

RECURSIVE EQUATIONS FOR ARBITRARY SCATTERING PROCESSES AT TREE ORDER AND BEYOND*

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(Received May 20, 2007)

The usefulness of recursive equations to compute scattering matrix elements for arbitrary processes is reviewed. The importance of the recently presented reduction method at the integrand level that opens the possibility to use efficiently recursive equations for one-loop amplitude and cross sections, is also briefly discussed.

PACS numbers: 11.10.-z, 12.38.Bx, 12.15.Lk, 13.85.Hd

1. Introduction

Recursive equations to compute scattering matrix elements have been used extensively over the last years in order to obtain results for multi-leg amplitudes. Their history started essentially with the work of Berends and Giele [1], who were able to prove the conjectured simple all- n form of Parke and Taylor [2] for the MHV amplitudes in QCD. The recognition of their usefulness has been expanded recently by the discovery of a new class of recursive equations, by Britto, Cachazo and Feng [3] and Witten [4].

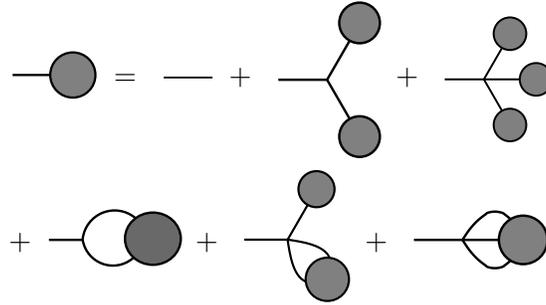
In this paper we are considering the Dyson–Schwinger (DS) recursive approach [5–8], and show how this can be used as a general framework for matrix elements computation. We are also briefly reviewing the possibility of extending the approach at the one loop level.

* Presented at the Cracow Epiphany Conference on Precision Physics and Monte Carlos for LHC, Cracow, Poland, 4–6 January 2007.

2. The Dyson–Schwinger approach

The traditional representation of the scattering amplitude in terms of Feynman graphs results to a computational cost that grows like the number of those graphs, therefore as $n!$ (at tree order), where n is the number of particles involved in the scattering process.

An alternative¹ to the Feynman graph representation is provided by the Dyson–Schwinger approach [7]. Dyson–Schwinger equations express recursively the n -point Green’s functions in terms of the $1-, 2-, \dots, (n-1)$ -point functions. In the framework of a theory with three- and four-point vertices the DS equations are rather simple and their diagrammatic representation is given below, for $1 \rightarrow n$ [12–17] amplitude:



Omitting the contribution of the second line in the above formula is equivalent to restrict ourselves at tree order. In order to get an idea of the actual mathematical form of these equations, let us consider the simplest case where we are interested to “count graphs”, so by dropping all propagators, couplings, wave-functions, *etc.*, we end up with the following equation:

$$a(n) = \delta_{n,1} + \frac{1}{2!} \sum \frac{n!}{n_1!n_2!} a(n_1)a(n_2)\delta_{n_1+n_2,n} \\ + \frac{1}{3!} \sum \frac{n!}{n_1!n_2!n_3!} a(n_1)a(n_2)a(n_3)\delta_{n_1+n_2+n_3,n}$$

with the initial condition $a(0) = 0$; $a(n)$ is nothing more than the number of Feynman graphs, contributing to the $1 \rightarrow n$ matrix element.

The computational cost of DS equations grows like $\sim 3^n$, which essentially counts the steps used to solve the recursive equations. Obviously for large n there is a tremendous saving of computational time, compared to the $n!$ growth of the Feynman graph approach.

¹ For other alternatives see [10,11].

Color representation or color decomposition of the amplitude is a major issue when dealing with multi-parton processes. Let us consider n -gluon scattering with external momenta $\{p_i\}_1^n$, helicities $\{\varepsilon_i\}_1^n$ and colors $\{a_i\}_1^n$ of gluons $i = 1, \dots, n$. As is well known the total amplitude can be expressed as a sum of single trace terms [18]:

$$\mathcal{M}(\{p_i\}_1^n, \{\varepsilon_i\}_1^n, \{a_i\}_1^n) = 2ig^{n-2} \sum_{I \in P(2, \dots, n)} \text{Tr}(t^{a_1} t^{a_{\sigma_I(2)}} \dots t^{a_{\sigma_I(n)}}) \mathcal{A}_I(\{p_i\}_1^n, \{\varepsilon_i\}_1^n),$$

where $\sigma_I(2 : n)$ represent the I -th permutation of the set $\{2, \dots, n\}$ and $\text{Tr}(t^{a_1} t^{a_{\sigma_I(2)}} \dots t^{a_{\sigma_I(n)}})$ represents a trace of generators of the $SU(N_c)$ gauge group in the fundamental representation. For processes involving quarks a similar but much more cumbersome expression can be derived [18].

One of the most interesting aspects of this decomposition is the fact that the $\mathcal{A}_I(\{p_i\}_1^n, \{\varepsilon_i\}_1^n)$ functions (called dual, partial or color-ordered amplitudes), which contain all the kinematic information, depend on the permutation and are gauge invariant and cyclically symmetric in the momenta and helicities of gluons. The color ordered amplitudes are simpler than the full amplitude because they only receive contributions from diagrams with a particular cyclic ordering of the external gluons (planar graphs).

Of course to get the full amplitude one has to square the matrix element

$$\sum_{\{a_i\}_1^n \{\varepsilon_i\}_1^n} |\mathcal{M}(\{p_i\}_1^n, \{\varepsilon_i\}_1^n, \{a_i\}_1^n)|^2 = g^{2n-4} \sum_{\varepsilon} \sum_{ij} \mathcal{A}_I \mathcal{C}_{IJ} \mathcal{A}_J^*,$$

where the $(n-1)! \times (n-1)!$ dimensional color matrix can be written in the most general form as follows:

$$\mathcal{C}_{IJ} = \sum_{1 \dots N_c} \text{Tr}(t^{a_1} t^{a_{\sigma_I(2)}} \dots t^{a_{\sigma_I(n)}}) \text{Tr}(I \leftrightarrow J)^*. \tag{1}$$

There exists a much simpler approach, in fact far superior from the point of view of an automatized numerical calculation, where the matrix element is represented as follows [7, 8, 19, 20]

$$\mathcal{M}(\{p_i\}_1^n, \{\varepsilon_i\}_1^n, \{c_i, a_i\}_1^n) = 2ig^{n-2} \sum_{I \in P(2, \dots, n)} D_I \mathcal{A}_I(\{p_i\}_1^n, \{\varepsilon_i\}_1^n)$$

with c_i, a_i the color and anticolor indices for each external particle, *i.e.* $(c, 0)$ for quarks, $(0, a)$ for antiquarks, (c, a) for gluons and $(0, 0)$ for non-colored particles, and

$$D_I = \delta_{c_1, a_{\sigma_I(1)}} \delta_{c_2, a_{\sigma_I(2)}} \dots \delta_{c_n, a_{\sigma_I(n)}}$$

or in a more abstract notation

$$D_I = \delta_{1,\sigma_I(1)} \delta_{2,\sigma_I(2)} \cdots \delta_{n,\sigma_I(n)},$$

where $\sigma_I(1 : n)$ represent the I -th permutation of the set $\{1, 2, \dots, n\}$. The sequence of numbers $i, \sigma_I(i)$, $i = 1 \dots n$, is identified as a color-connection configuration, describing the way the color connection is structured. In that sense, no explicit reference to “real” color indices is made. Finally the color matrix takes a very simple form

$$\mathcal{C}_{IJ} = N_c^{m(\sigma_I, \sigma_J)}, \quad (2)$$

where $1 \leq m(\sigma_I, \sigma_J) \leq n$ counts how many common cycles the permutations σ_I and σ_J have. For a detailed description, see [8].

Recursive equations can be written both for the full amplitude, \mathcal{M} , and for the color ordered, \mathcal{A} . In the latter case the DS equations are identical to the Berends–Giele ones.

For numerical applications the computation of the color ordered amplitudes suffers from the $n!$ growth related to the number of color-flow or color-connection configurations. In such cases it is preferable to write down DS equations for the full amplitude $\mathcal{M}(\{p_i\}_1^n, \{\varepsilon_i\}_1^n, \{c_i, a_i\}_1^n)$ and then perform the incoherent sum

$$\sum_{c_i, a_i=1\dots 3} |\mathcal{M}(\{p_i\}_1^n, \{\varepsilon_i\}_1^n, \{c_i, a_i\}_1^n)|^2$$

by Monte-Carlo methods. We have recently extended HELAC so that a Monte-Carlo over “real” colors, or color-configurations can be performed. A color configuration is identified by the sequence of numbers $\{c_i, a_i\}_1^n$, $c_i, a_i = 1 \dots 3$. The details are given in [8].

Besides the problem related to the color treatment, the summation over different flavors is also a very important problem when the flavor of partons at the final state is unidentified, as usually. In that case a Monte Carlo treatment over flavor degrees of freedom has been proposed some time ago [21], showing that the purely gluonic contribution falls from 45.7% for 3-jet, to 26.6% for 8-jet production [21].

HELAC/PHEGAS [6, 9] is a computer package², which is an implementation of the Dyson–Schwinger method and is able to compute helicity amplitudes and cross sections (based on multi-channel phase space generator PHEGAS [9]) for arbitrary scattering process in the Standard Model at the parton level. The current version includes also a summation over all subprocesses, relevant for processes like $pp \rightarrow A + n\text{jets}$ or $p\bar{p} \rightarrow A + n\text{jets}$, where

² <http://www.cern.ch/helac-phegas>

A represents any exclusive final state besides jets (*i.e.* light quarks), consisting of arbitrary combination of leptons, gauge bosons or heavy quarks. The output is written in the latest standard Les Houches Accord format [22] and an interface to PYTHIA [23] including optionally UPVETO for merging consistently with the parton shower algorithm, has been developed.

3. Beyond tree-order

Recursive equations can also simplify the calculation of one-loop amplitudes. In a recent paper [24], a reduction technique for arbitrary one-loop amplitudes at *the integrand level* by solving numerically the set of kinematical equations for the integration momentum has been presented. The method requires a minimal information about the form of the one-loop amplitude and therefore it is well suited for a numerical implementation. The method applies for any set of internal and/or external masses, so that one is capable to study the full electroweak model, without being limited to massless theories.

As is well known the main issue in one-loop calculations is to reduce, using computer algebra, generic one-loop integrals into a minimal set of scalar integrals (and remaining pieces, the so called rational terms). This can be achieved by tensor reduction [25–27]. For multi-particle processes though this method becomes quite cumbersome. This is due to the large number of terms generated and the appearance of numerical instabilities due to the zeros of Gram determinants. On the other hand, several numerical or semi-numerical methods aim for a direct numerical computation of the tensor integrals [28]. Although purely numerical methods can in principle deal with any configuration of masses and also allow for a direct computation of both non-rational and rational terms, their applicability remains limited due to the high demand of computational resources and the non-existence of an efficient automatization.

In a different approach, the one-loop amplitude rather than individual integrals are evaluated using the unitarity cut method [29], which relies on tree amplitudes and avoids the computation of Feynman diagrams. In another development, the four-dimensional unitarity cut method has been used for the calculation of QCD amplitudes [30], using twistor-based approaches [31]. Moreover, a generalization of the the unitarity cut method in d dimensions, has been pursued recently [32].

The starting point of the reduction at the integrand level method, that combines the traditional algebraic approach with unitarity based ideas, is the general expression for the *integrand* of a generic m -point one-loop amplitude [24]

$$A(\bar{q}) = \frac{N(q)}{\bar{D}_0 \bar{D}_1 \dots \bar{D}_{m-1}}$$

with

$$\bar{D}_i = (\bar{q} + p_i)^2 - m_i^2,$$

where a bar is used to denote objects living in $n = 4 + \varepsilon$ dimensions, and $\bar{q}^2 = q^2 + \tilde{q}^2$ ³. The numerator $N(q)$ depends on the 4-dimensional denominators $D_i = (q + p_i)^2 - m_i^2$ and can be written as follows

$$\begin{aligned} N(q) = & \sum_{i_0 < i_1 < i_2 < i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} D_i \\ & + \sum_{i_0 < i_1 < i_2}^{m-1} \left[c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2) \right] \prod_{i \neq i_0, i_1, i_2}^{m-1} D_i \\ & + \sum_{i_0 < i_1}^{m-1} \left[b(i_0 i_1) + \tilde{b}(q; i_0 i_1) \right] \prod_{i \neq i_0, i_1}^{m-1} D_i \\ & + \sum_{i_0}^{m-1} \left[a(i_0) + \tilde{a}(q; i_0) \right] \prod_{i \neq i_0}^{m-1} D_i \\ & + \tilde{P}(q) \prod_i^{m-1} D_i. \end{aligned}$$

This is the master equation. After integration over the loop momentum, the coefficients d, c, b, a will multiply the known one-loop scalar functions, whereas the terms proportional to $\tilde{d}, \tilde{c}, \tilde{b}, \tilde{a}$, the so-called spurious terms, will give vanishing contributions. Such a separation is always possible, as shown in Ref. [24], and, with this choice, the former set of coefficients is therefore immediately interpretable as the ensemble of the coefficients of all possible 4, 3, 2, 1-point one-loop functions contributing to the amplitude.

Once the master equation above is established, the task of computing the one-loop amplitude is then reduced to the algebraical problem of determining the coefficients d, c, b, a by evaluating the function $N(q)$ a sufficient number of times, at different values of q , and then inverting the system. That can be achieved quite efficiently by singling out particular choices of q such that, systematically, 4, 3, 2 or 1 among all possible denominators D_i vanishes. Then the system of equations is solved iteratively. First one determine all possible 4-point functions, then the 3-point functions and so on.

³ \tilde{q}^2 is ϵ -dimensional and $\tilde{q} \cdot q = 0$.

The method is opening a new direction in one loop computations, offering the possibility of a purely numerical approach which on the same time is fully algebraic, in the sense that no integration over loop-related quantities, *i.e.* loop momentum or Feynman parameters, is needed. The only ingredient is the calculation of the one-loop amplitudes (residues) on a set of specific loop momenta dictated by the zeros of the propagator functions.

4. Outlook

Recursive equations have been proven to be the framework for an efficient matrix element computation for arbitrary scattering processes. They are the basic ingredients towards the construction of an automatized generator including NLO corrections. The fusion with parton-shower generators and the understanding of the working of this fusion, will be one of the main tasks in the near future. Precision calculations will offer the solid basis needed for discoveries in future high-energy colliders.

C.G.P acknowledges support by the EU Transfer of Knowledge programme MTKD-CT-2004-014319.

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