CHAOTIC INFLATION WITH A QUADRATIC POTENTIAL IN ALL DIMENSIONS

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(Received January 22, 2007)

We study chaotic inflation with a quadratic potential in all dimensions. The slow-roll parameters, the spectral indices of scalar and tensor perturbations and also their running have been calculated in all dimensions.

PACS numbers: 04.50.+h; 98.80.Cq; 98.80.–k

Following the advent of string theory and its implication that space may have more than the usual three dimensions, we here study chaotic inflation with a quadratic potential, $V(\varphi) = m^2\varphi^2/2$, in all dimensions.

Take the metric in constant $(D+1)$-dimensional spacetime in the following form (we use natural units or high-energy physics units in which the fundamental constants are $\hbar = c = k_{\rm B} = 1$, $G = \ell_{\rm Pl}^2 = 1/m_{\rm Pl}^2$)

$$ds^2 = -N^2(t)dt^2 + a^2(t)d\Sigma_k^2\,, \tag{1}$$

where $N(t)$ denotes the lapse function and $d\Sigma_k^2$ is the line element for a D-manifold of constant curvature $k = +1, 0, -1$, corresponding to the closed, flat and hyperbolic spacelike sections, respectively. The Ricci scalar is given by [1]

$$R = \frac{D}{N^2}\left\{\frac{2\ddot{a}}{a} + (D-1)\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{N^2k}{a^2}\right] - \frac{2\dot{a}\dot{N}}{aN}\right\}. \tag{2}$$

The Einstein–Hilbert action for the pure gravity is given by [1]

$$S_G = \frac{1}{2\kappa_{(D+1)}}\int d^{(D+1)}x\sqrt{-g}R\,, \tag{3}$$

and the action for a perfect fluid may be expressed by

$$S_M = -\int d^{(D+1)}x\sqrt{-g}\rho\,. \tag{4}$$

In Eq. (3), the gravitational coupling constant in $(D+1)$-dimensional spacetime, $\kappa_{(D+1)}$, is related to the $(D+1)$-dimensional gravitational constant $G_{(D+1)}$ by [2]

$$\kappa_{(D+1)} = (D-1)S^{[D]}G_{(D+1)}\,, \tag{5}$$

where

$$S^{[D]} = \frac{2\pi^{D/2}}{\Gamma\left(\frac{D}{2}\right)}\,, \tag{6}$$

where $S^{[D]}$ is the surface area of the unit sphere in D-dimensional spaces. In the case $(3+1)$, $(4+1)$ and $(5+1)$-dimensional spacetime we have $\kappa_{(3+1)} = 8\pi G_{(3+1)}$ (*i.e.* $\kappa = 8\pi G$), $\kappa_{(4+1)} = 6\pi^2 G_{(4+1)}$ and $\kappa_{(5+1)} = \frac{32\pi^2}{3}G_{(5+1)}$, respectively. Using (5) and (6) we have (see Appendix)

$$\kappa_{(D+1)} = \frac{2(D-1)\pi^{D/2}G_{(D+1)}}{\Gamma\left(\frac{D}{2}\right)}\,. \tag{7}$$

Using (3) and (4), we obtain the following Lagrangian [1]

$$L := -\frac{a^D}{2\kappa_{(D+1)}}\frac{D(D-1)}{N}\left[\left(\frac{\dot{a}}{a}\right)^2 - \frac{N^2k}{a^2}\right] - \rho N a^D. \tag{8}$$

Varying the above Lagrangian with respect to N and a, we find the following equations of motion in the gauge $N=1$, respectively:

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{2\kappa_{(D+1)}\rho}{D(D-1)}\,, \tag{9}$$

$$\frac{\ddot{a}}{a} + \left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right]\left(-1+\frac{D}{2}\right) + \frac{\kappa_{(D+1)}p}{D-1} = 0\,. \tag{10}$$

Using (9) and (10), one gets the continuity equation

$$\frac{d}{dt}\left(\rho a^D\right) + p\frac{d}{dt}\left(a^D\right) = 0\,. \tag{11}$$

Inserting $k=0$ for a flat universe and the energy density and the pressure of a homogeneous inflation field

$$\rho \equiv \frac{1}{2}\dot{\varphi}^2 + V(\varphi)\,, \tag{12}$$

$$p \equiv \frac{1}{2}\dot{\varphi}^2 - V(\varphi) \tag{13}$$

in Eqs. (9) and (11), we are led to

$$H^2 = \frac{2\kappa_{(D+1)}}{D(D-1)}\left(\frac{1}{2}\dot{\varphi}^2 + V(\varphi)\right), \tag{14}$$

$$\ddot{\varphi} + DH\dot{\varphi} = -V'(\varphi), \tag{15}$$

where $H = \frac{\dot{a}}{a}$. The slow-roll conditions in D-dimensional spaces read [3]

$$\dot{\varphi}^2 \ll V(\varphi), \qquad \ddot{\varphi} \ll DH\dot{\varphi}, \qquad -\dot{H} \ll H^2. \tag{16}$$

Using these conditions, Eqs. (14) and (15) can be rewritten

$$H^2 = \frac{2\kappa_{(D+1)}V(\varphi)}{D(D-1)}, \tag{17}$$

$$DH\dot{\varphi} \simeq -V'(\varphi). \tag{18}$$

During inflation, H is slowly varying in the sense that its change per Hubble time, $\varepsilon \equiv -\dot{H}/H^2$ is less than one. The slow-roll condition $\eta \ll 1$ is actually a consequence of the condition $\varepsilon \ll 1$ plus the slow-roll approximation $DH\dot{\varphi} \simeq -V'(\varphi)$. Indeed, differentiating (18) one finds

$$\frac{\ddot{\varphi}}{H\dot{\varphi}} = \varepsilon - \eta, \tag{19}$$

where the slow-roll parameters in any constant space dimension are defined by

$$\varepsilon \equiv -\frac{\dot{H}}{H^2} = \frac{(D-1)}{4\kappa_{(D+1)}}\left(\frac{V'}{V}\right)^2, \tag{20}$$

$$\eta \equiv \frac{V''}{DH^2} = \frac{(D-1)}{2\kappa_{(D+1)}}\left(\frac{V''}{V}\right). \tag{21}$$

The number of e-foldings between t_i and t_f is given by

$$\mathcal{N} = \int_{t_i}^{t_f} H(t)dt = \ln\left(\frac{a_f}{a_i}\right) \simeq -\frac{2\kappa_{(D+1)}}{(D-1)}\int_{\phi_i}^{\phi_f} \frac{V}{V'}d\varphi. \tag{22}$$

The amplitudes of scalar and tensor perturbations generated in inflation can be expressed by [4–6]

$$A_{\mathrm{S}}^2 = \left(\frac{H}{2\pi}\right)^2\left(\frac{H}{\dot{\varphi}}\right)^2, \tag{23}$$

$$A_{\mathrm{T}}^2 = \frac{\kappa_{(D+1)}}{(D-1)}\left(\frac{H}{2\pi}\right)^2. \tag{24}$$

These amplitudes are equal to $\frac{25}{4}$ times the amplitudes as given in Ref. [5]. The amplitudes of scalar and tensor perturbations generated in inflation can be determined by substituting (17) and (18) into (23) and (24)[1]

$$A_{\rm S}^2 = \frac{2\kappa_{(D+1)}^3 V^3}{D(D-1)^3\pi^2 V'^2}\,, \tag{25}$$

$$A_{\rm T}^2 = \frac{\kappa_{(D+1)}^2 V}{2\pi^2 D(D-1)^2}\,. \tag{26}$$

These expressions are evaluated at the horizon crossing time when $k = aH$. Since the value of Hubble constant does not change too much during inflationary epoch, we can obtain $dk = Hda$ and $d\ln k = Hdt = da/a$. Using the slow-roll condition in $(D+1)$-dimensional spacetime

$$\frac{d}{d\ln k} = -\frac{V'}{DH^2}\frac{d}{d\varphi}\,, \tag{27}$$

and also a lengthy but straightforward calculation by using (17), (18), (20), (21) and (23)–(27) we find

$$n_{\rm S} - 1 \equiv \frac{d\ln A_S^2}{d\ln k} = -6\varepsilon + 2\eta\,, \tag{28}$$

$$n_{\rm T} \equiv \frac{d\ln A_T^2}{d\ln k} = -2\varepsilon\,, \tag{29}$$

where $n_{\rm S}$ and $n_{\rm T}$ are the spectral indices of scalar and tensor perturbations, respectively. If $n_{\rm S}$ and $n_{\rm T}$ are expressed as a function of e-folding $\mathcal{N}$, one can use the fact that $\frac{d}{d\ln k} = -\frac{d}{d\mathcal{N}}$ to obtain the desired derivatives even more easily. To calculate the running of the scalar and tensor spectral indices in all dimensions, we use Eqs. (20), (21) and (27). Therefore we have in all dimensions

$$\frac{d\varepsilon}{d\ln k} = -2\varepsilon\eta + 4\varepsilon^2\,, \tag{30}$$

$$\frac{d\eta}{d\ln k} = 2\varepsilon\eta - \xi\,, \tag{31}$$

[1] It is worth mentioning that the authors of Refs. [3,6] have studied chaotic inflation in higher dimensions and also in a model universe with time variable space dimensions by taking $\kappa = 8\pi G$ for all dimensions. Our results above improve the results given in Refs. [3, 6] because we here consider the gravitational coupling constant in all dimensions as a function of spatial dimensions, as given in Eq. (7).

where the third slow-roll parameter is defined by[2]

$$\xi \equiv \frac{(D-1)^2}{4\kappa^2_{(D+1)}}\left(\frac{V'V'''}{V^2}\right). \tag{32}$$

Using the above equations, running of the spectral indices of scalar and tensor perturbations in higher dimensions we have these explicit expressions

$$\frac{dn_{\mathrm{S}}}{d\ln k} = 16\varepsilon\eta - 24\varepsilon^2 - 2\xi\,, \tag{33}$$

$$\frac{dn_{\mathrm{T}}}{d\ln k} = 4\varepsilon\eta - 8\varepsilon^2\,. \tag{34}$$

For the chaotic inflation with a quadratic potential, $m^2\varphi^2/2$, the solution of Eqs. (17) and (18) are given by

$$\varphi(t) = \varphi_i - m\sqrt{\frac{D-1}{D\kappa_{(D+1)}}}t\,, \tag{35}$$

$$a(t) = a_i \exp\left(\frac{\kappa_{(D+1)}}{2(D-1)}\left[\varphi_i^2 - \varphi^2(t)\right]\right). \tag{36}$$

Using the slow-roll parameters

$$\varepsilon = \eta = \frac{(D-1)}{\kappa_{(D+1)}\varphi^2}\,, \tag{37}$$

and the failure of the slow-roll conditions

$$\max\{\varepsilon_f; |\eta_f|\} \simeq 1\,, \tag{38}$$

one concludes that

$$\varphi_f = \sqrt{\frac{(D-1)}{\kappa_{(D+1)}}}\,. \tag{39}$$

Substituting this value of φ_f into (22), one gets

$$\mathcal{N} = \frac{\kappa_{(D+1)}}{2(D-1)}\varphi_i^2 - \frac{1}{2}\,. \tag{40}$$

[2] This expression for ξ in all dimensions improve Eq. (56) of Ref. [6] in which $\kappa = 8\pi G$ has been considered for all dimensions.

One can also obtain the spectral indices of scalar and tensor perturbations and their running in $(D+1)$-spacetime dimension

$$n_{\mathrm{S}} - 1 = -4\varepsilon = -\frac{4(D-1)}{\kappa_{(D+1)}\varphi^2}\,, \tag{41}$$

$$n_{\mathrm{T}} = -2\varepsilon = -\frac{2(D-1)}{\kappa_{(D+1)}\varphi^2}\,, \tag{42}$$

$$\frac{dn_{\mathrm{S}}}{d\ln k} = -8\varepsilon^2 = -\frac{8(D-1)^2}{\kappa^2_{(D+1)}\varphi^4}\,, \tag{43}$$

$$\frac{dn_{\mathrm{T}}}{d\ln k} = -4\varepsilon^2 = -\frac{4(D-1)^2}{\kappa^2_{(D+1)}\varphi^4}\,. \tag{44}$$

Appendix

Gravitational coupling constant in higher dimensions

In $(3+1)$-dimensional spacetime, the gravitational coupling constant is given by $\kappa = 8\pi G$. Looking for the roots of the factor of 8π in κ we across the relation

$$R_{00} = (D-2)\nabla^2_D\phi\,, \tag{45}$$

where ∇_D is the ∇ operator in D-dimensional space. In $(3+1)$-dimensional spacetime, the Poisson equation is given by

$$\nabla^2\phi = 4\pi G\rho\,. \tag{46}$$

Applying Gauss law for a D-dimensional volume, we find the Poisson equation for arbitrary fixed dimension

$$\nabla^2_D\phi = S^{[D]}G_{(D+1)}\rho\,, \tag{47}$$

where $S^{[D]}$ is the surface area of a unit sphere in D-dimensional spaces, see Eq. (6). On the other hand we get

$$R_{00} = \left(\frac{D-2}{D-1}\right)\kappa_{(D+1)}\rho\,. \tag{48}$$

Using (6), (45), (47) and (48), we are led to the gravitational coupling constant in $(D+1)$-dimensional spacetime as given in (5) and (7).

F.N. thanks Hurieh Husseinian and A.A. Nasseri for noble helps and also thanks Amir and Shahrokh for truthful helps.

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