NUMERICAL EVALUATION OF SOME PARAMETERS FOR A MODEL OF NEUTRAL KAONS

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(Received January 11, 2007; revised version received April 29, 2007)

Using the Mathematica program we calculate numerically the difference of the diagonal matrix elements of the time dependent effective Hamiltonian for the neutral K meson complex. We consider the exactly solvable neutral K meson model based on the one-pole approximation for the mass density. The so-called Khalfin's Theorem is numerically examined. Some characteristic parameters for this system are also calculated. The results of all calculations are presented in the graphical form. The calculations are made assuming the total system is CPT-invariant and CP-noninvariant.

PACS numbers: 03.65.Ca, 11.30.Er, 11.10.St, 14.40.Aq

1. Introduction

The neutral kaon system is probably one of the most interesting complexes of elementary particles. Using this system it was found in 1964 that the CP symmetry is violated [1]. The description of CP violation effects in this system is based on the approximation proposed by Lee, Oehme and Yang (LOY) in [2]. This theory was then developed and intensively studied by Lee (see [3]) and by many other authors (see *e.g.* [4]). Within the LOY approach, a non–hermitian Hamiltonian $H_{\|}$ is used to study the properties of the particle–antiparticle unstable system [2–5]

$$H_{\|} \equiv M - \frac{i}{2}\Gamma, \tag{1}$$

where

$$M = M^{+} \ , \ \Gamma = \Gamma^{+} \tag{2}$$

are (2×2) matrices acting in $\mathcal{H}_{\|}$, where $\mathcal{H}_{\|}$ is a two–dimensional subspace of the total Hilbert space of states $\mathcal{H}$, spanned by the state vectors of K^0,

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$\bar{K}^0$ mesons. The M-matrix is called the mass matrix and Γ is the decay matrix. LOY derived their approximate effective Hamiltonian $H_{\|} \equiv H_{\rm LOY}$ by adapting the one-dimensional Weisskopf–Wigner (WW) method to the two-dimensional case corresponding to the neutral kaon system. Almost all properties of this system can be described by solving the Schrödinger-like equation

$$i\frac{\partial}{\partial t}|\psi;t\rangle_{\|} = H_{\|}|\psi;t\rangle_{\|}, \quad (t \geq t_0 > -\infty)\,, \tag{3}$$

where we have used $\hbar = c = 1$, and $|\psi;t\rangle_{\|} \in \mathcal{H}_{\|}$.

Within the LOY theory the physical states of neutral kaons are superpositions of $|K^0\rangle$ and $|\bar{K}^0\rangle$. They are the eigenvectors of $H_{\rm LOY}$,

$$|K_{\rm S}\rangle = p|K^0\rangle + q|\bar{K}^0\rangle, \quad |K_{\rm L}\rangle = p|K^0\rangle - q|\bar{K}^0\rangle, \tag{4}$$

$$H_{\|}|K_{\rm S(L)}\rangle = \mu_{\rm S(L)}|K_{\rm S(L)}\rangle, \tag{5}$$

and correspond to short-living (the vector $|K_{\rm S}\rangle$) and long-living (the vector $|K_{\rm L}\rangle$) states of neutral kaons. We will use the following notations further on in this paper: $|K^0\rangle \equiv |\mathbf{1}\rangle$, $|\bar{K}^0\rangle \equiv |\mathbf{2}\rangle$.

One of standard results of the LOY approach is the following: In a CPT invariant system, *i.e.* when

$$\Theta H \Theta^{-1} = H, \tag{6}$$

(where $\Theta = {\rm CPT}$, and H is the total self-adjoint Hamiltonian for the system containing neutral koans considered), there is

$$h_{11}^{\rm LOY} = h_{22}^{\rm LOY} \tag{7}$$

and

$$M_{11}^{\rm LOY} = M_{22}^{\rm LOY}\,, \tag{8}$$

where: $M_{jj}^{\rm LOY} = {\rm Re}(h_{jj}^{\rm LOY})$ and ${\rm Re}(z)$ denotes the real part of a complex number z (${\rm Im}(z)$ is the imaginary part of z), and $h_{jj}^{\rm LOY} = \langle \boldsymbol{j}|H_{\rm LOY}|\boldsymbol{j}\rangle$, ($j = 1,2$). Another important prediction of the LOY theory is that the ratio

$$r(t) \stackrel{\rm def}{=} \frac{p^2}{q^2} \equiv \frac{A_{12}(t)}{A_{21}(t)} = {\rm const.} \quad \text{and} \quad |r(t)| = |\frac{p^2}{q^2}| \neq 1\,, \tag{9}$$

when CP symmetry is violated, $[{\rm CP}, H] \neq 0$, [3–8]. Here

$$A_{jk}(t) = \langle \boldsymbol{j}|U_{\|}(t)|\boldsymbol{k}\rangle \equiv \langle \boldsymbol{j}|e^{-itH_{\|}}|\boldsymbol{k}\rangle\,, \quad (j,k = 1,2)\,, \tag{10}$$

and $U_{\|}(t)$ is the evolution operator for the subspace $\mathcal{H}_{\|}$.

The important result indicating some limitations of the LOY approach was obtained by Khalfin [6–12]. Khalfin found that in the exact theory there must be

$$\text{if} \qquad r(t) = \frac{A_{12}(t)}{A_{21}(t)} = \text{const.}\,, \qquad \text{then} \qquad |r(t)| = 1\,, \tag{11}$$

where

$$A_{jk}(t) = \langle \boldsymbol{j}|e^{-itH}|\boldsymbol{k}\rangle\,, \quad (j,k=1,2)\,. \tag{12}$$

Result (11) is known in the literature as "Khalfin's Theorem". Using this result Khalfin hypothesized that beyond the LOY approximation one should expect new CP-violation effects [7, 8] and he tried to obtain some model estimations of the possible magnitude of these effects. He found that the order of these effects should be 10^{-3} (see [8]). He obtained his estimation using the spectral language for the description of $K_{\rm S}, K_{\rm L}$ and K^0, $\bar{K}^0$, by introducing a hermitian Hamiltonian, H, with a continuous spectrum of decay products labeled by α, β, *etc.*,

$$H|\phi_\alpha(m)\rangle = m\,|\phi_\alpha(m)\rangle\,, \quad \langle\phi_\beta(m')|\phi_\alpha(m)\rangle = \delta_{\alpha\beta}\delta(m'-m)\,. \tag{13}$$

Here H is the above mentioned total Hamiltonian for the system. H includes all interactions and has absolutely continuous spectrum. We have

$$|K_{\rm S}\rangle = \int\limits_{\text{Spec }(H)} dm\, \sum_\alpha \omega_{{\rm S},\alpha}(m)|\phi_\alpha(m)\rangle\,, \tag{14}$$

$$|K_{\rm L}\rangle = \int\limits_{\text{Spec }(H)} dm\, \sum_\beta \omega_{{\rm L},\alpha}(m)|\phi_\beta(m)\rangle\,, \tag{15}$$

and

$$|\boldsymbol{j}\rangle = \int\limits_{\text{Spec }(H)} dm\, \sum_\alpha \omega_{j,\alpha}(m)|\phi_\alpha(m)\rangle\,, \tag{16}$$

where $j=1,2$. Thus, the exact $A_{jk}(t)$ can be written as the Fourier transform of the density $\rho_{jk}(m)$, $(j,k=1,2)$,

$$A_{jk}(t) = \int\limits_{-\infty}^{+\infty} dm\; e^{-imt}\rho_{jk}(m)\,, \tag{17}$$

where

$$\rho_{jk}(m) = \sum_\alpha \omega^*_{j,\alpha}(m)\,\omega_{k,\alpha}(m)\,. \tag{18}$$

The minimal mathematical requirement for $\rho_{jk}(m)$ is the following: $\int_{-\infty}^{+\infty} dm\,|\rho_{jk}(m)| < \infty$. Other requirements for $\rho_{jk}(m)$ are determined by basic physical properties of the system. The main property is that the energy (*i.e.* the spectrum of H) should be bounded from below, $\mathrm{Spec}(H) = [m_g,\,\infty)$ and $m_g > -\infty$.

Starting from densities $\rho_{jk}(m)$ one can calculate $A_{jk}(t)$. In order to find these densities from relation (18) one should know the expansion coefficients $\omega_{j,\alpha}(m)$. Using physical states $|K_{\rm S}\rangle, |K_{\rm L}\rangle$ and relations (4) they can be expressed in terms of the expansion coefficients $\omega_{{\rm S},\alpha}(m), \omega_{{\rm S},\alpha}(m)$. Thus, assuming the form of coefficients $\omega_{{\rm S},\alpha}(m), \omega_{{\rm S},\alpha}(m)$ defining physical states of neutral kaons one can compute all $A_{jk}(t)$, $(j,k=1,2)$.

The model considered by Khalfin is based on the assumption that (see formula (35) in [8])

$$\omega_{{\rm S},\beta}(m) = \sqrt{\frac{\Gamma_{\rm S}}{2\pi}}\;\frac{\xi_{{\rm S},\beta}(m)}{|\xi_{{\rm S},\beta}(m_{\rm S} - i\frac{\Gamma_{\rm S}}{2})|}\,\frac{a_{{\rm S},\beta}(K_{\rm S}\to\beta)}{m - m_{\rm S} + i\frac{\Gamma_{\rm S}}{2}}\,, \tag{19}$$

$$\omega_{{\rm L},\beta}(m) = \sqrt{\frac{\Gamma_{\rm L}}{2\pi}}\,\frac{\xi_{{\rm L},\beta}(m)}{|\xi_{{\rm S},\beta}(m_{\rm L} - i\frac{\Gamma_{\rm L}}{2})|}\,\frac{a_{{\rm L},\beta}(K_{\rm L}\to\beta)}{m - m_{\rm L} + i\frac{\Gamma_{\rm L}}{2}}\,, \tag{20}$$

where $a_{{\rm S},\beta}$ and $a_{{\rm L},\beta}$ are the decay (transition) amplitudes and $\xi_{{\rm S(L)},\beta}(m)$ are, in general, some nonsingular "preparation functions". Khalfin found his above mentioned estimation choosing, for simplicity, the trivial form of the "preparation functions", $\xi_{{\rm S(L)},\beta}(m) = 1$.

The discussion about the validity of the Khalfin's estimation of his new CP violation effect can be found in the literature (see *e.g.* [6,12]). Our attention will be concentrated on the attempt to verify the size of the Khalfin's estimation performed in [12]. The calculation performed in [12] uses Khalfin's assumption that $\xi_{{\rm S(L)},\beta}(m) = 1$, strictly speaking, they use the assumption that in (19), (20) there is

$$\begin{aligned}\frac{\xi_{{\rm S(L)},\beta}(m)}{|\xi_{{\rm S(L)},\beta}(m_{\rm S(L)} - i\frac{\Gamma_{\rm S(L)}}{2})|} \;&\equiv\; g(m-m_g) = [g(m-m_g)]^2 \\ &\stackrel{\rm def}{=} \begin{cases} 1 & \text{if } m \geq m_g\,, \\ 0 & \text{if } m < m_g\,. \end{cases}\end{aligned} \tag{21}$$

Within this assumption one obtains, for example, that

$$\mathcal{A}_{\rm SS}(t) \stackrel{\rm def}{=} \langle K_{\rm S}|e^{-itH}|K_{\rm S}\rangle = \int\limits_{-\infty}^{+\infty} dm\;\rho_{\rm SS}(m)\,e^{-itm}\,, \tag{22}$$

where

$$\rho_{\mathrm{SS}}(m) \;=\; g(m-m_g)\,\frac{\mathit{\Gamma}_{\mathrm{S}}}{(m-m_{\mathrm{S}})^2+\frac{\mathit{\Gamma}_{\mathrm{S}}^2}{4}}\,\frac{S}{2\pi}\,, \tag{23}$$

$$S=\sum_{\alpha}|a_{S,\alpha}(K_{\mathrm{S}}\to\alpha)|^2\,, \tag{24}$$

and so on.

The form of density $\rho_{\mathrm{SS}}(m)$ defined by (23) is not the most general one. In more realistic models functions $\omega_{j,\alpha}(m)$ and of type $\omega_{\mathrm{S},\beta}(m), \omega_{L,\beta}(m)$ lead to the densities $\rho(m)$ of general form similar to (23) with $g(m-m_g)$ and $S \equiv S(m)$ having more involved form [13, 14]. In the general case the threshold factor $g(m-m_g)$ describes the behavior of $\rho(m)$ for small m, (*i.e.*, for $m \simeq m_g$) and it is responsible for the long time properties of amplitudes of type $A_{jk}(t)$ and $\mathcal{A}_{\mathrm{SS}}(t)$. The second factor in the formulae of type (23) having the Breit–Wigner form results from the pole structure of functions of type $\omega_{\mathrm{S},\beta}(m), \omega_{\mathrm{L},\beta}(m)$ (see (19), (20)) defining densities $\rho(m)$ and it is responsible for the form of $A_{jk}(t), \mathcal{A}_{\mathrm{SS}}(t)$ *etc.* for the intermediate times (*i.e.* it is responsible for the exponential part of the survival probabilities). The third factor, *i.e.* the factor corresponding to $S(m)$ ensures the suitable behavior of $\rho(m)$ for $m \to \infty$.

For simplicity, it is assumed in [12] that $m_g = 0$. So all integrals of type (22) and (17) are taken between the limits $m = 0$ and $m = +\infty$. All these assumptions made it possible to express amplitudes of type $A_{jk}(t)$ in [12] in terms of known special functions. The same assumptions were used in [15] (see [15], relations (37)–(39) and (42)–(47)) and will be used in this paper. Note that putting $g(m-m_g) \equiv 1$ in (22) leads to strictly exponential form of amplitudes of type $\mathcal{A}_{\mathrm{SS}}(t)$ as functions of time t. On the other hand, keeping $g(m)$ in the assumed simplest physically admissible form (21) results in the presence of additional nonoscillatory terms in amplitudes of type $\mathcal{A}_{\mathrm{SS}}(t), \mathcal{A}_{\mathrm{LL}}(t)$ *etc.* and thus in amplitudes $A_{jk}(t)$ as well (see [12,15]).

In [15] the analytical formulae for $A_{jk}(t)$ obtained in [12] were used as the starting point to find analytical expressions for matrix elements of the effective Hamiltonian for this model for $t = \tau_{\mathrm{L}}$ and then to obtain a numerical value for the possible consequence of the Khalfin's Theorem analyzed in [16]. It is found there that, contrary to the standard LOY result (7), the diagonal matrix elements of the exact effective Hamiltonian for neutral meson complex cannot be equal if CPT symmetry holds but CP symmetry is violated. We found in [15] that

$$\mathrm{Re}(h_{11}(t \sim \tau_{\mathrm{L}}) - h_{22}(t \sim \tau_{\mathrm{L}})) \simeq -4.771 \times 10^{-18}\ \mathrm{MeV}\,, \tag{25}$$
$$\mathrm{Im}(h_{11}(t \sim \tau_{\mathrm{L}}) - h_{22}(t \sim \tau_{\mathrm{L}})) \simeq 7.283 \times 10^{-16}\ \mathrm{MeV} \tag{26}$$

and

$$\frac{|\,\mathrm{Re}(h_{11}(t \sim \tau_{\mathrm{L}}) - h_{22}(t \sim \tau_{\mathrm{L}}))|}{m_{\mathrm{average}}} \equiv \frac{m_{K^0} - m_{\bar{K^0}}}{m_{\mathrm{average}}} \sim 10^{-21}\,, \tag{27}$$

where $h_{jk}(t) = \langle j|H_{\|}(t)|k\rangle$ $(j,k = 1,2)$ and $H_{\|}(t)$ is the effective Hamiltonian. These results were obtained analytically for the considered model for the neutral kaon system in the case when the total system is CPT-invariant but CP-non-invariant (equations (68), (69) and (70) in [15]). The estimations (25)–(27) were obtained by inserting (20) and related $m_{\mathrm{S}} \simeq m_{\mathrm{L}} \simeq m_{\mathrm{average}} = 497.648$ MeV, $\Delta m = 3.489 \times 10^{-12}$ MeV, $\tau_{\mathrm{S}} = 0.8935 \times 10^{-10}$ s, $\tau_{\mathrm{L}} = 5.17 \times 10^{-8}$ s, $\gamma_{\mathrm{L}} = 1.3 \times 10^{-14}$ MeV, $\gamma_{\mathrm{S}} = 7.4 \times 10^{-12}$ MeV in formulae of type (19). In this paper we will use the same experimental data. We will also use the same notations and definitions as in [15]:

$$\gamma_{\mathrm{S}} \equiv \frac{\Gamma_{\mathrm{S}}}{2}\,, \qquad \gamma_{\mathrm{L}} \equiv \frac{\Gamma_{\mathrm{L}}}{2}\,, \qquad \Delta m \equiv m_{\mathrm{L}} - m_{\mathrm{S}}, \tag{28}$$

and so on. Note that results (25)–(27) agree with the general result obtained in [16].

The detailed analysis of the matrix elements of the effective Hamiltonian for the $K^0 - \bar{K^0}$ system shows that the non-zero difference between the diagonal matrix elements of the effective Hamiltonian in the considered model is caused by the nonzero contribution into $\rho_{jk}(m)$, (18), coming from expressions for $\langle K_{\mathrm{S}}|e^{-itH}|K_{\mathrm{L}}\rangle$ and $\langle K_{\mathrm{L}}|e^{-itH}|K_{\mathrm{S}}\rangle$ and by the non-oscillatory terms in the formulae for the amplitudes of type (17) for transitions: $K^0 \longleftrightarrow K^0$, $\bar{K^0} \longleftrightarrow \bar{K^0}$ $K^0 \longleftrightarrow \bar{K^0}$. It is not difficult to verify that neglecting the mentioned nonzero contribution and dropping all these non-oscillatory terms leads to the zero difference of the diagonal matrix elements of the effective Hamiltonian in the considered case. This is because, in fact, dropping these non–oscillatory terms is equivalent to replacing in (17) densities $\rho_{jk}(m)$ defined by (18)–(21) with densities defined by the new function $g_{\mathrm{WW}}(m)$ instead of $g(m)$ given by (21) such that $g_{\mathrm{WW}}(m) = 1$ for all $-\infty \leq m \leq +\infty$. Thus, the integrals, *e.g.* in formulae of type (22), are taken between the limits $m = -\infty$ and $m = +\infty$ with densities of type (23) having the Breit–Wigner form (and not truncated for $m < m_g = 0$) which leads to strictly exponential form of, *e.g.* $|\mathcal{A}_{\mathrm{SS}}(t)|^2$ and the like. The effective Hamiltonian $H_{\|}(t)$ obtained in such a case is the LOY effective Hamiltonian, H_{LOY}.

In this paper we continue searching for the properties of the model analyzed in [15]. The aim is to show how the difference of the diagonal matrix elements $(h_{11}(t) - h_{22}(t))$ discussed in [15] and some other parameters describing neutral kaons (including $r(t)$, (11)) change in time t in the case of preserved CPT and violated CP symmetries. The paper is organized as follows. In Section 2 we collect the formulae for the matrix elements of the effective Hamiltonian necessary for further analysis. Section 3 contains a numerical verification of the Khalfin's Theorem (11). We show there how this Theorem "acts". In Section 4 the value of the difference of the diagonal elements $(h_{11}(t) - h_{22}(t))$ is shown as calculated at $t = \tau_{\mathrm{L}}$ with the use of the Mathematica. Also, in this section the time dependence of the real and imaginary parts of the diagonal matrix elements of the effective Hamiltonian $(h_{11}(t) - h_{22}(t))$ in graphical form is given. In Section 5 the eigenvalues of the effective Hamiltonian $\mu_{\mathrm{L}}(t), \mu_{\mathrm{S}}(t)$ are calculated and the parameters of the violation of the CP symmetry $\varepsilon_{\mathrm{L}}(t), \varepsilon_{\mathrm{S}}(t)$ are estimated. We also show graphically the time dependence of all the calculated quantities there. In Section 6 we check the correctness of our results by verifying the relation $(\mu_{\mathrm{L}}(t) + \mu_{\mathrm{S}}(t) = h_{11}(t) + h_{22}(t))$ known from the literature. Section 7 contains a discussion of the results obtained in Sections 3–6 and some remarks concerning the experiments with neutral kaons.

2. Matrix elements of the effective Hamiltonian

General conclusions concerning properties of matrix elements of the effective Hamiltonian $H_{\|}(t)$ can be drawn using the following identity [16]

$$H_{\|}(t) \equiv i \frac{\partial \boldsymbol{A}(t)}{\partial t} [\boldsymbol{A}(t)]^{-1}, \tag{29}$$

where all matrix elements $A_{jk}(t)$, $(j,k = 1,2)$ of the matrix $\boldsymbol{A}(t)$ can be calculated, *e.g.* by means of (17).

Using (29), one can calculate all the matrix elements of the effective Hamiltonian $H_{\|}$ which may now be written as

$$h_{11}(t) = \frac{i}{\det \boldsymbol{A}(t)} \left(\frac{\partial A_{11}(t)}{\partial t} A_{22}(t) - \frac{\partial A_{12}(t)}{\partial t} A_{21}(t) \right), \tag{30}$$

$$h_{12}(t) = \frac{i}{\det \boldsymbol{A}(t)} \left(-\frac{\partial A_{11}(t)}{\partial t} A_{12}(t) + \frac{\partial A_{12}(t)}{\partial t} A_{11}(t) \right), \tag{31}$$

$$h_{21}(t) = \frac{i}{\det \boldsymbol{A}(t)} \left(\frac{\partial A_{21}(t)}{\partial t} A_{22}(t) - \frac{\partial A_{22}(t)}{\partial t} A_{21}(t) \right), \tag{32}$$

$$h_{22}(t) = \frac{i}{\det \boldsymbol{A}(t)} \left(-\frac{\partial A_{21}(t)}{\partial t} A_{12}(t) + \frac{\partial A_{22}(t)}{\partial t} A_{11}(t) \right), \tag{33}$$

where

$$\det \boldsymbol{A}(t) = A_{11}(t)\, A_{22}(t) - A_{12}(t)\, A_{21}(t)\,. \tag{34}$$

So, having these relations and inserting analytical expressions for $A_{jk}(t)$, (17), calculated in [15] within the assumptions (6) and $[\mathrm{CP}, H] \neq 0$, one obtains all matrix elements $h_{jk}(t)$ for the model considered. Next, such obtained analytical formulae for $h_{jk}(t)$ can be used for numerical calculations of some parameters characterizing neutral kaons for instants of time changing in given time intervals.

3. Numerical examination of the Khalfin's Theorem

It seems to be interesting to verify how the Khalfin's Theorem (11) acts in the system on neutral mesons. To see this, we can use amplitudes $A_{12}(t)$ and $A_{21}(t)$ calculated within the model considered in [15] in the case of conserved CPT and violated CP symmetries. It is not difficult to calculate the modulus of the ratio $\frac{A_{12}(t)}{A_{21}(t)}$ using numerical methods. The results of such calculations are presented below in Fig. 1(a) and (b). There are $y(x) = |\frac{A_{12}(x)}{A_{21}(x)}|$ and $x = (\gamma_{\mathrm{L}}/\hbar)t$ in these figures.

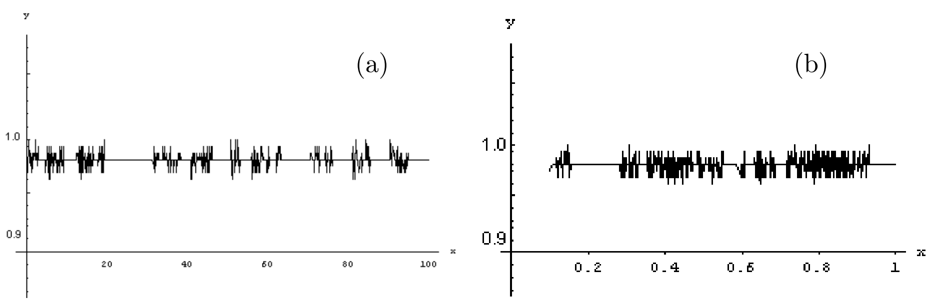

Fig. 1. The time dependence of the absolute value of $y(x) = |r(t)| \equiv |\frac{A_{12}(t)}{A_{21}(t)}|$ in (a) $x \in (0.01, 103)$ and (b) $x \in (0.1, 1)$. Here and in all other figures: $x = (\gamma_{\mathrm{L}}/\hbar)t$.

These figures show that if one is able to measure the modulus of the ratio $\frac{A_{12}(t)}{A_{21}(t)}$ only up to the accuracy of the order of 10^{-15} then one sees this ratio as a constant function of time: for $x \in (0.01, 103)$ we find that

$$y_{\max}(x) - y_{\min}(x) = 3.33067 \times 10^{-16}\,, \tag{35}$$

where

$$\begin{aligned} y_{\max}(x) &= |r(t)|_{\max}\,, \\ y_{\min}(x) &= |r(t)|_{\min}\,. \end{aligned} \tag{36}$$

4. The difference of the diagonal matrix elements $(h_{11}(t) - h_{22}(t))$

Assuming that the CPT symmetry is conserved in the system under considerations ($[\mathrm{CPT}, H]=0$) and using the necessary relations from [12,15,16] one finds the general form of the difference of the diagonal matrix elements of the effective Hamiltonian. It has the following form

$$h_{11}(t) - h_{22}(t) = \frac{X(t)}{\det A(t)}\,, \tag{37}$$

where

$$X(t) = i\left(\frac{\partial A_{21}(t)}{\partial t}\,A_{12}(t) - \frac{\partial A_{12}(t)}{\partial t}\,A_{21}(t)\right) \tag{38}$$

and $\det \boldsymbol{A}(t)$ is defined in (34).

Our analytical result in the one-pole approximation obtained in [15] for $t = \tau_{\mathrm{L}}$ can be written as

$$h_{11}(\tau_{\mathrm{L}}) - h_{22}(\tau_{\mathrm{L}}) \simeq (-4.771 \times 10^{-18} + i\,7.283 \times 10^{-16})\ \mathrm{MeV}\,. \tag{39}$$

The numerical result for $t = \tau_{\mathrm{L}}$ in the one-pole approximation obtained using the Mathematica has the following form

$$h_{11}(\tau_{\mathrm{L}}) - h_{22}(\tau_{\mathrm{L}}) \simeq (-7.129 \times 10^{-17} + i\,1.986 \times 10^{-13})\ \mathrm{MeV}\,. \tag{40}$$

It is seen, that the difference between (39) and (40) is small and it may be attributed to finite accuracy of numerical calculations performed by Mathematica. No approximations have been used in the analytical calculations.

Putting $A_{jk}(t)$ $(j, k = 1, 2)$ given by (17) into (37) and using the energy density $\rho_{jk}(m)$ found in [15], the difference $(h_{11}(t) - h_{22}(t))$ can be calculated as the function of time t. The results of our calculations are presented in

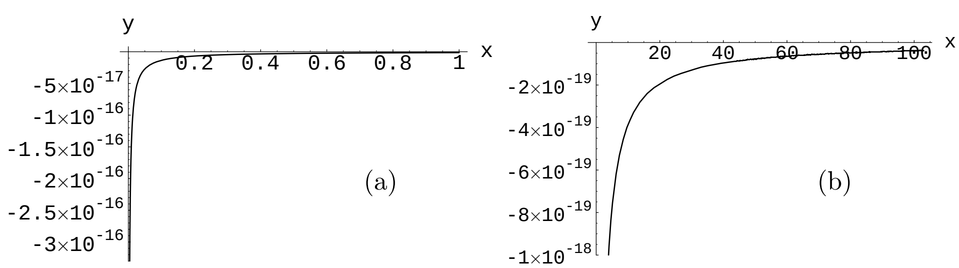

Fig. 2. The time dependence of the real part of the diagonal matrix elements of the effective Hamiltonian $y = \mathrm{Re}(h_{11}(x) - h_{22}(x))$ in the range (a) $x \in (0.001, 1)$ and (b) $x \in (1, 103)$.

a graphical form. The figures below show the time dependence of the real and imaginary parts of the diagonal matrix elements of the effective Hamiltonian $(h_{11}(t) - h_{22}(t))$.

At this point it should be explained that a more accurate analysis of the results of the calculations which lead to Fig. 2(a) and (b) and the use of a larger scale show that the obtained curves are not so smooth as can be seen in Fig. 3(a) and (b) but they are similar to curves in Fig. 3(a) and (b). Note that in Fig. 2(a) we have $\mathrm{Re}(h_{11}(t) - h_{22}(t)) = \mathrm{Re}(h_{11}(\tau_{\mathrm{L}}) - h_{22}(\tau_{\mathrm{L}}))$ for $x = 1$.

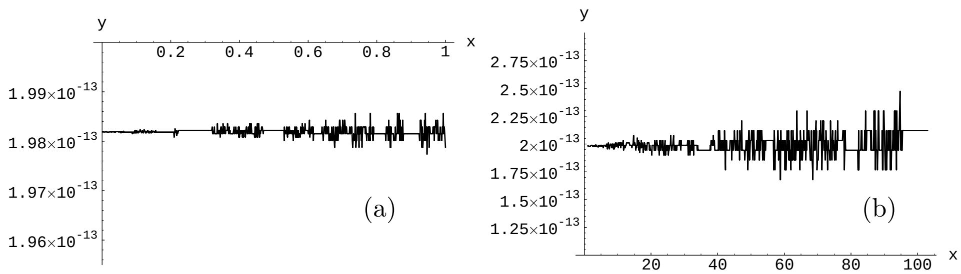

Fig. 3. The time dependence of the imaginary part of the diagonal matrix elements of the effective Hamiltonian $y = \mathrm{Im}(h_{11}(x) - h_{22}(x))$ in the interval (a) $x \in (0.001, 1)$ and (b) $x \in (1, 103)$.

5. Calculation of $\mu_{\mathrm{L}}(t), \mu_{\mathrm{S}}(t)$ and $\varepsilon_{\mathrm{L}}(t), \varepsilon_{\mathrm{S}}(t)$

The eigenvalues of the effective Hamiltonian $\mu_{\mathrm{L}}(t), \mu_{\mathrm{S}}(t)$ can be written as [18]

$$\mu_{\mathrm{L}}(t) = h_0(t) - h(t)\,, \tag{41}$$

$$\mu_{\mathrm{S}}(t) = h_0(t) + h(t)\,, \tag{42}$$

where

$$h_0(t) = \tfrac{1}{2}\left(h_{11}(t) + h_{22}(t)\right), \tag{43}$$

$$h(t) = \sqrt{h_z^2(t) + h_{12}(t)\, h_{21}(t)} \tag{44}$$

and

$$h_z(t) = \tfrac{1}{2}\left(h_{11}(t) - h_{22}(t)\right). \tag{45}$$

From (41) and (42) we have

$$\mu_{\mathrm{S}}(t) + \mu_{\mathrm{L}}(t) = h_{11}(t) + h_{22}(t) \equiv \mathrm{Tr}\left(H_{\|}(t)\right). \tag{46}$$

Relation (46) does not depend on any approximations and it is always true for every (2×2) matrix. Inserting (30)–(33) into (41) and (34) and then using (17) and performing all integrations of type (17) one can obtain for $t = \tau_L$

$$\mu_L(\tau_L) \simeq (497.648 - i\,4.458 \times 10^{-13}) \text{ MeV}\,, \tag{47}$$

and

$$\mu_S(\tau_L) \simeq (497.648 - i\,2.471 \times 10^{-13}) \text{ MeV}\,. \tag{48}$$

The general formula for $\mu_{L(S)}(t)$ can also be written as follows

$$\mu_{L(S)}(t) = m_{L(S)}(t) - \frac{i}{2}\;\gamma_{L(S)}(t)\,. \tag{49}$$

The results of our calculations for the real part and imaginary part in (47) and (48) are rounded to the third decimal place. It should be noted that the real part in (47) and the real part in (48) differ in the fourteenth decimal place. The above mentioned result corresponds with the fact, that $m_S \neq m_L$ and there is $|m_L - m_S| \sim |\gamma_S|$, [5, 20].

The time dependence of $\mu_L(t)$ and $\mu_S(t)$ is given below. Expansion of scale in Fig. 4(a) and Fig. 4(b) shows that continuous fluctuations with amplitudes of the order of 10^{-14} appear.

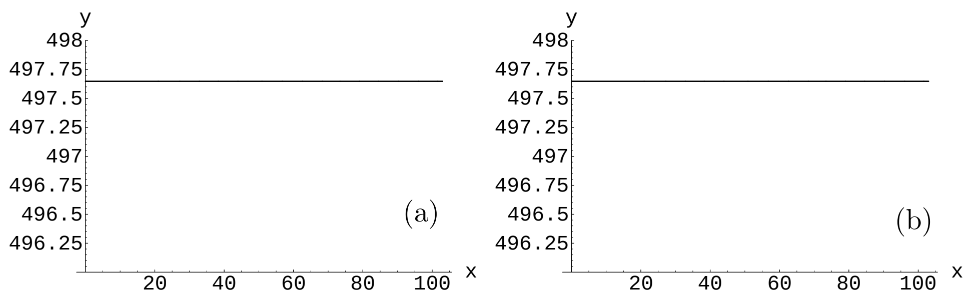

Fig. 4. The time dependence of the real part of (a) $\mu_L(x) : y = \text{Re}(\mu_L(x))$ and (b) $\mu_S(x) : y = \text{Re}(\mu_S(x))$ in the interval $x \in (0.001, 103)$.

We have the following formulae (see, *e.g.* [18])

$$\varepsilon_L(t) = -\frac{h_{21}(t) - h_{22}(t) + \mu_L(t)}{h_{21}(t) + h_{22}(t) - \mu_L(t)}\,, \tag{50}$$

$$\varepsilon_S(t) = -\frac{h_{21}(t) + h_{22}(t) - \mu_S(t)}{h_{21}(t) - h_{22}(t) + \mu_S(t)} \tag{51}$$

and we get for $t = \tau_L$

$$\varepsilon_L(\tau_L) \simeq -1.0000000000184743' + i\,0.0' \tag{52}$$

and

$$\varepsilon_S(\tau_L) \simeq 1.0000157759810688' - i\,0.0'\,. \tag{53}$$

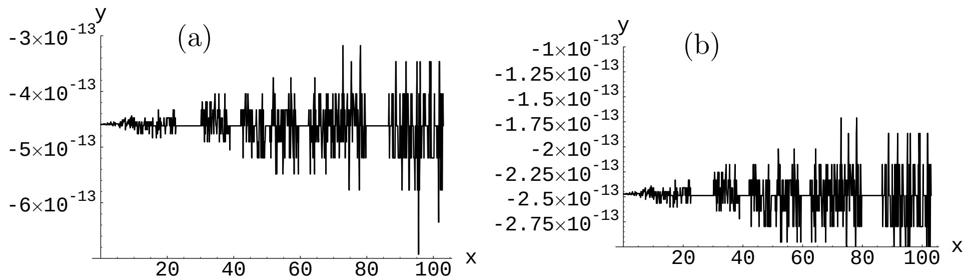

Fig. 5. The time dependence of the imaginary part of (a) $\mu_{\rm L}(x)$: $y = \mathrm{Im}(\mu_{\rm L}(x))$ and (b) $\mu_{\rm S}(x)$: $y = \mathrm{Im}(\mu_{\rm S}(x))$ in the interval $x \in (0.001, 103)$.

The time dependences of $\varepsilon_{\rm L}(t)$ and $\varepsilon_{\rm S}(t)$ are presented below.

Expansion of scale on Fig. 6(a) and Fig. 6(b) shows, that continuous fluctuations with amplitudes of the order of 10^{-12} appear here.

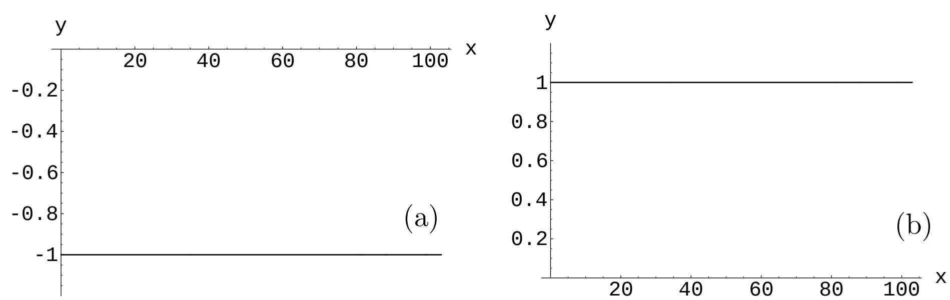

Fig. 6. The time dependence of the real part of (a) $\varepsilon_{\rm L}(x)$: $y = \mathrm{Re}(\varepsilon_{\rm L}(x))$ and (b) $\varepsilon_{\rm S}(x)$: $y = \mathrm{Re}(\varepsilon_{\rm S}(x))$ in the interval $x \in (0.001, 103)$.

From the formula

$$\varepsilon(t) = \tfrac{1}{2}\left(\varepsilon_{\rm L}(t) + \varepsilon_{\rm S}(t)\right) \tag{54}$$

we have for $t = \tau_{\rm L}$

$$\varepsilon(\tau_{\rm L}) \simeq 7.888 \times 10^{-6} - i\, 0.0' \,. \tag{55}$$

The absolute value of $\varepsilon(\tau_{\rm L})$

$$|\varepsilon(\tau_{\rm L})| \simeq 7.888 \times 10^{-6} \,. \tag{56}$$

The figures below present the time dependence of the absolute value of $\varepsilon(t)$.

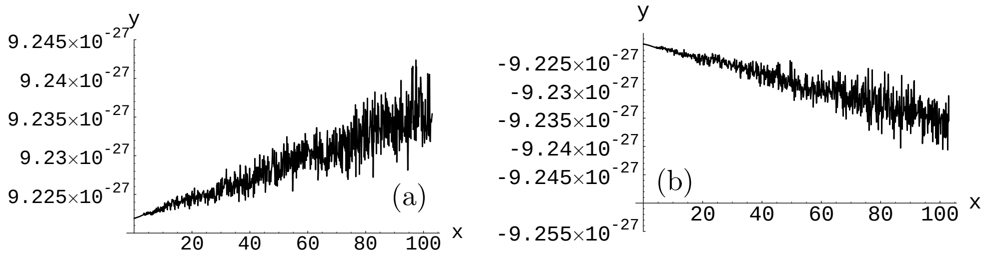

Fig. 7. The time dependence of the imaginary part of (a) $\varepsilon_{\rm L}(x) : y = {\rm Im}(\varepsilon_{\rm L}(x))$ and (b) $\varepsilon_{\rm S}(x) : y = {\rm Im}(\varepsilon_{\rm S}(x))$ in the interval $x \in (0.001, 103)$.

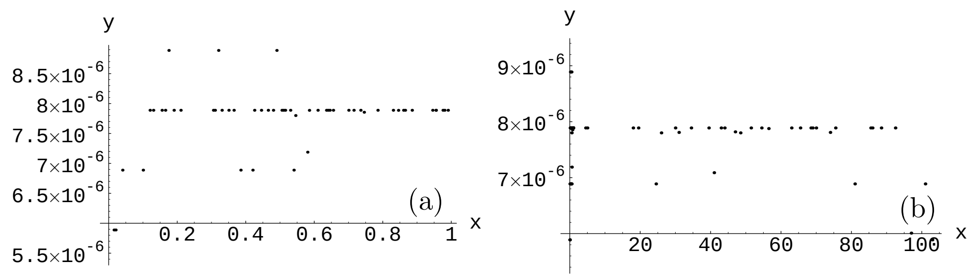

Fig. 8. The time dependence of the absolute value of (a) $\varepsilon(t)$: $y = |\varepsilon(t)|$ in the interval $x \in (0.001, 103)$ and (b) $\varepsilon(t)$: $y = |\varepsilon(t)|$ in the interval $x \in (1, 103)$.

6. Verification of the relation $\mu_{\rm L}(t) + \mu_{\rm S}(t) = h_{11}(t) + h_{22}(t)$

All results in this section have been rounded to the decimal third place. In accordance with formulae ((47), (48)) for $t = \tau_{\rm L}$ we have

$$\begin{aligned} \mu_{\rm L}(\tau_{\rm L}) &\simeq (497.648 - i\,4.458 \times 10^{-13})\ {\rm MeV}, \\ \mu_{\rm S}(\tau_{\rm L}) &\simeq (497.648 - i\,2.471 \times 10^{-13})\ {\rm MeV} \end{aligned}$$

and the corresponding matrix elements of the effective Hamiltonian (formulae (30)–(33)) can be written for $t = \tau_{\rm L}$ as

$$h_{11}(\tau_{\rm L}) \simeq (497.648 - i\,2.471 \times 10^{-13})\ {\rm MeV}\,, \tag{57}$$
$$h_{12}(\tau_{\rm L}) \simeq (1.787 \times 10^{-23} - i\,6.401 \times 10^{-24})\ {\rm MeV}\,, \tag{58}$$
$$h_{21}(\tau_{\rm L}) \simeq (-1.799 \times 10^{-23} - i\,5.127 \times 10^{-24})\ {\rm MeV}\,, \tag{59}$$
$$h_{22}(\tau_{\rm L}) \simeq (497.648 - i\,4.458 \times 10^{-13})\ {\rm MeV}\,. \tag{60}$$

For $t = \tau_{\rm L}$ we get

$$\mu_{\rm L}(\tau_{\rm L})+\mu_{\rm S}(\tau_{\rm L}) = h_{11}(\tau_{\rm L})+h_{22}(\tau_{\rm L}) \simeq (995.296-i\,6.929\times10^{-13})\ \text{MeV}\,. \quad (61)$$

Relation (46) is also fulfilled at $t = \tau_{\rm S}$

$$\mu_{\rm L}(\tau_{\rm S})+\mu_{\rm S}(\tau_{\rm S}) = h_{11}(\tau_{\rm S})+h_{22}(\tau_{\rm S}) \simeq (995.296-i\,9.623\times10^{-14})\ \text{MeV}\,. \quad (62)$$

Comparing (61) with (62) we can see that

$$\mu_{\rm L}(\tau_{\rm L}) + \mu_{\rm S}(\tau_{\rm L}) \neq \mu_{\rm L}(\tau_{\rm S}) + \mu_{\rm S}(\tau_{\rm S})\,. \quad (63)$$

It is interesting to notice that

$$\begin{aligned}
\mu_{\rm L}(\tau_{\rm L}) + \mu_{\rm S}(\tau_{\rm S}) &\simeq (995.296 - i\,5.234\times10^{-13})\ \text{MeV}\,, & (64)\\
\mu_{\rm L}(\tau_{\rm S}) + \mu_{\rm S}(\tau_{\rm L}) &\simeq (995.296 - i\,2.658\times10^{-13})\ \text{MeV}\,, & (65)\\
h_{11}(\tau_{\rm S}) + h_{22}(\tau_{\rm L}) &\simeq (995.296 + i\,4.644\times10^{-13})\ \text{MeV}\,, & (66)\\
h_{11}(\tau_{\rm L}) + h_{22}(\tau_{\rm S}) &\simeq (995.296 + i\,3.247\times10^{-13})\ \text{MeV}\,. & (67)
\end{aligned}$$

From our calculations it follows that the real parts of formulae (64)–(67) differ in the twelfth or thirteenth decimal place. This means that

$$\begin{aligned}
\mu_{\rm L}(\tau_{\rm S}) + \mu_{\rm S}(\tau_{\rm L}) &\neq \mu_{\rm L}(\tau_{\rm S}) + \mu_{\rm S}(\tau_{\rm S})\,, & (68)\\
\mu_{\rm L}(\tau_{\rm S}) + \mu_{\rm S}(\tau_{\rm L}) &\neq \mu_{\rm L}(\tau_{\rm L}) + \mu_{\rm S}(\tau_{\rm L})\,, & (69)\\
\mu_{\rm L}(\tau_{\rm S}) + \mu_{\rm S}(\tau_{\rm L}) &\neq \mu_{\rm L}(\tau_{\rm L}) + \mu_{\rm S}(\tau_{\rm S})\,, & (70)\\
\mu_{\rm L}(\tau_{\rm S}) + \mu_{\rm S}(\tau_{\rm L}) &\neq h_{11}(\tau_{\rm S}) + h_{22}(\tau_{\rm S})\,, & (71)\\
\mu_{\rm L}(\tau_{\rm L}) + \mu_{\rm S}(\tau_{\rm S}) &\neq h_{11}(\tau_{\rm L}) + h_{22}(\tau_{\rm L})\,, & (72)\\
\mu_{\rm L}(\tau_{\rm L}) + \mu_{\rm S}(\tau_{\rm S}) &\neq h_{11}(\tau_{\rm L}) + h_{22}(\tau_{\rm S})\,. & (73)
\end{aligned}$$

and so on.

7. Final remarks

First, as it was pointed out in [12], let us notice that in the considered model some relations assumed there and allowing to perform integrations of type (17) are not valid in the K^0–$\bar{K}^0$ system (see a comment between formulae (5.10) and (5.11) of [12], Sec. 5). These relations were also used in [12] and [15]. Next, a drawback of our model is that at $t = 0$ we obtain $h_{11}(t = 0) = \infty$ and $h_{22}(t = 0) = \infty$. However, this model allows to study all the consequences of Khalfin's Theorem and theorems considered in [16]. Bearing in mind the limitations of our model mentioned above one should not expect that our calculations based on this model will result in an exact reconstruction of all experimental parameters characterizing the neutral K meson system.

Results presented in Sec. 3 show how Khalfin's Theorem works. We can conclude that the effect of this Theorem should be visible in experiments with the neutral kaon complex in which the modulus of $r(t)$, (11), can be measured with the accuracy of order 10^{-16} or better. On the other hand, these results are in perfect agreement with the supposition formulated in [19]. Experimental results give $|1 - |r(t)|\,| \sim 10^{-3} = \text{const.}$ with some limited accuracy (see, *e.g.* [20]). The explanation of this fact proposed in [19] is based on the assumption that

$$r(t) = r_{\text{LOY}} + d(t)\,,$$

where r_{LOY} stands for $r(t)$ calculated within the LOY theory, and $d(t)$ is assumed to be a function varying in time t such that $|d(t)| < 10^{-11}$.

Results obtained in Sec. 4 suggest that the real part of the difference $(h_{11}(t) - h_{22}(t))$ is different from zero for very large times t: from $t \sim 0,1\tau_{\text{L}}$ up to $t \sim 100\tau_{\text{L}}$. Moreover, after division by m_{average}, this difference is only a little smaller than the corresponding experimental value [20]. The imaginary part of $(h_{11}(t) - h_{22}(t))$ turned out to be different from zero as well. However, this part oscillates about 2×10^{-13} MeV very fast. Note that from the results contained in [16] it follows that these differences should differ from zero for all $t > 0$. Within the standard treatment of the neutral K system the measurement of the difference of masses $(m_{K^0} - m_{\bar{K}^0})$ is considered as the CPT invariance test. This interpretation of such tests is based on the properties (7), (8) of the LOY approach: Within the LOY theory CPT symmetry is conserved only if $(m_{K^0} - m_{\bar{K}^0}) = 0$. The results obtained in Sec. 4 and in [15–17] show that such an interpretation of this test is true only for the LOY approximation and beyond LOY approximation properties (7), (8) do not occur. It seems to be obvious that the description of neutral K complex using the more accurate formalism than the LOY approximation leads to a more realistic description of such a complex. So, if within the more accurate theory one obtains $(m_{K^0} - m_{\bar{K}^0}) \neq 0$ when CPT symmetry holds and CP symmetry is violated then one is forced to conclude that such a property must be valid in the real CPT invariant systems. Therefore, taking into account results obtained in Sec. 4 (and in [15–17]) the conclusion that the measurement of the mass difference, $(m_{K^0} - m_{\bar{K}^0})$, should not be considered as CPT invariance test seems to be correct.

Results in Sec. 5 show that the imaginary parts of parameters $\mu_{\text{L}}, \mu_{\text{S}}$ and $\varepsilon_{\text{L}}, \varepsilon_{\text{S}}$ vary in time too. We can say the same about their real parts. The oscillation amplitude is of the order of 10^{-13} for $\text{Im}\,(\mu_{L(S)})$ and it is smaller than 10^{-27} for $\text{Im}\,(\varepsilon_{(\text{S})})$. The real parts $\text{Re}\,(\mu_{\text{L(S)}})$ and $\text{Re}\,(\varepsilon_{\text{L(S)}})$ oscillate in a similar way as their imaginary parts and this is the reason why they cannot be shown in our figures. The parameter ε, (54), is also a quantity varying in time as we can see from the graphical results (see Fig. 8(a) and Fig. 8(b)). The absolute value of $|\varepsilon|$ oscillates around the value 8×10^{-6}.

In Sec. 6, relation (46) was investigated. We know, from general considerations, that it has to be fulfilled for every time t irrespective of whether we consider an approximate model of neutral K meson system or if we investigate the exactly solvable model of this system. The considered mathematical results show that the left side of equation (46) is the same as its right side for $t = \tau_l$, $t = \tau_{\mathrm{S}}$. A similar mathematical result was obtained for other times. From relation (63) it follows that the left side of equation (46), as well as its rights side, are not constant in the time.

Relations (64)–(73) show that equation (46) is no longer fulfilled when the separate components of sums appearing in its left and right sides are taken at different moments of time and then inserted into this equation. From this observation and from results in Sec. 4, we can draw an important conclusion concerning the methods of experimental data registering and experimental data processing.

Let us note that an experimental system containing detectors which register the neutral K meson decay products can be schematically presented as in Fig. 9. This figure presents a longitudinal section of a cylindrical vacuum chamber. A K meson stream is fired into this chamber along a horizontal axis l on the left side. Detectors D surrounding the chamber form its walls. These detectors register the neutral K meson decay products. In the first region of this chamber (I), we observe a great amount of the decay products of neutral kaons into two pions ($K_{\mathrm{S}} \longrightarrow 2\pi$). In the second region of this chamber (II), we usually observe a great amount of the decay products of neutral kaons into three pions ($K_{\mathrm{L}} \longrightarrow 3\pi$). We can interpret the l axis as a path in an uniform straight-line motion of neutral kaons, whose decay products are registered by detectors D. We can write $l_{\mathrm{I}} = v_{K_{\mathrm{S}}} \tau_{\mathrm{S}}$ in the first

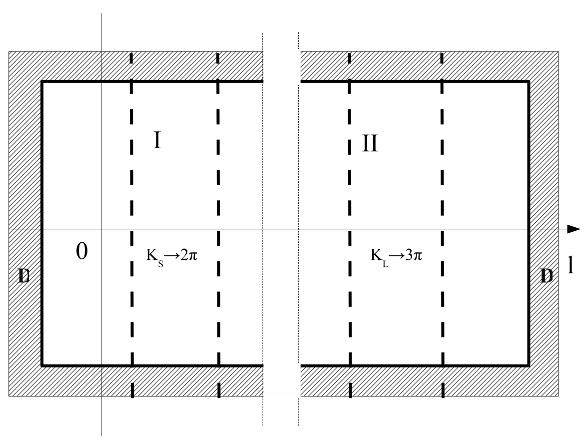

Fig. 9. A scheme of the experimental set for the experiment with the neutral K meson described in this section.

region (I) and $l_{\rm II} = v_{K_{\rm L}}\tau_{\rm L}$ in the second region (II) (where $v_{K_{\rm S(L)}}$ is the kaon $K_{\rm S(L)}$ speed). One can also obtain (for comparison): $l'_I = c\,\tau_{\rm S} = 0.026805$ m and $l'_{II} = c\,\tau_{\rm L} = 15.51$ m ($c = 3\times 10^8$ s — the speed of light in vacuum, $\tau_{\rm S} = 0.8935\times 10^{-10}$ s a $\tau_{\rm L} = 5.17\times 10^{-8}$ s).

Let us now return to the above mentioned conclusion. From (64)–(73) and from results presented in Sec. 5, it follows that only these parameters can correctly reflect real properties of neutral K system which are calculated using only data obtained from a ring of detectors limited by distances $(l,\ l+\Delta l)$. Since $t\sim l$, the events are registered between $t,\ t+\Delta t$ from the initial instant. Of course, Δl should be as small as possible. In other words, one should not use the experimental data obtained from the registration of the neutral K meson decay products in the calculations if these neutral K meson decay products come from different and distant parts of the measurement set of the type shown in Fig. 9. For example, one should not use calculations which were registered by the detector D in the region where $t\sim\tau_{\rm S}$ simultaneously with the data which were registered by the detector D in the region where $t\sim\tau_{\rm L}$. One should not mix the data coming from different and distant parts of the measurement. If this rule is not observed, it may turn out that the obtained values $\mu_{\rm L(S)}$ (obtained on the basis of parameters measured in this way) will not satisfy the consistency check given by (46). Of course, in order to check the consistency of the experimental results with (46) the experiment should be conducted in such a way that both sides of (46) can be found in independent measurements. Then one will obtain independently each other the left side and the right side of this equality.

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