

ENERGY-MOMENTUM PROBLEM IN GENERAL RELATIVITY AND TELE-PARALLEL GRAVITY FOR STRING AND DOMAIN WALL

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This study is purposed to elaborate the problem of energy and momentum distribution of string and domain wall in the context of two different approaches of gravity such as general relativity and tele-parallel gravity. In this connection, we have calculated energy-momentum of domain wall and string by using the energy-momentum definitions of Einstein, Bergmann–Thomson and Landau–Lifshitz in general relativity and tele-parallel gravity. In our analysis we obtained that *(i)* general relativity and tele-parallel gravity are equivalent theories for string and domain wall *(ii)* different energy-momentum complexes do not provide the same energy and momentum densities neither in general relativity nor in tele-parallel gravity for domain wall.

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1. Introduction

It is still a challenging problem to know the exact physical situation at very early stages of the formation of our universe. At the very early stages of evolution of universe, it is generally assumed that during the phase transition (as the universe passes through its critical temperature) the symmetry of the universe is broken spontaneously.

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Spontaneous symmetry breaking is an old idea, described within the particle physics context in terms of the Higgs field. The symmetry is called spontaneously broken if the ground state is not invariant under the full symmetry of the Lagrangian density. Thus, the vacuum expectation value of the Higgs field is nonzero. In quantum field theories, broken symmetries are restored at high enough temperatures.

Spontaneous symmetry breaking can give rise to topologically stable defects. Depending on whether the symmetry is local (gauged) or global (rigid), topological defects are called local or global. They are expected to be remnants of phase transitions that may have occurred in the early universe. Topological defects [1, 2] are stable field configurations that arise in field theories with spontaneously broken discrete or continuous symmetries. Depending on the topology of the vacuum manifold M they are usually identified as domain walls [2] when $M = Z_2$, as strings [3] and one-dimensional textures [4, 5] when $M = S^1$, as monopoles [6–8] and two dimensional textures when $M = S^2$ and three dimensional textures when $M = S^3$. They also form in various condensed matter systems which undergo low temperature transitions [9].

The basic purpose of this paper is to obtain the energy-momentum complexes of domain wall and string space times in the context of general relativity and tele-parallel theory by using Einstein, Bermann–Thomson and Landau–Lifshitz prescriptions.

In the gravitation theories of general relativity and the tetrad theory of gravity, the formulation of energy and momentum distributions is one of the oldest, interesting and controversial problems. It is well known that one of the most interesting and challenging problems of general relativity is the energy and momentum localization. Energy-momentum is an important conserved quantity in any physical theory whose definition has been under investigation for a long time from the General Relativity viewpoint. The problem is to find an expression which is physically meaningful. The point is that the gravitational field can be made locally vanish and so one is always able to find the frame in which the energy-momentum of gravitational field is zero while in the other frames, it is not true. Unfortunately, there is still no generally accepted definition of energy-momentum for gravitational field. The problem arises with the expression defining the gravitational field energy part.

Schrödinger showed that the pseudo-tensor can be made vanish outside the Schwarzschild radius using a suitable choice of coordinates. There have been many attempts in order to find a more suitable quantity for describing the distribution of energy and momentum due to matter, non-gravitational and gravitational fields. The proposed quantities which actually fulfill the conservation law of matter plus gravitational parts are called gravitational

field pseudo-tensors. The choice of the gravitational field pseudo-tensor is not unique. Because of this, quite a few definitions of these pseudo-tensors have been proposed. The notion of energy-momentum prescriptions was severely criticized for a number reasons. Firstly, the nature of symmetric and locally conserved object is non-tensorial one; thus its physical interpretation appeared obscure [11]. Secondly, different energy-momentum complexes could yield different energy-momentum distributions for the same gravitational background [12]. Finally, energy-momentum complexes were local objects while it was generally believed that the suitable energy-momentum of the gravitational field was only total, *i.e.* it cannot be localized [13].

In order to obtain a meaningful expression for energy, momentum and angular momentum for a general relativistic system, Einstein himself proposed an expression. After Einstein's energy-momentum complex [14], many complexes have been found, for instance, Landau–Lifshitz [15], Tolman [16], Papapetrou [17], Møller [18, 19], Weinberg [20] and Bergmann [21]. Some of these definitions are coordinate dependent while others are not.

In this paper, we will proceed according to the following scheme. In Section 2 and Section 3, the simple definitions of Einstein, Bergmann–Thomson and Landau–Lifshitz energy-momentum prescriptions are given in general relativity and the form of tele-parallel gravity. In Section 4, the energy and momentum distributions of domain wall and string are obtained in general relativity and tele-parallel gravity by using Einstein, Bergmann–Thomson and Landau–Lifshitz energy prescriptions. Finally, we summarize and discuss our results. Throughout this paper, we use units where $G = \hbar = c = 1$. Greek and Latin indices run from 0 to 3 and represent the vector components and the vector numbers, respectively.

2. Some energy-momentum prescriptions in the General Relativity

2.1. Bergmann–Thomson energy-momentum prescription in GR

The energy-momentum prescription of Bergmann–Thomson [21] is given by

$${}^{\text{GR}}B^{\mu\nu} = \frac{1}{16\pi} \Pi^{\mu\nu\beta}_{,\beta}, \quad (1)$$

where

$$\Pi^{\mu\nu\beta} = g^{\mu\alpha} V_{\alpha}^{\nu\beta}, \quad (2)$$

with

$$V_{\beta}^{\nu\alpha} = -V_{\beta}^{\alpha\nu} = \frac{g_{\beta\xi}}{\sqrt{-g}} \left[-g \left(g^{\nu\xi} g^{\alpha\rho} - g^{\alpha\xi} g^{\nu\rho} \right) \right]_{,\rho}. \quad (3)$$

${}^{\text{GR}}B_0^0$ is the energy density, ${}^{\text{GR}}B_\mu^0$ are the momentum density components, and ${}^{\text{GR}}B_0^\mu$ are the components of the energy current density. The energy and momentum components are given by

$$P^\mu = \int \int \int {}^{\text{GR}}B^{\mu 0} dx dy dz. \quad (4)$$

Further Gauss's theorem furnishes

$$P^\mu = \frac{1}{16\pi} \int \int \Pi^{\mu 0\alpha} \kappa_\alpha dS, \quad (5)$$

where κ_α stands for the 3-components of unit vector over an infinitesimal surface element dS . The quantities P^i for $i=1,2,3$ are the momentum components, while P^0 is the energy.

2.2. Einstein energy-momentum prescription in GR

The energy-momentum complex as defined by Einstein [19] is given by

$${}^{\text{GR}}E_\mu^\nu = \frac{1}{16\pi} H_{\mu,\alpha}^{\nu\alpha}, \quad (6)$$

where

$$H_\mu^{\nu\alpha} = \frac{g_{\mu\beta}}{\sqrt{-g}} \left[-g(g^{\nu\beta} g^{\alpha\xi} - g^{\alpha\beta} g^{\nu\xi}) \right]_{,\xi}. \quad (7)$$

${}^{\text{GR}}E_0^0$ is the energy density, ${}^{\text{GR}}E_\alpha^0$ are the momentum density components, and ${}^{\text{GR}}E_0^\alpha$ are the components of energy current density. The Einstein energy and momentum components are given by

$$P^\mu = \int \int \int {}^{\text{GR}}E_\mu^0 dx dy dz. \quad (8)$$

Further Gauss's theorem furnishes

$$P^\mu = \frac{1}{16\pi} \int \int H_\mu^{0\alpha} \eta_\alpha dS. \quad (9)$$

η_α stands for the 3-components of unit vector over an infinitesimal surface element dS . The quantities P^i for $i=1,2,3$ are the momentum components, while P^0 is the energy.

2.3. Landau–Lifshitz energy-momentum prescription in GR

Energy-momentum prescription of Landau–Lifshitz [15] is given by

$${}^{\text{GR}}L^{\mu\alpha} = \frac{1}{16\pi} S^{\mu\nu\alpha\beta}_{,\nu\beta}, \quad (10)$$

where

$$S^{\mu\nu\alpha\beta} = -g(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\beta}g^{\nu\alpha}). \quad (11)$$

${}^{\text{GR}}L^0_0$ is the energy density, ${}^{\text{GR}}L^0_\mu$ are the momentum density components, and ${}^{\text{GR}}L^\mu_0$ are the components of energy current density. The energy and momentum components are given by

$$P^\mu = \int \int \int {}^{\text{GR}}L^{\mu 0} dx dy dz. \quad (12)$$

Further Gauss's theorem furnishes

$$P^\mu = \frac{1}{16\pi} \int \int S^{\mu\alpha 0\nu}_{,\nu} \eta_\alpha dS, \quad (13)$$

where η_α stands for the 3-components of unit vector over an infinitesimal surface element dS . The quantities P^i for $i=1,2,3$ are the momentum components, while P^0 is the energy.

Above three energy-momentum prescriptions defined in GR satisfy the local conservations laws.

3. Some energy-momentum prescriptions in the tele-parallel gravity

3.1. Einstein, Bergmann–Thomson and Landau–Lifshitz energy-momentum prescriptions in TG

The tele-parallel gravity is an alternative approach to gravitation and corresponds to a gauge theory for the translation group based on Weitzenböck geometry [22]. In the theory of the tele-parallel gravity, gravitation is attributed to torsion [23], which plays the role of a force [24], and the curvature tensor vanishes identically. The essential field is acted by a nontrivial tetrad field, which gives rise to the metric as a by-product. The translational gauge potentials appear as the nontrivial item of the tetrad field, so induces on space-time a tele-parallel structure which is directly related to the presence of the gravitational field. The interesting place of tele-parallel gravity is that, due to its gauge structure, it can reveal a more appropriate approach to consider some specific problem. This is the situation, for example, in the energy and momentum problem, which becomes more transparent when considered from the tele-parallel point of view.

The Einstein (${}^{\text{TG}}E^\mu_\nu$), Bergmann–Thomson (${}^{\text{TG}}B^{\mu\nu}$) and Landau–Lifshitz's (${}^{\text{TG}}L^{\mu\nu}$) energy-momentum complexes in tele-parallel gravity [26] are respectively:

$$h {}^{\text{TG}}E^\mu_\nu = \frac{1}{4\pi} \partial_\lambda (U_\nu{}^{\mu\lambda}), \quad (14)$$

$$h {}^{\text{TG}}B^{\mu\nu} = \frac{1}{4\pi} \partial_\lambda (g^{\mu\beta} U_\beta{}^{\nu\lambda}), \quad (15)$$

$$h {}^{\text{TG}}L^{\mu\nu} = \frac{1}{4\pi} \partial_\lambda (h g^{\mu\beta} U_\beta{}^{\nu\lambda}), \quad (16)$$

where $U_\beta{}^{\nu\lambda}$ is the Freud's super-potential, which is given by:

$$U_\beta{}^{\nu\lambda} = h S_\beta{}^{\nu\lambda}, \quad (17)$$

where $h = \det(h^a{}_\mu)$ and $S^{\mu\nu\lambda}$ is the tensor

$$S^{\mu\nu\lambda} = m_1 T^{\mu\nu\lambda} + \frac{m_2}{2} (T^{\nu\mu\lambda} - T^{\lambda\mu\nu}) + \frac{m_3}{2} (g^{\mu\lambda} T^{\beta\nu}_\beta - g^{\nu\mu} T^{\beta\lambda}_\beta), \quad (18)$$

with m_1 , m_2 and m_3 the three dimensionless coupling constants of tele-parallel gravity [27]. For the tele-parallel equivalent of general relativity the specific choice of these three constants are:

$$m_1 = \frac{1}{4}, \quad m_2 = \frac{1}{2}, \quad m_3 = -1. \quad (19)$$

To calculate this tensor, firstly we must calculate Weitzenböck connection:

$$\Gamma^\alpha{}_{\mu\nu} = h_a{}^\alpha \partial_\nu h^a{}_\mu \quad (20)$$

and torsion of the Weitzenböck connection:

$$T^\mu{}_{\nu\lambda} = \Gamma^\mu{}_{\lambda\nu} - \Gamma^\mu{}_{\nu\lambda}. \quad (21)$$

The energy-momentum complexes of Einstein, Bergmann–Thomson and Landau–Lifshitz in the tele-parallel gravity are given by the following equations, respectively:

$$P^\text{E}_\mu = \int_\Sigma h {}^{\text{TG}}E^0_\mu dx dy dz, \quad (22)$$

$$P^\text{B}_\mu = \int_\Sigma h {}^{\text{TG}}B^0_\mu dx dy dz, \quad (23)$$

$$P^\text{L}_\mu = \int_\Sigma h {}^{\text{TG}}L^0_\mu dx dy dz. \quad (24)$$

P_μ is called the momentum four-vector, P_i give momentum components P_1 , P_2 , P_3 and P_0 gives the energy and the integration hyper-surface Σ is described by $x^0 = t = \text{constant}$.

4. Some energy and momentum of the domain wall in GR and TG

4.1. Bergmann–Thomson, Einstein and Landau–Lifshitz's energies and momentum of the domain wall in GR

We can write the domain wall metric [1] as

$$ds^2 = -(1 - \kappa|x|)^2 dt^2 + dx^2 + (1 - \kappa|x|)^2 e^{2\kappa t} [dy^2 + dz^2], \quad (25)$$

where $\kappa = 2\pi G\sigma$ and σ is the domain wall tension. From Eqs. (2) and (3), we have obtained the required components of $\Pi^{\mu\nu\beta}$ for the line element (Eq. (25)) to get Bergmann–Thomson energy and momentum of domain wall

$$\begin{aligned} \Pi^{001} &= 4\kappa e^{2\kappa t}, \\ \Pi^{010} &= -4\kappa e^{2\kappa t}. \end{aligned} \quad (26)$$

Using the above component in Eq. (1), we obtain Bergmann–Thomson's energy and momentum of domain wall in GR as follows

$$\begin{aligned} {}^{\text{GR}}B^{00} &= 0, \\ {}^{\text{GR}}B^{10} &= -\frac{1}{4\pi}\kappa^2 e^{2\kappa t}, \\ {}^{\text{GR}}B^{20} &= {}^{\text{GR}}B^{30} = 0. \end{aligned} \quad (27)$$

It is obtained the required components of $H_\mu^{\nu\alpha}$ for the line element (Eq. (25)) to get Einstein energy and momentum of domain wall

$$\begin{aligned} H_0^{01} &= -4\kappa e^{2\kappa t}(1 - \kappa|x|)^2, \\ H_0^{10} &= 4\kappa e^{2\kappa t}(1 - \kappa|x|)^2. \end{aligned} \quad (28)$$

Substituting the Eq. (28) in Eq. (6), we obtain Einstein's energy and momentum of domain wall in GR as follows

$$\begin{aligned} {}^{\text{GR}}E^{00} &= -\frac{1}{2\pi}\kappa^2 e^{2\kappa t}(1 - \kappa|x|)^{-1}, \\ {}^{\text{GR}}E^{10} &= -\frac{1}{4\pi}\kappa^2 e^{2\kappa t}, \\ {}^{\text{GR}}E^{20} &= {}^{\text{GR}}E^{30} = 0. \end{aligned} \quad (29)$$

From Eq. (11), we get the required components of $S^{\mu\nu\alpha\beta}$ for the line element (Eq. (25)) to obtain Landau–Lifshitz energy and momentum of domain wall

$$\begin{aligned}
S^{0101} &= -e^{4\kappa t}(1 - \kappa|x|)^4, \\
S^{0202} &= S^{0303} = -e^{2\kappa t}(1 - \kappa|x|)^2, \\
S^{1001} &= e^{4\kappa t}(1 - \kappa|x|)^4, \\
S^{2002} &= S^{3003} = e^{2\kappa t}(1 - \kappa|x|)^2.
\end{aligned} \tag{30}$$

Using the Eq. (30) in Eq. (10), we obtain Landau–Lifshitz’s energy and momentum of domain wall in GR as follows

$$\begin{aligned}
{}^{\text{GR}}L^{00} &= -\frac{3}{4\pi}\kappa^2 e^{4\kappa t}(1 - \kappa|x|)^2, \\
{}^{\text{GR}}L^{10} &= -\frac{1}{\pi}\kappa^2 e^{4\kappa t}(1 - \kappa|x|)^3, \\
{}^{\text{GR}}L^{20} &= {}^{\text{GR}}L^{30} = 0.
\end{aligned} \tag{31}$$

4.2. Bergmann–Thomson, Einstein and Landau–Lifshitz’s energy and momentum of the domain wall in TG

If we would like to calculate Einstein, Bergmann–Thomson and Landau–Lifshitz’s energy and momentum distributions of the metric in TG, it is needed to calculate tetrad components of the line element (Eq. (25)). So, we obtain the tetrad components (h_i^μ) of line element (Eq. (25)) as

$$\begin{bmatrix} (1 - \kappa|x|) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (1 - \kappa|x|)e^{\kappa t} & 0 \\ 0 & 0 & 0 & (1 - \kappa|x|)e^{\kappa t} \end{bmatrix}. \tag{32}$$

From Eqs. (20) and (32), we have obtained the components of Weitzenböck connections as

$$\begin{aligned}
\Gamma_{21}^2 &= \Gamma_{31}^3 = \Gamma_{01}^0 = -\frac{\kappa}{(1 - \kappa|x|)}, \\
\Gamma_{20}^2 &= \Gamma_{30}^3 = \kappa.
\end{aligned} \tag{33}$$

Using the Eq. (33) in Eq. (21), we have calculated the torsion of the Weitzenböck connections as follows

$$\begin{aligned}
T_{12}^2 &= T_{13}^3 = T_{10}^0 = -T_{21}^2 = -T_{31}^3 = -T_{01}^0 = -\frac{\kappa}{(1 - \kappa|x|)}, \\
T_{02}^2 &= T_{03}^3 = -T_{20}^2 = -T_{30}^3 = \kappa.
\end{aligned} \tag{34}$$

Using the Eqs. (17)–(21) and Eqs. (32)–(34), the required components of Freus's super-potential are obtained as

$$\begin{aligned} U_1^{10} &= -U_1^{01} = 2U_2^{20} = -2U_2^{02} = 2U_3^{30} = -2U_3^{03} = -\kappa e^{2\kappa t}(1 - \kappa|x|), \\ U_2^{12} &= -U_2^{21} = U_3^{13} = -U_3^{31} = 2U_0^{10} = -U_0^{01} = \kappa e^{2\kappa t}(1 - \kappa|x|)^2. \end{aligned} \quad (35)$$

Substituting the Eq. (35) in Eqs. (14)–(16), the Einstein, Bergmann–Thomson and Landau–Lifshitz's energy and momentum densities are obtained, as follows in TG, respectively:

$$\begin{aligned} h^{\text{TG}} E^{00} &= -\frac{1}{2\pi} \kappa^2 e^{2\kappa t} (1 - \kappa|x|)^{-1}, \\ h E^{10} &= -\frac{1}{4\pi} \kappa^2 e^{2\kappa t}, \\ h^{\text{TG}} E^{20} &= h^{\text{TG}} E^{30} = 0, \end{aligned} \quad (36)$$

$$\begin{aligned} h^{\text{TG}} B^{10} &= -\frac{1}{4\pi} \kappa^2 e^{2\kappa t}, \\ h^{\text{TG}} B^{00} &= h^{\text{TG}} B^{20} = h^{\text{TG}} B^{30} = 0 \end{aligned} \quad (37)$$

and

$$\begin{aligned} h^{\text{TG}} L^{00} &= -\frac{3}{4\pi} \kappa^2 e^{4\kappa t} (1 - \kappa|x|)^2, \\ h^{\text{TG}} L^{10} &= -\frac{1}{\pi} \kappa^2 e^{4\kappa t} (1 - \kappa|x|)^3, \\ h^{\text{TG}} L^{20} &= h^{\text{TG}} L^{30} = 0. \end{aligned} \quad (38)$$

5. Some energy and momentum of the string in GR and TG

5.1. Bergmann–Thomson, Einstein and Landau–Lifshitz's energy and momentum of the string in GR

We can write the string metric [1] as

$$ds^2 = -dt^2 + dz^2 + dr^2 + (1 - 8G\mu)r^2 d\theta^2, \quad (39)$$

where $G\mu$ is the dimensionless parameter and plays an important role in physics of cosmic strings. Its magnitude can be estimated the form of $G\mu \sim (\frac{\eta}{m_{\text{Pl}}})^2$, here η and m_{Pl} are the string symmetry breaking scale and Planck mass, respectively.

It is well known that the energy-momentum complexes give meaningful result if calculations are performed in quasi-Cartesian coordinates. The line element Eq. (39) may be transformed to quasi-Cartesian coordinates:

$$ds^2 = - dt^2 + \left\{ \frac{x^2 + y^2 a^2}{x^2 + y^2} \right\} dx^2 + \left\{ \frac{y^2 + x^2 a^2}{x^2 + y^2} \right\} dy^2 \\ + dz^2 - \left\{ \frac{2xy(a^2 - 1)}{x^2 + y^2} \right\} dx dy, \quad (40)$$

where $a = \sqrt{1 - 8G\mu}$ and the coordinates r, θ, z, t in Eq. (39) and x, y, z, t are related through

$$r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1} \left(\frac{y}{x} \right), \quad z = z, \quad t = t. \quad (41)$$

From Eqs. (2) and (3) we get the required components of $\Pi^{\mu\nu\beta}$ for the line element (Eq. (40)) to obtain Bergmann–Thomson energy and momentum of string

$$\Pi^{001} = - \frac{x(a^2 - 1)}{a(x^2 + y^2)}, \\ \Pi^{002} = - \frac{y(a^2 - 1)}{a(x^2 + y^2)}. \quad (42)$$

Using the above component in Eq. (1), we obtain Bergmann–Thomson's energy and momentum of string in GR as follows

$${}^{\text{GR}}B^{00} = {}^{\text{GR}}B^{10} = {}^{\text{GR}}B^{20} = {}^{\text{GR}}B^{30} = 0. \quad (43)$$

We have obtained the required components of $H_\mu^{\nu\alpha}$ for the line element (Eq. (40)) by using Eq. (7) to get Einstein energy and momentum of string

$$H_0^{01} = \frac{x(a^2 - 1)}{a(x^2 + y^2)}, \\ H_0^{02} = \frac{y(a^2 - 1)}{a(x^2 + y^2)}. \quad (44)$$

Using the Eq. (44) in Eq. (6), we obtain Einstein's energy and momentum of string in GR as follows

$${}^{\text{GR}}E^{00} = {}^{\text{GR}}E^{10} = {}^{\text{GR}}E^{20} = {}^{\text{GR}}E^{30} = 0. \quad (45)$$

From Eq. (11), we get the required components of $S^{\mu\nu\alpha\beta}$ for the line element (Eq.(40)) to obtain Landau–Lifshitz energy and momentum of string

$$\begin{aligned} S^{0101} &= -S^{1001} = -\frac{a^2x^2 + y^2}{x^2 + y^2}, \\ S^{0202} &= -S^{2002} = -\frac{a^2y^2 + x^2}{x^2 + y^2}, \\ S^{0201} &= -S^{1002} = -S^{2001} = -\frac{xy(a^2 - 1)}{x^2 + y^2}, \\ S^{0303} &= -S^{3003} = -a^2. \end{aligned} \quad (46)$$

Using the Eq. (46) in Eq. (10), we obtain Landau–Lifshitz's energy and momentum of string in GR as follows

$${}^{\text{GR}}L^{00} = {}^{\text{GR}}L^{10} = {}^{\text{GR}}L^{20} = {}^{\text{GR}}L^{30} = 0. \quad (47)$$

5.2. Bergmann–Thomson, Einstein and Landau–Lifshitz's energy and momentum of the string in TG

If we would like to calculate Einstein, Bergmann–Thomson and Landau–Lifshitz's energy and momentum distributions of the metric in TG, it is needed to calculate tetrad components of the line element (Eq. (40)). So, we obtain the tetrad components (h_i^μ) of line element (Eq. (40)) as

$$\begin{bmatrix} a\sqrt{\frac{x^2+y^2}{a^2x^2+y^2}} & 0 & 0 & 0 \\ \frac{xy(a^2-1)}{\sqrt{(y^2+a^2x^2)(x^2+y^2)}} & -\sqrt{\frac{a^2x^2+y^2}{x^2+y^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (48)$$

From Eqs. (20) and (48), we obtained the components of Weitzenböck connections of string as

$$\begin{aligned} \Gamma_{11}^1 &= -\Gamma_{11}^1 = -\frac{xy^2(a^2 - 1)}{(x^2 + y^2)(y^2 + a^2x^2)}, \\ \Gamma_{12}^1 &= -\Gamma_{22}^2 = \frac{x^2y(a^2 - 1)}{(x^2 + y^2)(y^2 + a^2x^2)}, \\ \Gamma_{11}^2 &= -\frac{y(y^2 - x^2)(a^2 - 1)}{(x^2 + y^2)(y^2 + a^2x^2)}, \\ \Gamma_{12}^2 &= \frac{x(y^2 - x^2)(a^2 - 1)}{(x^2 + y^2)(y^2 + a^2x^2)}. \end{aligned} \quad (49)$$

Using the Eq. (49) in Eq. (21), we have calculated the torsion of the Weitzenböck connections as follows

$$\begin{aligned} T_{12}^1 &= -T_{21}^1 = -\frac{x^2 y (a^2 - 1)}{(y^2 + a^2 x^2)(x^2 + y^2)}, \\ T_{12}^2 &= -T_{21}^2 = -\frac{x^3 (a^2 - 1)}{(x^2 + y^2)(y^2 + a^2 x^2)}. \end{aligned} \quad (50)$$

Using the Eqs. (17)–(21) and Eqs. (48)–(50), we have calculated the required components of Freus's super-potential of string as

$$\begin{aligned} U_0^{01} &= -\frac{ax^3(a^2 - 1)}{2(x^2 + y^2)(y^2 + a^2 x^2)}, \\ U_0^{02} &= -\frac{ax^2 y (a^2 - 1)}{2(x^2 + y^2)(y^2 + a^2 x^2)}. \end{aligned} \quad (51)$$

Substituting the Eq. (51) in Eqs. (14)–(16), we have calculated the Einstein, Bergmann–Thomson and Landau–Lifshitz's energy and momentum densities of string in TG, respectively:

$${}^{\text{TG}}E^{00} = {}^{\text{TG}}E^{10} = {}^{\text{TG}}E^{20} = {}^{\text{TG}}E^{30} = 0, \quad (52)$$

$${}^{\text{TG}}B^{00} = {}^{\text{TG}}B^{10} = {}^{\text{TG}}B^{20} = {}^{\text{TG}}B^{30} = 0, \quad (53)$$

and

$${}^{\text{TG}}L^{00} = {}^{\text{TG}}L^{10} = {}^{\text{TG}}L^{20} = {}^{\text{TG}}L^{30} = 0. \quad (54)$$

6. Summary and discussion

In this study we have calculated the energy-momentum distributions of domain wall and string in GR and TG by using Bergmann–Thomson, Einstein and Landau–Lifshitz definitions. Our results are briefly given for domain wall and string in Table I and II, respectively.

TABLE I

The energy and momentum densities in Bergmann–Thomson, Einstein and Landau–Lifshitz Definitions for Domain Wall in GR and TG.

In the case of domain wall, we have used different energy-momentum complexes, Bergmann–Thomson, Einstein and Landau–Lifshitz, to calculate energy-momentum of domain wall. We have found that these definitions do not provide the same energy and momentum densities in GR. We were hoping that the theory of tele-parallel gravity would solve this problem. Unfortunately, these definitions do not provide also the same results for the energy and momentum densities of domain wall. Because, although we get that the total energy density is vanishing everywhere for domain wall in GR and TG by using Bergmann–Thomson definition, we find the same energy distributions which are different from zero for domain wall in GR and TG by using Einstein and Landau–Lifshitz definitions.

Prescription	Energy-Momentums in GR	Energy-Momentums in TG
Bergmann–Thomson	${}^{\text{GR}}B^{00}=0$	$h^{\text{TG}}B^{00}=0$
	${}^{\text{GR}}B^{10}=-\frac{1}{4\pi}\kappa^2e^{2\kappa t}$	$h^{\text{TG}}B^{10}=-\frac{1}{4\pi}\kappa^2e^{2\kappa t}$
	${}^{\text{GR}}B^{20}={}^{\text{GR}}B^{30}=0$	$h^{\text{TG}}B^{20}=h^{\text{TG}}B^{30}=0$
Einstein	${}^{\text{GR}}E^{00}=-\frac{1}{2\pi}\kappa^2e^{2\kappa t}(1-\kappa x)^{-1}$	$h^{\text{TG}}E^{00}=-\frac{1}{2\pi}\kappa^2e^{2\kappa t}(1-\kappa x)^{-1}$
	${}^{\text{GR}}E^{10}=-\frac{1}{4\pi}\kappa^2e^{2\kappa t}$	$h^{\text{TG}}E^{10}=-\frac{1}{4\pi}\kappa^2e^{2\kappa t}$
	${}^{\text{GR}}E^{20}={}^{\text{GR}}E^{30}=0$	$h^{\text{TG}}E^{20}=h^{\text{TG}}E^{30}=0$
Landau–Lifshitz	${}^{\text{GR}}L^{00}=-\frac{3}{4\pi}\kappa^2e^{4\kappa t}(1-\kappa x)^2$	$h^{\text{TG}}L^{00}=-\frac{3}{4\pi}\kappa^2e^{4\kappa t}(1-\kappa x)^2$
	${}^{\text{GR}}L^{10}=-\frac{1}{\pi}\kappa^2e^{4\kappa t}(1-\kappa x)^3$	$h^{\text{TG}}L^{10}=-\frac{1}{\pi}\kappa^2e^{4\kappa t}(1-\kappa x)^3$
	${}^{\text{GR}}L^{20}={}^{\text{GR}}L^{30}=0$	$h^{\text{TG}}L^{20}=h^{\text{TG}}L^{30}=0$

TABLE II

The energy and momentum densities in Bergmann–Thomson, Einstein and Landau–Lifshitz Definitions for String in GR and TG.

In the string case, although the energy and momentum definitions of Bergmann–Thomson, Einstein and Landau–Lifshitz are different in GR and TG, we obtained that energy and momentum distributions of string are the same and zero in these different gravitation theories. The results that the total energy and momentum densities are vanishing everywhere for strings in GR and TG elucidate the importance of the energy-momentum definitions. Since string forms two-surface (it means that the congruences u^a and x^a are two-surface forming), the energy contributions from matter and gravitational field inside an arbitrary two-surface cancel each other [28]. Also, the result that although the energy and momentum definitions of Bergmann–Thomson, Einstein and Landau–Lifshitz are different in GR and TG, energy distributions are the same for strings in these different gravitation theories elucidates an important point. Because, it is independent of the tele-parallel dimensionless coupling constant, which means that it is valid not only in tele-parallel equivalent of general relativity but also in any tele-parallel model.

Prescription	Energy-Momentums in GR	Energy-Momentums in TG
Bergmann–Thomson	${}^{\text{GR}}B^{00}=0$ ${}^{\text{GR}}B^{10} = {}^{\text{GR}}B^{20} = {}^{\text{GR}}B^{30} = 0$	${}^{\text{TG}}B^{00}=0$ ${}^{\text{TG}}B^{10} = {}^{\text{TG}}B^{20} = {}^{\text{TG}}B^{30} = 0$
Einstein	${}^{\text{GR}}E^{00}=0$ ${}^{\text{GR}}E^{10} = {}^{\text{GR}}E^{20} = {}^{\text{GR}}E^{30} = 0$	${}^{\text{TG}}E^{00}=0$ ${}^{\text{TG}}E^{10} = {}^{\text{TG}}E^{20} = {}^{\text{TG}}E^{30} = 0$
Landau–Lifshitz	${}^{\text{GR}}L^{00}=0$ ${}^{\text{GR}}L^{10} = {}^{\text{GR}}L^{20} = {}^{\text{GR}}L^{30} = 0$	${}^{\text{TG}}L^{00}=0$ ${}^{\text{TG}}L^{10} = {}^{\text{TG}}L^{20} = {}^{\text{TG}}L^{30} = 0$

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