{1,2}$ are the electric charges of the initial and final fermion (with W exchanges accounted, they may be different), $Y_{1,2}$ are their hyper-charges and $C_F^{SU(2)} = (N^2 - 1)/2N$, with $N = 2$. We have used in Eq. (25) the standard notations g and g' for the $SU(2)$ and $U(1)$ EW couplings. The

structure of the exponent in Eq. (25) is quite clear: the first, μ dependent term comes from the factorization of soft photons like the exponent in Eq. (9) while other terms correspond to the W and Z factorization; the factor in the squared brackets is the sum of the $SU(2)$ and $U(1)$ Casimirs, with the photon Casimir being subtracted to avoid the double counting. In the limit $\mu = M$ the group factor in the exponent is just the Casimir of $SU(2) \otimes U(1)$.

3. Application of IREE to the polarized Deep-Inelastic Scattering

Cross-sections of the polarized DIS are described by the structure functions $g_{1,2}$. They appear from the standard parametrization of the spin-dependent part $W_{\mu\nu}$ of the hadronic tensor:

$$W_{\mu\nu} = i\varepsilon_{\mu\nu\lambda\rho}q_\lambda \frac{m}{pq} \left[S_\rho g_1(x, Q^2) + \left(S_\rho - p_\rho \frac{Sq}{pq} \right) g_2(x, Q^2) \right], \quad (26)$$

where p , m and S are the momentum, mass and spin of the incoming hadron; q is the virtual photon momentum; $Q^2 = -q^2$; $x = Q^2/2pq$. Obviously, $Q^2 \geq 0$ and $0 \leq x \leq 1$.

Unfortunately, $g_{1,2}$ cannot be calculated in a straightforward model-independent way because it would involve QCD at long distances. To avoid this problem, $W_{\mu\nu}$ is regarded as a convolution of $\Phi_{q,g}$ -probabilities to find a polarized quark or gluon and the partonic tensors $\tilde{W}_{\mu\nu}^{(q,g)}$ parameterized identically to Eq. (26). In this approach $\tilde{W}_{\mu\nu}^{(q,g)}$ involve only QCD at short distances, *i.e.* the Perturbative QCD while long-distance effects are accumulated in $\Phi_{q,g}$. As $\Phi_{q,g}$ are unknown, they are mimicked by the initial quark and gluon densities δq , δg . They are fixed a posteriori from phenomenological considerations. So, the standard description of DIS is:

$$W_{\mu\nu} \approx W_{\mu\nu}^{(q)} \otimes \delta q + W_{\mu\nu}^{(g)} \otimes \delta g. \quad (27)$$

The standard theoretical instrument to calculate g_1 is DGLAP [11] complemented with standard fits [12] for δq , δg . We call it **Standard Approach** (SA). In this approach

$$g_1(x, Q^2) = C_q(x/z) \otimes \Delta q(z, Q^2) + C_g(x/z) \otimes \Delta g(z, Q^2), \quad (28)$$

where $C_{q,g}$ are coefficient functions and $\Delta q(z, Q^2)$, $\Delta g(z, Q^2)$ are called the evolved (with respect to Q^2) quark and gluon distributions. They are found as solutions to DGLAP evolution equations

$$\begin{aligned} \frac{d\Delta q}{d \ln Q^2} &= \frac{\alpha_s(Q^2)}{2\pi} [P_{qq}\Delta q + P_{qg}\Delta g], \\ \frac{d\Delta g}{d \ln Q^2} &= \frac{\alpha_s(Q^2)}{2\pi} [P_{gq}\Delta q + P_{gg}\Delta g], \end{aligned} \quad (29)$$

where P_{ab} are the splitting functions. The Mellin transforms γ_{ab} of P_{ab} are called the DGLAP anomalous dimensions. They are known in the leading order (LO) where they are $\sim \alpha_s$ and in the next-to-leading order (NLO), *i.e.* $\sim \alpha_s^2$. Similarly, $C_{q,g}$ are known in LO and NLO. Details on this topic can be found in the literature (*e.g.* see review [13]). Structure function g_1 has the flavor singlet and non-singlet components, g_1^S and g_1^{NS} . Expressions for g_1^{NS} are simpler, so we will use mostly them in the present paper when possible. It is convenient to write g_1 in the form of the Mellin integral. In particular

$$g_1^{\text{NS DGLAP}}(x, Q^2) = (e_q^2/2) \int_{-i\infty}^{i\infty} \frac{d\omega}{2\pi i} \left(\frac{1}{x}\right)^\omega C_{\text{NS}}(\omega) \delta q(\omega) \times \exp \left[\int_{\mu^2}^{Q^2} \frac{dk_\perp^2}{k_\perp^2} \gamma_{\text{NS}}(\omega, \alpha_s(k_\perp^2)) \right], \quad (30)$$

where μ^2 is the starting point of the Q^2 -evolution; C_{NS} and γ_{NS} are the non-singlet coefficient function and anomalous dimension. In LO

$$\begin{aligned} \gamma_{\text{NS}}(\omega, Q^2) &= \frac{\alpha_s(Q^2) C_F}{2\pi} \left[\frac{1}{\omega(1+\omega)} + \frac{3}{2} + S_2(\omega) \right], \\ C_{\text{NS}}^{\text{LO}}(\omega) &= 1 + \frac{\alpha_s(Q^2) C_F}{2\pi} \left[\frac{1}{\omega^2} + \frac{1}{2\omega} + \frac{1}{2\omega+1} - \frac{9}{2} \right. \\ &\quad \left. + \left(\frac{3}{2} - \frac{1}{\omega(1+\omega)} \right) (S_1(\omega) + S_1^2(\omega) - S_2(\omega)) \right], \end{aligned} \quad (31)$$

with $S_r(\omega) = \sum_{j=1}^{\omega} 1/j^r$. The initial quark and gluon densities in Eq. (30) are defined through fitting experimental data. For example, the fit for δq taken from the first paper in Ref. [12] is

$$\delta q(x) = N x^{-\alpha} \left[(1-x)^\beta (1+\gamma x^\delta) \right], \quad (32)$$

with N being the normalization, $\alpha = 0.576$, $\beta = 2.67$, $\gamma = 34.36$ and $\delta = 0.75$.

DGLAP equations were suggested for describing DIS in the region

$$x \lesssim 1, \quad Q^2 \gg \mu^2 \quad (33)$$

(μ stands for a mass scale, $\mu \gg \Lambda_{\text{QCD}}$) and there is absolutely no theoretical grounds to apply them in the small- x region, however, being complemented with the standard fits they are commonly used at small- x . It is known that

SA provides a good agreement with available experimental data but the price is invoking a good deal of phenomenological parameters. The point is that DGLAP, summing up leading $\ln^k Q^2$ to all orders in α_s , cannot do the same with leading $\ln^k(1/x)$. The later is not important in the region (33) where $\ln^k(1/x) \ll 1$ but becomes a serious drawback of the method at small- x . The total resummation of DL contributions to g_1 in the region

$$x \ll 1, \quad Q^2 \gg \mu^2 \tag{34}$$

was done in Refs. [14]. The weakest point in those papers was keeping α_s as a parameter, *i.e.* fixed at an unknown scale. Accounting for the most important part of single-logarithmic contributions, including the running coupling effects were done in Refs. [15]. In these papers μ^2 was treated as the starting point of the Q^2 -evolution and as the IR cut-off at the same time. The structure function g_1 was calculated with composing and solving IREE in the following way. It is convenient to compose IREE not for g_1 but for forward (with $|t| \lesssim \mu^2$) Compton amplitude M related to g_1 as follows:

$$g_1 = \frac{1}{\pi} M. \tag{35}$$

It is also convenient to use for amplitude M the asymptotic form of the Sommerfeld–Watson transform:

$$M = \int_{-i\infty}^{i\infty} \frac{d\omega}{2\pi i} \left(\frac{s}{\mu^2}\right)^\omega \xi^{(-)}(\omega) F(\omega, Q^2/\mu^2), \tag{36}$$

where $\xi^{(-)}(\omega) = [e^{-i\pi\omega} - 1]/2 \approx -i\pi\omega/2$ is the signature factor. The transform of Eq. (36) and is often addressed as the Mellin transform but one should remember that it coincides with the Mellin transform only partly. IREE for Mellin amplitudes $F(\omega, Q^2)$ look quite simple.

For example, the IREE for the non-singlet Mellin amplitude F^{NS} related to g_1^{NS} by Eqs. (35), (36) is depicted in Fig. 3. In the Mellin space it takes the simple form:

$$[\omega + \partial/\partial y] F^{\text{NS}} = (1 + \omega/2) H_{\text{NS}} F^{\text{NS}}, \tag{37}$$

where $y = \ln(Q^2/\mu^2)$. Eq. (37) involves a new object (the lowest blob in the last term in Fig. 3): the non-singlet anomalous dimension H_{NS} accounting for the total resummation of leading logarithms $1/x$. Like in DGLAP, the anomalous dimension does not depend on Q^2 but, in contrast to DGLAP, H_{NS} can be found with the same method. The IREE for it is algebraic:

$$\omega H_{\text{NS}} = A(\omega) C_{\text{F}}/8\pi^2 + (1 + \omega/2) H_{\text{NS}}^2 + D(\omega)/8\pi^2. \tag{38}$$

The system of Eqs. (37), (38) can be easily solved but before doing it let us comment on them. The left-hand sides of Eqs. (37), (38) are obtained with applying the operator $-\mu^2\partial/\partial\mu^2$ to Eq. (36). The Born contribution in Fig. 3 does not depend on μ and therefore vanishes. The last term in Fig. 3 (r.h.s. of Eq. (37)) is the result of a new, t channel factorization which does not exist in the hard kinematics defined in Eq. (1).

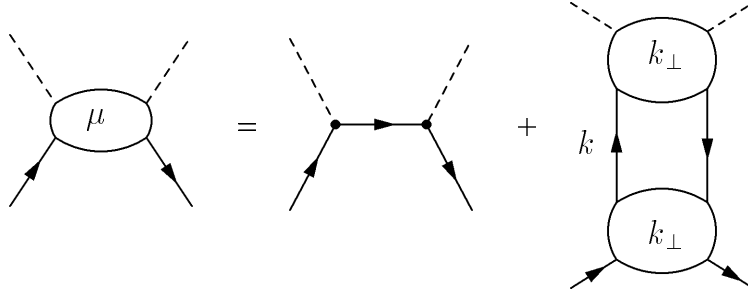


Fig. 3. The IREE for the non-singlet component of the spin structure function g_1 .

In order to compose the IREE for the Compton amplitude M , in accordance with the prescription in the previous section we should first introduce the cut-off μ . Then **Step 2** is to tag the softest particles. In the case under discussion we do not have soft external particles. Had the softest particle been a gluon, it could be factorized in the same way like in Section 2. However, the only option now is to attach the softest propagator to the external quark lines and get $\ln(t/\mu^2) = 0$ from integration over β (*cf.* Eq. (7)). So, the softest gluon does not yield DL contributions. The other option is to find a softest quark. The softest t channel quark pair factorizes amplitude M into two amplitudes (the last term in Fig. 3) and yield DL contributions. The IREE for H_{NS} is different:

- (i) H_{NS} does not depend on Q^2 , so there is not a derivative in the l.h.s. of Eq. (37).
- (ii) The Born term depends on μ and contributes to the IREE (term A in Eq. (37)).
- (iii) As all external particles now are quarks, the softest virtual particle can be both a quark and gluon. The case when it is the t channel quark pair, corresponds to the quadratic term in the r.h.s. of Eq. (37). The case of the softest gluon yields the term D , with

$$D(\omega) = \frac{2C_F}{b^2N} \int_0^\infty d\eta e^{-\omega\eta} \ln\left(\frac{\rho+\eta}{\eta}\right) \left[\frac{\rho+\eta}{(\rho+\eta)^2 + \pi^2} - \frac{1}{\eta} \right], \quad (39)$$

where $b = (33 - 2n_f)/12\pi$ and $\eta = \ln(\mu^2/\Lambda_{\text{QCD}}^2)$.

The term A in Eq. (37) stands instead of α_s . The point is that the standard parametrization $\alpha_s = \alpha_s(Q^2)$ cannot be used at $x \ll 1$ and should be changed (see Ref. [16] for detail). It leads to the replacement α_s by

$$A(\omega) = \frac{1}{b} \left[\frac{\eta}{\eta^2 + \pi^2} - \int_0^\infty \frac{d\rho e^{-\omega\rho}}{(\rho + \eta)^2 + \pi^2} \right]. \quad (40)$$

Having solved Eqs. (37), (38), we arrive at the following expression for g_1^{NS} in the region (34):

$$g_1^{\text{NS}}(x, Q^2) = (e_q^2/2) \int_{-i\infty}^{i\infty} \frac{d\omega}{2\pi i} (1/x)^\omega C_{\text{NS}}(\omega) \delta q(\omega) \exp(H_{\text{NS}}(\omega)y), \quad (41)$$

where the coefficient function $C_{\text{NS}}(\omega)$ is expressed through $H_{\text{NS}}(\omega)$:

$$C_{\text{NS}}(\omega) = \frac{\omega}{\omega - H_{\text{NS}}(\omega)}, \quad (42)$$

and $H_{\text{NS}}(\omega)$ is the solution of algebraic equation (43):

$$H_{\text{NS}} = (1/2) \left[\omega - \sqrt{\omega^2 - B(\omega)} \right], \quad (43)$$

where

$$B(\omega) = (4\pi C_F(1 + \omega/2)A(\omega) + D(\omega)) / (2\pi^2). \quad (44)$$

It is shown in Ref. [17] that the expression for g_1 in the region

$$x \ll 1, \quad Q^2 \lesssim \mu^2 \quad (45)$$

can be obtained from the expressions obtained in Refs. [15] for g_1 in the region (34) by the shift

$$Q^2 \rightarrow Q^2 + \mu_0^2, \quad (46)$$

where $\mu_0 = 1$ GeV for the non-singlet g_1 and $\mu_0 = 5.5$ GeV for the singlet.

4. Comparison of expressions (30) and (41) for g_1^{NS}

Eqs. (30) and (41) read that the non-singlet g_1 is obtained from δq with evolving it with respect to x (using the coefficient function) and with respect to Q^2 (using the anomalous dimension). Numerical comparison of Eqs. (30) and (41) can be done when δq is specified.

4.1. Comparison of small- x asymptotics, neglecting the impact of δq

In the first place let us compare the small- x asymptotics of for $g_1^{\text{NS DGLAP}}$ and g_1^{NS} , assuming that δq does not affect them. In other words, we compare the difference in the x -evolution at $x \rightarrow 0$. Applying the saddle-point method to Eqs. (30) and (41) leads to the following expressions:

$$g_1^{\text{NS DGLAP}} \sim \exp \left[\sqrt{\ln(1/x) \ln \ln(Q^2/\Lambda_{\text{QCD}}^2)} \right] \quad (47)$$

and

$$g_1^{\text{NS}} \sim (1/x)^{\Delta_{\text{NS}}} (Q^2/\mu^2)^{\Delta_{\text{NS}}/2}, \quad (48)$$

where $\Delta_{\text{NS}} = 0.42$ is the non-singlet intercept¹. Expression (47) is the well-known DGLAP asymptotics. Obviously, the asymptotics (48) is much steeper than the DGLAP asymptotics (30).

4.2. Numerical comparison of Eqs. (30) and (41), neglecting the impact of δq

A comparison between Eqs. (30) and (41) strongly depends on the choice of δq but also depends on the difference between the coefficient functions and anomalous dimensions. To clarify the latter we choose the simplest form of δq :

$$\delta q(\omega) = N_q. \quad (49)$$

It corresponds to the evolution from the bare quark where $\delta q(x) = N_q \delta(1 - \mu^2/s)$. Numerical results for $R = [g_1^{\text{NS}} - g_1^{\text{NS DGLAP}}]/g_1^{\text{NS DGLAP}}$ with δq chosen by Eq. (49) manifest (see Ref. [19] for detail) that R increases when x is decreases. In particular, $R > 0.3$ at $x \lesssim 0.05$. It means that the total resummation of leading $\ln^k(1/x)$ cannot be neglected at $x \lesssim 0.05$ and DGLAP cannot be used beyond $x \approx 0.05$. On the other hand, it is well-known that Standard Approach based on DGLAP works well at $x \ll 0.05$. To solve this puzzle, we have to consider the standard fit for δq in more detail.

4.3. Analysis of the standard fits for δq

There are known different fits for δq . We consider the fit of Eq. (32). Obviously, in the ω space Eq. (32) is a sum of pole contributions:

$$\delta q(\omega) = N\eta \left[(\omega - \alpha)^{-1} + \sum m_k (\omega + \lambda_k)^{-1} \right], \quad (50)$$

¹ The singlet intercept is much greater: $\Delta_{\text{S}} = 0.86$.

with $\lambda_k > 0$, so that the first term in Eq. (50) corresponds to the singular term $x^{-\alpha}$ of Eq. (32) and therefore the small- x asymptotics of f_{DGLAP} is given by the leading singularity $\omega = \alpha = 0.57$ of the integrand in Eq. (50) so that the asymptotics of $g_1^{\text{NS DGLAP}}(x, Q^2)$ is not given by the classic exponential of Eq. (47) but actually is the Regge-like:

$$g_1^{\text{NS DGLAP}} \sim C(\alpha)(1/x)^\alpha \left(\ln(Q^2/\Lambda^2) / \ln(\mu^2/\Lambda^2) \right)^{\gamma(\alpha)/b}, \quad (51)$$

with $b = (33 - 2n_f)/12\pi$. Comparison of Eq. (48) and Eq. (51) demonstrates that both DGLAP and our approach lead to the Regge behavior of g_1 , though the DGLAP prediction is more singular than ours. Then, they predict different Q^2 behavior. However, it is important that our intercept Δ_{NS} is obtained by the total resummation of the leading logarithmic contributions and without assuming singular fits for δq whereas the SA intercept α in Eq. (47) is generated by the phenomenological factor $x^{-0.57}$ of Eq. (32) which makes the structure functions grow when x decreases and mimics in fact the total resummation². In other words, the role of the higher-loop radiative corrections on the small- x behavior of the non-singlets is, actually, incorporated into SA phenomenologically, through the initial parton densities fits. It means that the singular factors can be dropped from such fits when the coefficient functions account for the total resummation of the leading logarithms and therefore fits for δq become regular in x in this case. They also can be simplified. Indeed, if x in the regular part $N \left[(1-x)^\beta (1+\gamma x^\delta) \right]$ of the fit (32) is not large, all x dependent terms can be neglected. So, instead of the rather complicated expression of Eq. (32), δq can be approximated by a constant or a linear form

$$\delta q(x) = N(1 + ax), \quad (52)$$

with 2 phenomenological parameters instead of 5 in Eq. (32).

5. Correcting misconceptions

The total resummation of $\ln^k(1/x)$ allows to correct several misconceptions popular in the literature. We list and correct them below.

Misconception 1: Impact of non-leading perturbative and non-perturbative contributions on the intercepts of g_1 is large.

Actually: Confronting our results and the estimates of the intercepts in Refs. [18] obtained from fitting available experimental data manifests that

² We remind that our estimates for the intercepts $\Delta_{\text{NS}}, \Delta_{\text{S}}$ were confirmed (see Refs. [18]) by analysis of the experimental data.

the total contribution of non-leading perturbative and non-perturbative contributions to the intercepts is very small, so the main impact on the intercepts is brought by the leading logarithms.

Misconception 2: Intercepts of g_1 should depend on Q^2 through the parametrization of the QCD coupling $\alpha_s = \alpha(Q^2)$.

Actually: This is groundless from the theoretical point of view and appears only if the parametrization of the QCD coupling $\alpha_s = \alpha(k_\perp^2)$ is kept in all ladder rungs. It is shown in Ref. [16] that this parametrization cannot be used at small- x and should be replaced by the parametrization of Eq. (40).

Misconception 3: Initial densities $\delta q(x)$ and $\delta g(x)$ are singular but they are defined at x not too small. Later, being convoluted with the coefficient functions, they become less singular.

Actually: It is absolutely wrong: Eq. (50) proves that the pole singularity $x^{-\alpha}$ in the fits does not become weaker with the x -evolution.

Misconception 4: Fits for the initial parton densities are complicated because they mimic unknown non-perturbative contributions.

Actually: Our results demonstrate that the singular factors in the fits mimic the total resummation of $\ln^k(1/x)$ and can be dropped when the resummation is accounted for. In the regular part of the fits the x -dependence is essential for large x only, so impact of non-perturbative contributions is weak at the small- x region.

Misconception 5: Total resummations of $\ln^k(1/x)$ may become of some importance at extremely small- x but not for x available presently and in a forthcoming future.

Actually: The efficiency of SA in the available small- x range is based on exploiting the singular factors in the standard fits to mimic the resummations. So, the resummations have always been used in SA at small- x in an inexplicit way, through the fits, but without being aware of it.

6. Combining the total resummation and DGLAP

The total resummation of leading logarithms of x considered in Section 4 is essential at small- x . When $x \sim 1$, all terms $\sim \ln^k(1/x)$ in the coefficient functions and anomalous dimensions cannot have a big impact compared to other terms. DGLAP accounts for those terms. It makes DGLAP be more precise at large x than our approach. So, there appears an obvious appeal to combine the DGLAP coefficient functions and anomalous dimensions with our expressions in order to obtain an approach equally good in the whole

range of $x : 0 < x < 1$. The prescription for such combining was suggested in Ref. [19]. Let us, for the sake of simplicity, consider here combining the total resummation and LO DGLAP. The generalization to NLO DGLAP can be done quite similarly. The prescription consists of the following points:

Step A: Take Eqs. (31) and replace α_s by A of Eq. (40), converting γ_{NS} into $\tilde{\gamma}_{\text{NS}}$ and $C_{\text{NS}}^{\text{LO}}$ into $\tilde{C}_{\text{NS}}^{\text{LO}}$.

Step B: Sum up the obtained expressions and Eqs. (42), (43):

$$\tilde{c}_{\text{NS}} = \tilde{C}_{\text{NS}}^{\text{LO}} + H_{\text{S}}, \quad \tilde{h}_{\text{NS}} = \tilde{\gamma}_{\text{NS}} + H_{\text{NS}}. \tag{53}$$

New expressions $\tilde{c}_{\text{NS}}, \tilde{h}_{\text{NS}}$ combine the total resummation and DGLAP but they obviously contain the double counting: some of the one-loop contributions are present both in Eqs. (31) and in Eqs. (42), (43). To avoid the double counting, let us expand Eqs. (42), (43) into series and retain in the series only the first loop contributions³:

$$H_{\text{NS}}^{(1)} = \frac{A(\omega C_{\text{F}})}{2\pi} \left[\frac{1}{\omega} + \frac{1}{2} \right], \quad C_{\text{NS}}^{(1)} = 1 + \frac{A(\omega C_{\text{F}})}{2\pi} \left[\frac{1}{\omega^2} + \frac{1}{2\omega} \right]. \tag{54}$$

Finally, there is **Step C:** Subtract the first-loop expressions (54) from Eq. (53) to get the combined or “synthetic” coefficient function c_{NS} and anomalous dimension h_{NS} :

$$c_{\text{NS}} = \tilde{c}_{\text{NS}} - C_{\text{NS}}^{(1)}, \quad h_{\text{NS}} = \tilde{h}_{\text{NS}} - H_{\text{NS}}^{(1)}. \tag{55}$$

Substituting Eqs. (55) in Eq. (41) leads to the expression for g_1^{NS} equally good at large and small- x . This description does not require singular factors in the fits for the initial parton densities. An alternative approach for combining DLA expression for g_1 was suggested in Ref. [20]. However, the parametrization of α_s in this approach was simply borrowed from DGLAP, which makes this approach unreliable at small- x .

7. Conclusion

We have briefly considered the essence of the IREE method together with examples of its application to different processes. They demonstrate that IREE is indeed the efficient and reliable instrument for all-orders calculations in QED, QCD and the Standard Model of EW interactions.

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³ For combining the total resummation with NLO DGLAP one more term in the series should be retained.

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