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UPSCALING IN DIFFUSION PROBLEMS IN DOMAINS WITH SEMIPERMEABLE BOUNDARIES

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The asymptotic behavior of the solutions of some nonlinear variational inequalities with highly oscillating coefficients modeling chemical reactive flows through the exterior of a domain containing periodically distributed reactive solid obstacles, with period ε, is analyzed. In this kind of boundary-value problems there are involved two distinct sources of oscillations, one coming from the geometrical structure of the domain and the other from the fact that the medium is heterogeneous. We focus on the only case in which a real interaction between both these sources appears, *i.e.* the case in which the obstacles are of the so-called *critical size* and we prove that the solution of such a boundary-value problem converges to the solution of a new problem, associated to an operator which is the sum of a standard homogenized one and extra zero order terms coming from the geometry and the nonlinearity of the problem.

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1. Introduction

In the last years, the theory of variational inequalities became an important mathematical tool for the study of partial differential equations, with a wide range of applications to unilateral mechanics, plasticity theory, physics, economics and engineering (see, for instance, [8] and [10]).

The goal of this paper is to get the effective behavior of a nonlinear diffusion problem through the exterior of a domain containing periodically distributed reactive solid grains (or reactive obstacles) with semipermeable boundaries. Also, this type of problem could arise in elasticity (Signorini's problem), in problems concerning heat transfer across a surface and in biology, in the modeling of chemical flows in cells surrounded by semipermeable membranes. All these problems may be rigorously formulated as variational inequalities.

The boundary-value problem which we shall address here, based on one of the most important models proposed in the literature for the study of diffusion problems in domains with semipermeable membranes, involves the existence of two distinct sources of oscillations, one coming from the geometrical structure of the domain and the other one from the fact that the medium is also an heterogeneous one. Moreover, we shall consider also another nonlinear term, modeling some chemical reactions, which will give rise to a new extra-term in the limit problem.

Let Ω be an open bounded set in $\mathbb{R}^n$ and let us insert in Ω a set of periodically distributed reactive obstacles. As a result, we obtain, outside the obstacles, an open set Ω^ε which will be referred to as being the *exterior domain*; ε represents a small parameter related to the characteristic size of the reactive obstacles.

Let us consider a family of inhomogeneous media occupying the region Ω, parameterized by ε and represented by $n \times n$ matrices $A^\varepsilon(x)$ of real-valued coefficients defined on Ω. The positive parameter ε will also define a length scale measuring how densely the inhomogeneities are distributed in Ω.

The nonlinear problem studied in this paper concerns the stationary reactive flow of a fluid confined in Ω^ε, of concentration u^ε, reacting inside Ω^ε. We assume that the boundaries of the obstacles are acting like semipermeable membranes, *i.e.* we shall impose a typical nonlinear interfacial condition arising in the case of semipermeable membranes, *i.e.* a relationship between the concentration and the normal flux across the boundaries of the grains.

With Ω^ε we associate the following nonempty closed convex subset of the Sobolev space $H^1(\Omega^\varepsilon)$:

$$K^\varepsilon = \left\{ v \in H^1(\Omega^\varepsilon) \mid v = 0 \text{ on } \partial\Omega,\ v \geq 0 \text{ on } S^\varepsilon \right\}, \tag{1}$$

where S^ε is the boundary of the obstacles and $\partial\Omega$ is the external boundary of Ω.

Our goal is to analyze the asymptotic behavior, as $\varepsilon \to 0$, of the solution of the following variational problem in Ω^ε :

$$\begin{cases} \text{Find } \ u^\varepsilon \in K^\varepsilon \text{ such that} \\ \displaystyle\int_{\Omega^\varepsilon} A^\varepsilon \nabla u^\varepsilon \nabla (v^\varepsilon - u^\varepsilon) dx + \int_{\Omega^\varepsilon} g(u^\varepsilon)(v^\varepsilon - u^\varepsilon) dx \geq \int_{\Omega^\varepsilon} f (v^\varepsilon - u^\varepsilon) dx \,, \\ \forall v^\varepsilon \in K^\varepsilon, \end{cases} \tag{2}$$

where f is a given function in $L^2(\Omega)$ (a given source term) and g is a continuous function, monotonously non-decreasing and such that $g(0) = 0$, modeling some chemical reactions taking place in Ω^ε (see Section 2).

The solution u^ε of (2) is also the solution of the following non-linear free boundary-value problem: find a smooth function u^ε and two subsets S_0^ε and

S^ε_+ such that $S^\varepsilon_0 \cup S^\varepsilon_+ = S^\varepsilon$, $S^\varepsilon_0 \cap S^\varepsilon_+ = \emptyset$, and

$$\begin{cases} -\mathrm{div}(A^\varepsilon \nabla u^\varepsilon) + g(u^\varepsilon) = f \text{ in } \Omega^\varepsilon\,, \\ u^\varepsilon = 0 \text{ on } S^\varepsilon_0\,,\ A^\varepsilon \nabla u^\varepsilon \cdot \nu \geq 0 \text{ on } S^\varepsilon_0\,, \\ u^\varepsilon > 0 \text{ on } S^\varepsilon_+\,,\ A^\varepsilon \nabla u^\varepsilon \cdot \nu = 0 \text{ on } S^\varepsilon_+\,, \end{cases}$$

where ν is the exterior unit normal to the surface S^ε. So, on S^ε there are two *a priori* unknown subsets S^ε_0 and S^ε_+ where u^ε satisfies complementary boundary conditions coming from the following global constraints:

$$u^\varepsilon,\ A^\varepsilon \nabla u^\varepsilon \cdot \nu \geq 0 \quad \text{and} \quad u^\varepsilon A^\varepsilon \nabla u^\varepsilon \cdot \nu = 0 \text{ on } S^\varepsilon\,.$$

This is a typical nonlinear interfacial condition arising in the case of semipermeable membranes, *i.e.* a relationship between the concentration and the normal flux across the semipermeable boundaries.

We shall deal with periodic structures, defined by $A^\varepsilon(x) = A(\frac{x}{\varepsilon})$. Here $A = A(y)$ is a continuous matrix-valued function on $\mathbb{R}^n$ which is Y-periodic and $Y = \left(-\frac{1}{2}, \frac{1}{2}\right)^n$ is the basic cell. We shall just consider periodic structures obtained by removing periodically from Ω, with period ε, an elementary obstacle T which has been appropriated rescaled. We use the symbol # to denote periodicity properties. We shall assume that

$$\begin{cases} A \in L^\infty_\#(\Omega)^{n\times n}\,, \\ A \text{ is a symmetric matrix}\,, \\ \text{For some } 0 < \alpha < \beta,\ \alpha\,|\xi|^2 \leq A(y)\xi\cdot\xi \leq \beta\,|\xi|^2 \quad \forall \xi,\ y \in \mathbb{R}^n\,. \end{cases} \tag{3}$$

Under the above hypotheses and the conditions fulfilled by K^ε, it is well-known by a classical existence and uniqueness result of Lions and Stampacchia [10] that (2) is a well-posed problem.

Since the period of the structure is small compared to the dimension of Ω, or in other words, since the nonhomogeneities are small compared to the global dimension of the structure, an asymptotic analysis becomes necessary. Two scales are important for a suitable description of the given structure: one which is comparable with the dimension of the period, called the *microscopic scale* and denoted by $y = x/\varepsilon$, and another one which is of the same order of magnitude as the global dimension of our system, called the *macroscopic scale* and denoted by x.

The main goal of the homogenization method is to pass from the microscopic scale to the macroscopic one; more precisely, using the homogenization method, we try to describe the macroscopic properties of our nonhomogeneous system in terms of the properties of its microscopic structure. Intuitively, the nonhomogeneous system is replaced by a fictitious homogeneous one, whose global characteristics represent a good approximation of the initial system. Hence, the homogenization method provides a general framework for obtaining these macroscale properties, eliminating the difficulties related to the explicit determination of a solution of the problem at the microscale and offering a less detailed description, but one which is applicable to much more complex systems.

The asymptotic behavior of the solution of such a problem depends on the size of the obstacles. We shall focus on the most interesting and difficult case, *i.e.* the case of grains of the order $\varepsilon^{n/(n-2)}$ (if $n \geq 3$). As it will be proved, in this critical case, the limit u of u^ε is the solution of a Dirichlet problem in Ω associated with a new operator which is the sum of the standard homogenized one and extra terms generated by the geometry and the nonlinearity of the problem. This solution u, which satisfies an equation with constant coefficients, approximates the exact solution u^ε, which is, in general, very difficult to find out. More precisely, we shall prove that the solution u^ε converges to the unique solution of the following variational equality:

$$\begin{cases} \text{Find } u \in H_0^1(\Omega) \text{ such that} \\ \displaystyle\int_\Omega A^0 \nabla u \nabla v dx + \int_\Omega g(u) v dx - \langle \mu_0 u^-, v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = \int_\Omega f v dx, \; \forall v \in H_0^1(\Omega) \, . \end{cases} \tag{4}$$

Here, A^0 is the classical homogenized matrix, whose entries are defined by:

$$a_{ij}^0 = \frac{1}{|Y|} \int_Y \left(a_{ij}(y) + a_{ik}(y) \frac{\partial \chi_j}{\partial y_k} \right) dy \, , \tag{5}$$

in terms of the functions χ_j, $j = 1, ..., n$, solutions of the so-called cell problems

$$\begin{cases} -\mathrm{div}_y (A(y) \nabla_y (y_j + \chi_j)) = 0 \;\; \text{in } Y \, , \\ \chi_j - Y \text{ periodic} \, , \end{cases} \tag{6}$$

and μ_0 is given by

$$\mu_0 = \inf_{\zeta \in H^1(\mathbb{R}^n)} \left\{ \int_{\mathbb{R}^n} A(0) \nabla \zeta \nabla \zeta dx \;\mid\; \zeta \geq 1 \text{ q.e. on } T \right\} , \tag{7}$$

T being the elementary obstacle.

In this limit problem, the periodic heterogeneous structure of the medium is reflected by the presence of the homogenized matrix A^0, the critical size of the obstacles is reflected by the appearance of a zero order term μ_0 and the nonlinearity of the problem produced by the presence of chemical reactions by the term given by g. Also, notice in (4) the spreading effect of the unilateral condition $u^\varepsilon \geq 0$ on S^ε which can be seen by the fact that the extra term only charges the negative part of u; it is just the negative part u^- that is penalized in the limit.

The proof is based on the so-called *method of oscillating test functions* introduced by Tartar [12] for studying homogenization problems, coupled with a general rearrangement technique due to De Giorgi [7] (see also [1]) and with monotonicity methods and results from the theory of semilinear problems [2, 6, 11]).

This paper has its starting points in the well-known work of Cioranescu and Murat [4], where the authors deal with the asymptotic behavior of solutions of Dirichlet problems in perforated domains, showing the appearance of a "strange" extra-term as the period of the perforations tends to zero and the obstacles are of critical size.

In [5], this framework was generalized to a class of Signorini's problem in heterogeneous media, involving just a positivity condition imposed on the boundary of the grains (see, also, Section 3). In the present paper, we generalize some of the results obtained in [5], by considering also the nonlinear term given by the function g, which gives rise in the limit to a new zero-order extra-term. We address here only the case in which the grains are of critical size. The case in which the grains are of the same size as the period and we still have nonlinear terms in our variational inequalities will be addressed in a forthcoming paper.

The structure of our paper is as follows: in Section 2, after some preliminaries, we formulate the main convergence result, the proof of which is given in Section 3. Finally, in the last section we summarize the results obtained in this paper.

2. Preliminaries and the main result

Let Ω be a bounded connected open subset of $\mathbb{R}^n$, $n \geq 3$, with $\partial\Omega \in C^2$ and let T be another open bounded subset of $\mathbb{R}^n$, with $\partial T \in C^2$. We shall refer to T as being *the elementary obstacle*. We assume that 0 belongs to T and that T is star-shaped with respect to 0. Also, without loss of generality, we shall assume that $\overline{T} \subset Y$, where $Y = (-\frac{1}{2}, \frac{1}{2})^n$ is the representative cell in $\mathbb{R}^n$.

Let ε be a real parameter taking values in a sequence of positive numbers converging to zero and let $r : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous map, which will represent the size of the obstacles. As mentioned in the Introduction, throughout this paper we shall just focus on the case in which the size $r(\varepsilon)$ of the obstacles is exactly of the order of $\varepsilon^{n/(n-2)}$. We shall refer to this case as being the "critical" one.

For each ε and for any integer vector $\boldsymbol{i} \in \mathbb{Z}^n$, we shall denote by $T^\varepsilon_{\boldsymbol{i}}$ the translated image of $r(\varepsilon)T$ by the vector $\varepsilon\boldsymbol{i}$, $\boldsymbol{i} \in \mathbb{Z}^n$:, $T^\varepsilon_{\boldsymbol{i}} = \varepsilon\boldsymbol{i} + r(\varepsilon)T$.

Also, let us denote by T^ε the set of all the obstacles contained in Ω, *i.e.* $T^\varepsilon = \bigcup \left\{ T^\varepsilon_{\boldsymbol{i}} \ \mid \ \overline{T^\varepsilon_{\boldsymbol{i}}} \subset \Omega, \ \boldsymbol{i} \in \mathbb{Z}^n \right\}$. Set $\Omega^\varepsilon = \Omega \setminus \overline{T^\varepsilon}$. Hence, Ω^ε is a periodic domain with grains of the size $r(\varepsilon)$. All of them have the same shape, the distance between two adjacent grains is of order ε and they do not overlap. Also, let us remark that the obstacles do not intersect the boundary $\partial\Omega$. Let $S^\varepsilon = \cup\{\partial T^\varepsilon_{\boldsymbol{i}} \mid \overline{T^\varepsilon_{\boldsymbol{i}}} \subset \Omega, \boldsymbol{i} \in \mathbb{Z}^n\}$, $\partial\Omega^\varepsilon = \partial\Omega \cup S^\varepsilon$.

Also, let us notice that throughout the paper, by C we shall denote a generic fixed strictly positive constant, whose value can change from line to line.

As already mentioned, we are interested in studying the asymptotic behavior of the solutions, in such periodic domains, of variational inequalities with highly oscillating obstacles constraints of the form (2). We shall consider the case of a general medium represented by a coercive periodic matrix with rapidly oscillating coefficients. Let $A \in L^\infty_\#(\Omega)^{n\times n}$ be a symmetric matrix whose entries are Y-periodic, bounded and measurable real functions. Let us suppose that A satisfies the assumptions (3). We shall denote by $A^\varepsilon(x)$ the value of $A(y)$ at the point $y = x/\varepsilon$, *i.e.* $A^\varepsilon(x) = A(\dfrac{x}{\varepsilon})$. We further assume that A is continuous with respect to y.

We shall consider that the function g in (2) is a continuous function, monotonously non-decreasing and such that $g(0) = 0$ and we shall take $G(v) = \int\limits_0^v g(s)ds$. Finally, we assume that there exist $C \geq 0$ and an exponent q, with $0 \leq q < n/(n-2)$, such that

$$|g(v)| \leq C(1 + |v|^q)\,. \tag{8}$$

This general situation is well illustrated by the following important practical examples:

(a) $\quad g(v) = \frac{\alpha v}{1+\beta v}, \quad \alpha, \beta > 0 \quad$ (Langmuir kinetics)

and

(b) $\quad g(v) = |v|^{p-1}v, \quad 0 < p < 1 \quad$ (Freundlich kinetics).

The exponent p is called *the order of the reaction.*

By classical results (see [2] and [10]), since K^ε is a nonempty convex set, for any given $f \in L^2(\Omega)$, we know that there exists a unique weak solution $u^\varepsilon \in V^\varepsilon \bigcap H^2(\Omega^\varepsilon)$, where

$$V^\varepsilon = \left\{ v \in H^1(\Omega^\varepsilon) \ \mid v = 0 \text{ on } \partial\Omega \right\}\,.$$

We shall be interested in getting the asymptotic behavior of this solution, when $\varepsilon \to 0$.

The main convergence result of this paper (given by Theorem 2.1 or, equivalently, by Theorem 2.2 below) involves any extension $\widehat{u^\varepsilon}$ of the solution u^ε of the variational inequality (2) inside the grains such that it depends continuously on u^ε and it is positive in T^ε (see [3]). For instance, we can extend u^ε inside the grains in such a way that

$$\begin{cases} \Delta\widehat{u^\varepsilon} = 0 \quad \text{in} \quad T^\varepsilon \,, \\ \widehat{u^\varepsilon} = u^\varepsilon \quad \text{on} \quad S^\varepsilon \,. \end{cases} \tag{9}$$

Then, since $u^\varepsilon \in K^\varepsilon$, by the maximum principle we have $\widehat{u^\varepsilon} \geq 0$ inside the grains and $\widehat{u^\varepsilon} \in H_0^1(\Omega)$. Also, $(\widehat{u^\varepsilon})^- = 0$ inside the grains and $(\widehat{u^\varepsilon})^- \in H_0^1(\Omega^\varepsilon)$.

The main result of this paper is the following one:

Theorem 2.1 *There exists an extension $\widehat{u^\varepsilon}$ of the solution u^ε of the variational inequality (2), positive inside the grains, such that*

$$\widehat{u^\varepsilon} \rightharpoonup u \qquad \text{weakly in } H_0^1(\Omega)\,,$$

where u is the unique solution of

$$\begin{cases} u \in H_0^1(\Omega)\,, \\ \displaystyle\int_\Omega g(u)v + \int_\Omega A^0\nabla u\nabla v dx - \langle \mu_0 u^-, v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} = \int_\Omega fv dx, \;\; \forall v \in H_0^1(\Omega)\,. \end{cases} \tag{10}$$

Here, A^0 is the classical homogenized matrix, whose entries were defined by (5)–(6) and μ_0 is given by (7). The constant matrix A^0 is symmetric and positive-definite.

Also, let us notice that u^ε is also the unique solution of the minimization problem:

$$\begin{cases} \text{Find } u^\varepsilon \in K^\varepsilon \text{ such that} \\ J^\varepsilon(u^\varepsilon) = \inf\limits_{v\in K^\varepsilon} J^\varepsilon(v)\,, \end{cases}$$

where

$$J^\varepsilon(v) = \frac{1}{2}\int_{\Omega^\varepsilon} A^\varepsilon\nabla v\nabla v dx + \int_{\Omega^\varepsilon} G(v)dx - \int_{\Omega^\varepsilon} fv dx\,.$$

If we introduce the following functional defined on the Sobolev space $H_0^1(\Omega)$:

$$J^0(v) = \frac{1}{2}\int_\Omega A^0\nabla v\nabla v dx + \int_\Omega G(v)dx + \frac{1}{2}\left\langle \mu_0, (u^-)^2\right\rangle - \int_\Omega fv dx\,,$$

then the main result of this paper can also be formulated as follows:

Theorem 2.2 *One can construct an extension $\widehat{u^\varepsilon}$ of the solution u^ε of the variational inequality (2) such that*

$$\widehat{u^\varepsilon} \rightharpoonup u \quad \text{weakly in } \; H_0^1(\Omega)\,,$$

where u is the unique solution of the minimization problem

$$\begin{cases} \text{Find } u \in H_0^1(\Omega) \text{ such that} \\ J^0(u) = \inf\limits_{v \in H_0^1(\Omega)} J^0(v)\,. \end{cases}$$

3. Proof of the main result

In order to prove Theorem 2.1, we shall remember a result proved in [5]. More precisely, in [5] we have analyzed the asymptotic behavior of the solution of the following variational problem in Ω^ε:

$$\begin{cases} \text{Find } \; u^\varepsilon \in K^\varepsilon \text{ such that} \\ \displaystyle\int_{\Omega^\varepsilon} A^\varepsilon \nabla u^\varepsilon \nabla (v^\varepsilon - u^\varepsilon) dx \geq \int_{\Omega^\varepsilon} f(v^\varepsilon - u^\varepsilon) dx, \ \forall v^\varepsilon \in K^\varepsilon\,, \end{cases} \tag{11}$$

where f is a given function in $L^2(\Omega)$. We proved that there exists an extension $\widehat{u^\varepsilon}$ of the solution u^ε of the variational inequality (11), positive inside the grains, such that

$$\widehat{u^\varepsilon} \rightharpoonup u \quad \text{weakly in } H_0^1(\Omega)\,,$$

where u is the unique solution of

$$\begin{cases} u \in H_0^1(\Omega), \\ \displaystyle\int_{\Omega} A^0 \nabla u \nabla v dx - \left\langle \mu_0 u^-, v \right\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = \int_{\Omega} f v dx \ \ \forall v \in H_0^1(\Omega)\,. \end{cases} \tag{12}$$

Here, A^0 is the classical homogenized matrix, whose entries were defined by (5)–(6) and μ_0 is given by (7).

The proof was based on the so-called *method of oscillating test functions* introduced by Tartar [12] for studying homogenization problems, coupled with a general rearrangement technique due to De Giorgi [7] (see also [1]).

In this paper, we have added to our previous problem (11) a quite general nonlinear term, given by g and modeling nonlinear chemical reactions taking place in Ω^ε.

As already mentioned, due to the special conditions imposed on g, the existence and uniqueness of a weak solution of (2) can be settled by using the classical theory of semilinear monotone problems.

Let $u^\varepsilon \in K^\varepsilon$ be the solution of the variational inequality (2) and let $\widehat{u^\varepsilon}$ the extension of u^ε inside the obstacles given by (9). It is easy to see that

$$\left\|\widehat{u^\varepsilon}\right\|_{H_0^1(\Omega)} \leq C\,.$$

Therefore, by passing to a subsequence, still denoted by $\widehat{u^\varepsilon}$, we can assume that there exists $u \in H_0^1(\Omega)$ such that

$$\widehat{u^\varepsilon} \rightharpoonup u \quad \text{weakly in } H_0^1(\Omega)\,.$$

It remains to identify the limit variational inequality satisfied by u, *i.e.* to pass to the limit, with $\varepsilon \to 0$, in the variational inequality (2), or, equivalently, in the functional J^ε.

Using our previously mentioned convergence results, the only thing that remains to be done in order to complete the proof of Theorem 2.1 is to show how we can pass to the limit in the new term containing g, *i.e.* in the second term of the functional J^ε.

To this end, let us notice that it is enough to prove that for any $z^\varepsilon \rightharpoonup z$ weakly in $H_0^1(\Omega)$, we get

$$G(z^\varepsilon) \to G(z) \quad \text{strongly in } L^{\overline{q}}(\Omega)\,, \tag{13}$$

where $\overline{q} = \dfrac{2n}{q(n-2)+n}$. Therefore, it will be easy to see that we get

$$\int\limits_{\Omega^\varepsilon} G(u^\varepsilon)dx = \int\limits_{\Omega} G(\widehat{u^\varepsilon})\chi_{\Omega^\varepsilon}\,dx \to \int\limits_{\Omega} G(u)dx\,. \tag{14}$$

So, it will suffice to prove (13). But this is just a consequence of the following well-known result (see [6] and [9]):

Theorem 3.1 *Let $F : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function, i.e.*

(a) for every v the function $F(\cdot, v)$ is measurable with respect to $x \in \Omega$.

(b) for every (a.e.) $x \in \Omega$, the function $F(x,\cdot)$ is continuous with respect to v.

Moreover, if we assume that there exists a positive constant C such that

$$|F(x,v)| \leq C\left(1 + |v|^{r/t}\right)\,,$$

with $r \geq 1$ and $t < \infty$, then the map $v \in L^r(\Omega) \mapsto F(x, v(x)) \in L^t(\Omega)$ is continuous in the strong topologies. ∎

Indeed, since

$$|G(v)| \leq C(1 + |v|^{q+1}),$$

applying the above theorem for $F(x,v) = G(v)$, $t = \overline{q}$ and $r = (2n/(n-2)) - r'$, with $r' > 0$ such that $q + 1 < r/t$ and using the compact injection $H^1(\Omega) \hookrightarrow L^r(\Omega)$ we easily get (13).

So, all the terms in the nonlinear functional J^ε pass to the limit and, hence, u is the unique solution of the minimization problem

$$\begin{cases} \text{Find } u \in H_0^1(\Omega) \text{ such that} \\ J^0(u) = \inf\limits_{v \in H_0^1(\Omega)} J^0(v) \,. \end{cases}$$

As u is uniquely determined, the whole sequence $\widehat{u^\varepsilon}$ converges to u and Theorem 2.1 is proved. ∎

Finally, let us remark that due to the compactness injection theorems in Sobolev spaces, it would be enough, with the same reasoning as in the paper, to assume that g satisfies the growth condition (8) for

$$0 \leq q < (n+2)/(n-2) \,.$$

Also, notice that we can treat in a similar manner the case in which the nonlinear function g is a multi-valued maximal monotone graph, including, hence, much more general nonlinear chemical reactions.

4. Conclusions

The general question which made the object of this paper was the homogenization of some nonlinear variational inequalities with highly oscillating coefficients in heterogeneous periodically domains. Such boundary-value problems involve the existence of two distinct sources of oscillations, one coming from the geometrical structure of the domain and the other one from the heterogeneity of the medium. It is shown how these sources interact to produce the limit behavior of the system. In the case of a critical size of the obstacles, the limit problem is a Dirichlet one, associated to a new operator which is the sum of a standard homogenized operator and two extra zero-order terms, coming from the special periodic geometrical structure of the heterogeneous domain and the nonlinearity of this problem.

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