# A CLASS OF SPACETIMES OF NON-RIGIDLY ROTATING DUST

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We find a class of exact solutions of differentially rotating dust in the framework of General Relativity. There exist asymptotically flat spacetimes of the flow with positive mass function that for radii sufficiently large is monotone and tends to zero at infinity. Some of the spacetimes may have non-vanishing total angular momentum.

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## 1. Introduction

The paper is devoted to investigation of asymptotically flat solutions of self-gravitating stationary and cylindrically symmetric dust flow on circular orbits along trajectories of locally non-rotating observers in the framework of General Relativity. The congruence of locally non-rotating observers used to define energy-momentum exist globally. The flow is non-rigid (differential) and non-expanding. It is also purely relativistic — it has no Newtonian limit. The resulting spacetimes are globally regular, apart from internal singularities where additional sources of matter and angular momentum may be located. We shall call it the $\mathcal{K}$ flow for brevity. The main deficiency of the flow is that the proper energy density is necessarily negative definite. Despite the fact, by matching the solutions onto asymptotically flat vacuum external solutions, one would obtain spacetimes with positive total mass.

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The result of our paper is the construction of a multipole sequence of asymptotically flat, stationary, and cylindrically symmetric external solutions of $\mathcal{K}$ flow to which a broad class of other asymptotically flat solutions of $\mathcal{K}$ flow can be decomposed. We construct also the corresponding multipole sequence of internal spacetimes. There exist also a continuum of basic solutions of which we do not examine here as they are nonanalytic on the axis of rotation. The stationary part of multipolar solutions that are asymptotically flat are called external multipoles, and the stationary part of multipolar solutions that are not asymptotically flat are called internal multipoles. We show that external multipoles and any asymptotically flat spacetimes of $\mathcal{K}$ flow are massless. Despite the fact, some of the spacetimes can have non-vanishing total angular momentum.

Even though the resulting spacetimes are solutions of Einstein equations of a differentially rotating dust, thus more interesting than rigid van Stockum–Bonnor flows [1,4,5], it seems that they are of no physical concern, as far as no exotic forms of matter are considered. Physically reliable matter should have nonnegative energy density, which is not the case for $\mathcal{K}$ flow. On the other hand, it would be interesting to consider a vacuum continuation of $\mathcal{K}$ flow (we leave open the question whether such a continuation is possible), as the resulting spacetimes would have positive mass and nonzero angular momentum and would be asymptotically indistinguishable from the Kerr solution. This is the physical motivation for considering $\mathcal{K}$ flow. Moreover, $\mathcal{K}$ flow is interesting *per se* as it presents itself a class of exactly solvable dust flows in general relativity with non-trivial spacetime geometry.

It turns out that a structure function of the flow, which is defined as the scalar product of time translation and axial symmetry Killing vectors and denoted by $K$, is the same as for the van-Stockum–Bonnor flow [5], nonetheless, both the flows and geometry of the resulting spacetimes are qualitatively different. An asymptotically flat van Stockum–Bonnor flow is rigid, non-expanding, and has positive energy density proportional to the square of the vorticity scalar which does not vanish. An asymptotically flat $\mathcal{K}$ flow is differential, non-expanding, locally non-rotating, and has negative definite energy density proportional to the square of the non-vanishing shear tensor. Asymptotically flat dust flow of van Stockum–Bonnor does not rotate with respect to asymptotic observers and moves rigidly, it has a point-dependent physical velocity as measured with respect to the locally dragged inertial frames and is proportional to $K$. Dust of $\mathcal{K}$ flow rotates differentially with respect to asymptotic stationary observers and vorticity tensor of the flow is identically zero. The angular velocity is proportional to $K$ and, by construction, is identical to the angular velocity of dragging of inertial frames $\omega$. The corresponding physical velocity, which by definition is measured with respect to dragged local inertial frames, is identically zero — no angular momentum is carried by elements of $\mathcal{K}$ flow.

As an example we construct a $z$-symmetric, cylindrically symmetric, and stationary for radii sufficiently large, asymptotically flat spacetime which is globally smooth (with the exception of singularities residing on the axis of rotation), and which contains dipole momentum in its series expansion in external multipoles. Consequently, the solution has nonzero total angular momentum and zero total mass.

In what follows we shall be using notation in which two vectors placed adjacent to each other denotes a scalar product. We use units in which $c = 1 = G$.

## 2. Setup

We assume global existence of the cylindrical symmetry space-like Killing vector $\eta$ with closed field lines, together with global existence of the time translation Killing vector $\boldsymbol{\xi}$ of which field lines are opened, time-like for radii sufficiently large, and asymptotically normalizable to unity. Having in mind stationary and asymptotically flat spacetimes with cylindrical symmetry, we shall proceed as in the standard theory of rotating stars in the relativistic astrophysics [3]. The assumption of asymptotic flatness allows for the unique determination of Killing vectors $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ by the defining properties. In addition one assumes that asymptotically $\boldsymbol{\xi\eta} \to 0$. Field lines of the Killing vectors, which clearly are frame independent objects, may be viewed as two of four coordinate lines in some particular coordinates in which the time coordinate $t$ runs along open lines of $\boldsymbol{\xi}$ and the cyclic coordinate $\phi$ along closed lines of $\boldsymbol{\eta}$, that is, $\boldsymbol{\xi} \equiv \boldsymbol{\partial}_t$ and $\boldsymbol{\eta} \equiv \boldsymbol{\partial}_\phi$ by definition. The other two, denoted by $\tilde{\rho}$ and $\tilde{z}$, are arbitrary internal coordinates in a two dimensional subspace orthogonal to $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$. In this coordinates the most general line element of a cylindrically symmetric and stationary spacetime is fully determined by four structure functions $\lambda(\tilde{\rho}, \tilde{z})$, $\psi(\tilde{\rho}, \tilde{z})$, $\omega(\tilde{\rho}, \tilde{z})$ and $\tilde{\mu}(\tilde{\rho}, \tilde{z})$ and reads [3]

$$ds^2 = e^{2\lambda} dt^2 - e^{2\psi} \left(d\phi - \omega dt\right)^2 - e^{2\tilde{\mu}} \left(d\tilde{\rho}^2 + d\tilde{z}^2\right) . \tag{2.1}$$

The particular coordinates, which we shall call Bardeen coordinates, are distinguished by the property that in these coordinates Killing vectors $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ attain particularly simple form

$$\xi^\mu = \delta^\mu_t, \qquad \eta^\mu = \delta^\mu_\phi .$$

In asymptotically flat spacetime one can introduce cylindrical coordinate system in which the line element at infinity reduces to

$$ds^2 \to dt^2 - d\rho^2 - \rho^2 d\phi^2 - dz^2 .$$

Axes of the coordinate frame are attached to 'fixed stars', otherwise the axes would rotate. This in turn would be in contradiction with the assumption that $\boldsymbol{\partial}_t$ can be asymptotically normalised to unity. One assumes, therefore, that asymptotically the condition that $e^{2\lambda} - \omega^2 e^{2\psi} > 0$ should hold. The latter is not the case *e.g.* in a uniformly rotating frame of reference. Put differently, Bardeen coordinates are asymptotically inertial (anyway, this metric can be treated more formally and be used to describe solutions that are not asymptotically flat).

### *2.1. The line element of $\mathcal{K}$ flow*

We derive the line element of the $\mathcal{K}$ flow step by step from the general form of the line element given in Eq. (2.1). We specify the structure functions uniquely by the requirement that dust spacetime trajectories are identical to field lines of the four-velocity field of locally non-rotating observers and that Einstein equations are satisfied.

#### 2.1.1. Locally non-rotating observers

It should be clear that irrespectively of any particular reference frame, a generic cylindrically symmetric and stationary flow on circular orbits is fully determined by a four-velocity field

$$\boldsymbol{U}(\Omega) = Z(\Omega)(\boldsymbol{\xi} + \Omega\boldsymbol{\eta}),$$

provided

$$Z(\Omega)^{-2} \equiv \boldsymbol{\xi\xi} + 2\Omega\boldsymbol{\xi\eta} + \Omega^2\boldsymbol{\eta\eta} > 0, \quad \pounds_{\boldsymbol{\xi}}\Omega = 0 = \pounds_{\boldsymbol{\eta}}\Omega .$$

In an asymptotically flat spacetime $\Omega$ has the interpretation of the angular velocity measured with respect to 'fixed stars'. In particular, the condition $\boldsymbol{U}(\Omega)\boldsymbol{\eta} = 0$ gives

$$\Omega = -\frac{\boldsymbol{\xi\eta}}{\boldsymbol{\eta\eta}} \equiv \omega .$$

The vorticity tensor[1] of velocity field $\boldsymbol{n} \equiv \boldsymbol{U}(\omega)$ vanishes identically, which can be verified by direct calculation in Bardeen coordinates (2.1). Therefore, the observers are called locally non-rotating, although they move differentially on circular orbits with respect to 'fixed stars' with angular velocity $\omega$ of dragging of inertial frames. The congruence of locally non-rotating

[1] See footnote 2.

observers distorts without changing proper volume[2]

$$\boldsymbol{\Omega}^2(\boldsymbol{n}) \equiv 0, \quad \boldsymbol{\Theta}(\boldsymbol{n}) \equiv 0, \quad \boldsymbol{\sigma}^2(\boldsymbol{n}) = -\frac{1}{4}\frac{(\boldsymbol{\eta\eta})^2(\boldsymbol{\nabla}\omega)^2}{\begin{vmatrix} \boldsymbol{\xi\eta} & \boldsymbol{\xi\xi} \\ \boldsymbol{\eta\eta} & \boldsymbol{\xi\eta} \end{vmatrix}}, \quad \omega = -\frac{\boldsymbol{\xi\eta}}{\boldsymbol{\eta\eta}}.$$

From the construction it follows that dust in asymptotically flat $\mathcal{K}$ flow moves differentially on circular orbits with angular velocity of dragging of inertial frames $\omega$ with respect to an asymptotic stationary observer. This flow has vanishing vorticity vector, that is, it does not rotate locally, and has vanishing physical velocity — it is at rest with respect to congruence of local standards of rest dragged with angular velocity $\omega$. It follows also that the geometric angular momentum of the flow equal to $-\boldsymbol{n\eta}$ is identically zero as $\boldsymbol{n}$ and $\boldsymbol{\eta}$ are orthogonal. Nevertheless, as we shall see later, total angular momentum of the flow is nonzero for some asymptotically flat spacetimes of $\mathcal{K}$ flow. This signalises the particular spacetimes must contain singularities that are sources of the angular momentum. We shall clarify this later.

#### 2.1.2. Specifying the line element

The energy-momentum tensor of dust matter moving along field lines of vector field $\boldsymbol{n}$ reads $\boldsymbol{T} = \mathcal{D}\boldsymbol{n} \otimes \boldsymbol{n}$, where

$$\boldsymbol{n}(\omega) = Z(\omega)(\boldsymbol{\xi} + \omega\boldsymbol{\eta}) \qquad \Rightarrow \qquad n^\mu = e^{-\lambda}[1, 0, \omega, 0] .$$

We stress the construction of the energy-momentum tensor of $\mathcal{K}$ flow makes sense globally as $\boldsymbol{n}$ is everywhere time-like $\boldsymbol{nn} = \boldsymbol{\xi\xi} + 2\omega\boldsymbol{\xi\eta} + \omega^2\boldsymbol{\eta\eta} \equiv 1 > 0$.

Einstein's equations $G_{\mu\nu} = 8\pi T_{\mu\nu}$ and the contracted Bianchi identities $\nabla^\mu G_{\mu\nu} = 0$ yield local conservation law $\nabla^\mu T_{\mu\nu} = 0$. In the particular case of dust with a four-velocity $U^\mu$ the latter gives continuous flow along geodesic paths

$$\nabla_\mu(\mathcal{D}U^\mu) = 0 \qquad \text{and} \qquad U^\nu \nabla_\nu U_\mu = 0 .$$

The continuity equation is satisfied identically for $\mathcal{K}$ flow on the power of Killing equations and of symmetries of energy density $\boldsymbol{\xi}\mathcal{D} = 0$ and $\boldsymbol{\eta}\mathcal{D} = 0$. In Bardeen coordinates $n^\nu \nabla_\nu n_\mu = \{0, -\partial_{\tilde{\rho}}\lambda, 0, -\partial_{\tilde{z}}\lambda\}$ which implies that $\lambda$ must be constant and we may set $\lambda = 0$.

[2] Dilation tensor $\boldsymbol{\Theta}(\boldsymbol{u})$ and (traceless) shear tensor $\boldsymbol{\sigma}(\boldsymbol{u})$ are defined for a velocity field $\boldsymbol{u}$ as $\Theta_{\mu\nu} = \nabla_\alpha u^\alpha h_{\mu\nu}$ and $\sigma_{\mu\nu} = \nabla_{(\alpha}u_{\beta)}h^\alpha{}_\mu h^\beta{}_\nu - \frac{1}{3}\nabla_\alpha u^\alpha h_{\mu\nu}$, where $h^\mu{}_\nu = \delta^\mu{}_\nu - u^\mu u_\nu$ is a projector $\boldsymbol{h}(\boldsymbol{u})$ onto the subspace orthogonal to $\boldsymbol{u}$. A vorticity tensor $\boldsymbol{\Omega}(\boldsymbol{u})$ of $\boldsymbol{u}$ is defined as $\omega_{\mu\nu} = \nabla_{[\alpha}u_{\beta]}h^\alpha{}_\mu h^\beta{}_\nu$ and it yields a derivative quantity that characterises vertex — the vorticity vector $\omega^\mu = \frac{1}{2}\frac{\varepsilon^{\mu\alpha\beta\gamma}}{\sqrt{-g}}u_\alpha\omega_{\beta\gamma}$. One defines also the square of dilation scalar $\boldsymbol{\sigma}^2(\boldsymbol{u}) = \frac{1}{2}\sigma^{\mu\nu}\sigma_{\mu\nu} \geq 0$ and the square of vorticity scalar $\boldsymbol{\Omega}^2(\boldsymbol{u}) = \frac{1}{2}\omega^{\mu\nu}\omega_{\mu\nu} \geq 0$, then $\boldsymbol{\Omega}^2(\boldsymbol{u}) = -\omega^\mu\omega_\mu$.

The existence of vectors $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$ (and so $\boldsymbol{n}$) allows for rewriting linear combinations of Einstein equations in a frame independent way as scalar identities. Let $\widetilde{T}_{\mu}{}^{\nu} = T_{\mu}{}^{\nu} - \frac{1}{2}T\delta_{\mu}{}^{\nu}$. Then for dust matter the scalar $R_{\mu\nu}X^{\mu}Y^{\mu} = 8\pi\widetilde{T}_{\mu\nu}X^{\mu}Y^{\mu}$ must be either zero or proportional to $\mathcal{D}$ for any vectors $\boldsymbol{X}$, $\boldsymbol{Y}$. By taking $(\boldsymbol{n},\boldsymbol{n})$ , $(\boldsymbol{\eta},\boldsymbol{n})$ and $(\boldsymbol{\eta},\boldsymbol{\eta})$ in place of $(\boldsymbol{X},\boldsymbol{Y})$ and by rearranging we obtain, respectively,

$$8\pi\mathcal{D} = -e^{2\psi}\widetilde{\nabla}_j\omega\widetilde{\nabla}^j\omega \tag{2.2}$$

$$0 = \widetilde{\nabla}_j\left(e^{3\psi}\widetilde{\nabla}^j\omega\right) \tag{2.3}$$

$$8\pi\mathcal{D} = -e^{2\psi}\widetilde{\nabla}_j\omega\widetilde{\nabla}^j\omega - 2e^{-\psi}\widetilde{\nabla}_j\widetilde{\nabla}^j e^{\psi}\,, \tag{2.4}$$

where $\widetilde{\nabla}_j$ is the covariant derivative operator in the 2-dimensional subspace $\mathcal{S}$ orthogonal to $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$.

As follows from Eq. (2.2) energy density is negative definite which is a generic feature of $\mathcal{K}$ flow. As we have seen this is a direct consequence of Einstein's equations and Bianchi identities applied to $\mathcal{K}$ flow. This fact is astonishing since for $\Omega = 0$ we would get van Stockum flow [1] with positive definite density while in astrophysical situations where $|\Omega| \gg |\omega|$ one would expect positive density, as well. This signalises that a generic flow on circular orbits in $GR$ is not structurally stable, it may change qualitatively for a family of solutions, say, $\Omega_\alpha = \alpha\omega$.

By subtracting (2.4) from (2.2) we obtain another constraint on structure functions

$$\widetilde{\nabla}_j\widetilde{\nabla}^j e^{\psi} = 0\,, \tag{2.5}$$

thus $e^{\psi}$ is a harmonic function on $\mathcal{S}$. Note, that the equation is valid only for pressure-free flow. By choosing any particular solution $\psi$ we define geometry and topology of the final spacetime. For example, in asymptotic flatness $\psi$ must not be bounded from above while for $\psi$ bounded from below circular orbits of $\mathcal{K}$ flow could not have arbitrarily small radii. On the other hand it seems this choice affects $\mathcal{K}$ flow spacetime's structure not entirely as shown by the construction which we shall come to in later. In any case, irrespectively of any particular choice of $\psi$, regular regions of the spacetimes must be filled with the nonphysical negative density matter.

We have not yet specified the arbitrary internal coordinates $\tilde{\rho}$ and $\tilde{z}$ on $\mathcal{S}$. Motivated by asymptotic flatness in cylindrical coordinates in which $e^{2\psi} \to \rho^2$ for radii sufficiently large, we can specify the unknown 'radial' coordinate by identifying it with $e^{\psi}$ which we can always do (at least in the vicinity of some point in $\mathcal{S}$), namely $\rho \equiv e^{\psi}$. The other coordinate can be chosen to be the conjugate harmonic function which we shall denote by $z$. In these new coordinates on $\mathcal{S}$ the line element reads $e^{2\mu}\left(d\rho^2 + dz^2\right)$, where $\mu(\rho, z)$ is to be specified yet. The existence of local coordinates on a two dimensional surface in which the line element assumes this symmetric form was proved by Gauss. Note, that locally this construction is independent on the particular choice of $\psi$ if only it is harmonic. The analysis of metrical properties of final spacetime could shed some light on the nature of $\psi$ (or a class of $\psi$'s) we have chosen implicitly by the above construction.

We have thus shown that $\mathcal{K}$ flow requires at most two structure functions $\omega$ and $\mu$. As $\boldsymbol{\xi\xi} = 1 - \omega^2\rho^2$ the resulting spacetime is stationary in regions where $\omega^2\rho^2 < 1$. Note that $\det[g] = -\rho^2 e^{4\mu}$ does not depend on $\omega$ explicitly, thus the boundary surface $\omega^2\rho^2 = 1$ which would demarcate regions in which $\boldsymbol{\xi}$ is time-like and space-like, is nonsingular. Killing vector $\boldsymbol{\eta}$ is always space-like $\boldsymbol{\eta\eta} \equiv -\rho^2$, thus the spacetime of $\mathcal{K}$ flow is globally cylindrically symmetric.

Let's define $K = \boldsymbol{\xi\eta}$, then in Bardeen coordinates $K = \rho^2\omega$, and $(K, \mu)$ can be used in place of $(\omega, \mu)$ as structure functions of $\mathcal{K}$ flow. Then the line element (2.1) reduces to

$$ds^2 = \left(1 - \frac{K^2(\rho,z)}{\rho^2}\right) dt^2 + 2K(\rho,z)dtd\phi - \rho^2 d\phi^2 - e^{2\mu(\rho,z)}\left(d\rho^2 + dz^2\right) . \quad (2.6)$$

We remind that in spacetimes described by the line element, world-lines of $\mathcal{K}$ flow are geodesics.

#### 2.1.3. Determining the other structure functions of $\mathcal{K}$ flow

We have shown that for $\mathcal{K}$ flow the general metric (2.1) can be reduced to (2.6). In what follows we shall derive equations for $K$ and $\mu$. Let $E_{\mu\nu} = G_{\mu\nu} - 8\pi T_{\mu\nu}$. Einstein's equations ${E^\mu}_\nu = 0$ imply from

$${E^\rho}_\rho = \frac{e^{-2\mu}}{\rho}\left(\rho^3\frac{\omega_{,z}^2 - \omega_{,\rho}^2}{4} - \mu_{,\rho}\right), \qquad {E^\rho}_z = -\frac{e^{-2\mu}}{\rho}\left(\rho^3\frac{\omega_{,\rho}\omega_{,z}}{2} + \mu_{,z}\right)$$

$({E^\rho}_\rho = -{E^z}_z,\ {E^\rho}_z = {E^z}_\rho)$ that

$$\mu_{,\rho} = \rho^3\frac{\omega_{,z}^2 - \omega_{,\rho}^2}{4}, \qquad \mu_{,z} = -\rho^3\frac{\omega_{,\rho}\omega_{,z}}{2} . \quad (2.7)$$

For $\mathcal{C}^2$ solutions the Schwarz identity $\mu_{,\rho z} = \mu_{,z\rho}$ imposes on $\omega$ the linear elliptic constraint

$$\omega_{,\rho\rho} + 3\rho^{-1}\omega_{,\rho} + \omega_{,zz} = 0 \quad \Leftrightarrow \quad K_{,\rho\rho} - \frac{K_{,\rho}}{\rho} + K_{,zz} = 0 . \quad (2.8)$$

As

$${E^t}_\phi = \frac{e^{-2\mu}}{2}\left(K_{,\rho\rho} - \frac{K_{,\rho}}{\rho} + K_{,zz}\right)$$

equation ${E^t}_\phi = 0$ is satisfied identically. The constraint is a particular case of Eq. (2.3). By calculating $\mu_{,\rho\rho}$ and $\mu_{,zz}$ from (2.7) and using (2.8), one may check that the component ${E^t}_t$ (then ${E^\phi}_t = K\rho^{-2}{E^t}_t$) reduces to ${E^t}_t = -e^{-2\mu}\rho^2\left(\omega_{,\rho}^2 + \omega_{,z}^2\right) - 8\pi\mathcal{D} = 0$ which is equivalent to (2.2), that is

$$8\pi\mathcal{D} = -e^{-2\mu}\rho^2\left(\omega_{,\rho}^2 + \omega_{,z}^2\right) . \tag{2.9}$$

in regular regions of $\mathcal{K}$ flow. The other components of $\boldsymbol{E}$ vanish identically by symmetry. As energy density is negative definite, $\mathcal{K}$ flow is nonphysical in the sense it cannot be made of ordinary matter. Once a solution of (2.8) is found, which is a simple task due to linearity, (2.9) gives the respective energy density and (2.7) can be easily integrated.

## 3. Some solutions

To obtain basic solutions we transform (2.8) to spherical coordinates $\rho \to r\sin\theta$, $z \to r\cos\theta$. By substituting $K(r,\theta) = R(r)Y(\theta)$, the separation of variables gives $R(r) = r^{-n}$ or $R(r) = r^{n+1}$, and the hypergeometric equation for $Y(x)$, where $x = \cos\theta$. We assume here $n \in \mathbb{N}$ and take only solutions that are analytic at $x = \pm 1$, by which the multipole series is established. The general formula for $z$-antisymmetric and $z$-symmetric external multipoles that are solutions of (2.8) and that give asymptotically flat spacetimes is

$$K(\rho,z) = \begin{cases} \frac{1}{(\rho^2+z^2)^{m-1/2}}\, {}_2F_1\left(m-\frac{1}{2}, -m; \frac{1}{2}, \frac{z^2}{\rho^2+z^2}\right), & m=1,2,3,\ldots \\ \frac{z}{(\rho^2+z^2)^{m+1/2}}\, {}_2F_1\left(m+\frac{1}{2}, -m; \frac{3}{2}, \frac{z^2}{\rho^2+z^2}\right), & m=1,2,3,\ldots \end{cases} \tag{3.1}$$

For $m = 0$ one obtains $K = \sqrt{\rho^2 + z^2}$ and the monopole $K = z/\sqrt{\rho^2 + z^2}$ which are asymptotically non-flat. The corresponding series of internal multipoles (singular at infinity) is

$$K(\rho,z) = \begin{cases} \left(\rho^2+z^2\right)^m\, {}_2F_1\left(m-\frac{1}{2}, -m; \frac{1}{2}, \frac{z^2}{\rho^2+z^2}\right), & m=0,1,2,\ldots \\ z\left(\rho^2+z^2\right)^m\, {}_2F_1\left(m+\frac{1}{2}, -m; \frac{3}{2}, \frac{z^2}{\rho^2+z^2}\right), & m=0,1,2,\ldots \end{cases} \tag{3.2}$$

### *3.1. An example*

As an example we construct a $z$-symmetric and bounded $K$ which gives an asymptotically flat spacetime. Such a solution can be found by calculating the integral $K_a(\rho,z) = -a^{-1}\int s ds \tilde{K}(\rho, z-s)$, $s \in (-a,a)$ where $\tilde{K}(\rho,z)$ is the monopole solution $z/\sqrt{\rho^2+z^2}$ ($|\tilde{K}| \le 1$), hence

$$K_a(\rho,z)= \frac{2\rho^2 z}{(z+a)\sqrt{\rho^2+(z-a)^2}+(z-a)\sqrt{\rho^2+(z+a)^2}}+\ldots$$
$$\ldots+\frac{\rho^2}{2a}\ln\left(\frac{z-a+\sqrt{\rho^2+(z-a)^2}}{z+a+\sqrt{\rho^2+(z+a)^2}}\right). \tag{3.3}$$

The solution can be represented as a series of external $z$-symmetric multipoles

$$K_a(\rho,z)=\frac{2}{3}\frac{\rho^2}{r^3}a^2-\frac{1}{5}\frac{\rho^2\left(\rho^2-4z^2\right)}{r^7}a^4+\frac{3}{28}\frac{\rho^2\left(\rho^4-12\rho^2z^2+8z^4\right)}{r^{11}}a^6+\ldots,$$

where $r=\sqrt{\rho^2+z^2}$. Solution $K_a$ is bounded, has single extremum $K_a(0,0)=a$ and is globally continuous. The conformal mapping $z+i\rho=a\cosh(u+iv)$, invertible except for points $(\rho,z)=(0,\pm a)$, defines new coordinates in which (3.3) reads

$$K_a(u,v)=a\sin^2(v)\left(\cosh(u)+\frac{1}{2}\sinh^2(u)\ln\left[\tanh^2\left(\frac{u}{2}\right)\right]\right).$$

The function is smooth everywhere with the exception of $u=0$ at which it has singularity of the type $u^2\ln u$. Consequently, $K_a(\rho,z)$ is smooth everywhere with the exception of the segment

$$S_a=\{(\rho,z):\rho=0,z\in[-a,a]\},$$

which corresponds to $u=0$ and $v\in[0,\pi]$. The resulting spacetime is asymptotically flat

$$K_a(r,\theta)\sim\frac{2}{3}\frac{a^2}{r}\sin^2\theta,\quad \omega_a(r,\theta)\sim\frac{2}{3}\frac{a^2}{r^3}\quad \mu_a(r,\theta)\sim\frac{a^4\sin^4\theta}{4r^4},\quad r\to\infty,$$

and

$$g_{tt}\sim 1-\tfrac{4}{9}\tfrac{a^4}{r^4}\sin^2\theta,\quad g_{t\phi}\sim\tfrac{2}{3}\tfrac{a^2}{r}\sin^2\theta\left(1+\tfrac{3}{20}\tfrac{a^2}{r^2}\left(3+5\cos 2\theta\right)\right),$$
$$g_{\phi\phi}=-r^2\sin^2\theta,\quad g_{rr}=\tfrac{g_{\theta\theta}}{r^2}\sim-1-\tfrac{a^4}{2r^4}\sin^4\theta.$$

Comparison with asymptotical expansion of the Kerr metric gives total mass of the spacetime, which is $M_a=0$, and total angular momentum, which is $J_a=a^2/3$. The same can be inferred from the Papapetrou conditions which determine asymptotic behaviour of metric functions [2]. As we have noted before, the angular momentum carried by any element of $\mathcal{K}$ flow is zero which signalises that the segment $S_a$ is the source of the total angular momentum. That it is so, can be proved by analysing the differential form $\boldsymbol{\eta} = \eta_\mu dx^\mu$. If only $\mathcal{M}$ is such that $\star d\boldsymbol{\eta}$ (by $\star$ we denote the Hodge star operator[3]) is continuously differentiable inside a region $\mathcal{M}$ and continuous on its boundary $\partial\mathcal{M}$, then the Stokes theorem can be applied giving

$$\int\limits_{\partial\mathcal{M}} \star d\boldsymbol{\eta} = \int\limits_{\mathcal{M}} d\star d\boldsymbol{\eta} \qquad \text{if} \quad \mathcal{M}\cap S_a = \emptyset\,.$$

Let $\mathcal{M}$ be a 3-dimensional subspace of constant time $t$ enclosed by 2-spheres $\mathcal{S}_1$ and $\mathcal{S}_2$ of radii $r_1$ and $r_2$, respectively, such that $a < r_1 < r_2$, then $\mathcal{M}\cap S_a = \emptyset$. As $d\star d\boldsymbol{\eta}$ vanishes identically in smooth regions of $\mathcal{K}$ flow, it vanishes outside $S_a$ for $K_a$, as so, the Stokes theorem can be applied to $\mathcal{M}\cup\partial\mathcal{M}$. This fact and the definition of total angular momentum $J$ of an asymptotically flat spacetime imply in the limit $r_2\to\infty$ that

$$J \equiv -\lim_{r_2\to\infty}\frac{1}{16\pi}\int\limits_{\mathcal{S}_2}\star d\boldsymbol{\eta} = -\frac{1}{16\pi}\int\limits_{\mathcal{S}_1}\star d\boldsymbol{\eta}$$

for any $a < r_1 < \infty$, provided $\mathcal{S}_1$ has the same orientation as $\mathcal{S}_2$. In particular, for $\omega = \rho^{-2}K_a$ we obtain in spherical coordinates

$$J = -\lim_{r\to\infty}\frac{1}{16\pi}\iint\left(r^2\sin^2\theta\partial_r\omega\right)r^2\sin\theta d\theta\wedge d\phi = \frac{a^2}{3} = J_a\,. \tag{3.4}$$

By shrinking and deforming $\mathcal{S}_1$ continously to coalesce finally with $S_a$ one infers that the singular segment is the source of the total angular momentum of the spacetime described by $K_a$.

Similarly, total mass $M$ of asymptotically flat spacetime is defined as a surface integral over sphere at infinity, and for the line element (2.6) written in spherical coordinates

$$M = \lim_{r\to\infty}\frac{1}{8\pi}\int\limits_{\mathcal{S}_r}\star d\boldsymbol{\xi} = -\lim_{r\to\infty}\frac{1}{8\pi}\iint\left(r^2\sin^2\theta\omega\partial_r\omega\right)r^2\sin\theta d\theta\wedge d\phi\,, \tag{3.5}$$

where $\mathcal{S}_r$ is the sphere of radius $r$ and $\boldsymbol{\xi} = \xi_\mu dx^\mu$. For $r$ finite we obtain a mass function which is positive and, for $r$ sufficiently large, behaves as $M(r)\sim 4a^4/\left(9r^3\right) > 0$. The function tends to 0 as $r\to\infty$, which confirms

[3] The Hodge operator acting in a spacetime on a differential form $d\boldsymbol{X}$, where $\boldsymbol{X} = X_\mu dx^\mu$, gives $\star d\boldsymbol{X} = \frac{1}{2}\sqrt{-g}\varepsilon_{\mu\nu\alpha\beta}\nabla^\mu X^\nu dx^\alpha\wedge dx^\beta$. If in addition $\boldsymbol{X}$ satisfies Killing equations $\nabla_\mu X_\nu + \nabla_\nu X_\mu = 0$, one can prove that $d\star d\boldsymbol{X} = \frac{1}{3}\sqrt{-g}\varepsilon_{\mu\alpha\beta\gamma}R^\mu_\nu X^\nu dx^\alpha\wedge dx^\beta\wedge dx^\gamma$.

our previous result that $M_a = 0$. The unusual behaviour of $M(r)$ is due to negative matter density of $\mathcal{K}$ flow. Indeed, $M(r+\varepsilon) - M(r) = (8\pi)^{-1} \int_{\Sigma_\varepsilon} d\star d\boldsymbol{\xi} = \int_{\Sigma_\varepsilon} \sqrt{-g}\mathcal{D} d^3x < 0$ for $\varepsilon > 0$, where $\Sigma_\varepsilon$ is the integration region of constant time and bounded by concentric spheres of radius $r$ and $r + \varepsilon$, respectively, and $r$ is such that all singularities of $K$ are contained inside the ball bounded by $\mathcal{S}_r$.

### *3.2. Asymptotically flat spacetimes of $\mathcal{K}$ flow are massless, some of them may have non-vanishing total angular momentum*

Asymptotical flatness for the line element (2.6) requires $\omega$ to fall off in spherical coordinates at least like $r^{-3}$. If $\omega \sim r^{-3}$, that is, if a given asymptotically flat solution contains a nonzero contribution from the dipole momentum, it necessarily has nonzero total angular momentum defined by (3.4). All asymptotically flat spacetimes of $\mathcal{K}$ flow have vanishing total mass defined in equation (3.5). This can be interpreted as being due to the screening of singular sources by regular flow regions of negative energy density. That such singular sources have to exist, where Einstein's equations are not smooth and must contain distributional sources, is best illustrated by examining the differential form $\star d\eta$ for the dipole solution

$$K = \frac{a^2 \sin^2\theta}{r^3}, \quad \mu = \frac{9a^4}{16}\frac{\sin^4\theta}{r^4}\,.$$

For the Stokes theorem to hold everywhere one needs either $d\star d\boldsymbol{\eta} \neq 0$ in $r = 0$, which must be interpreted as a singularity being the additional distributional source of angular momentum — this is also curvature singularity as the Ricci scalar

$$R = 9e^{-2\mu}\frac{a^4\sin^2\theta}{r^6}$$

is unbounded at $r = 0$ (for example, $R$ is divergent along lines of constant $\mu$ as $r \to 0$) — or one has to exclude the 'point' $r = 0$ from the spacetime, then the nonzero $J$ is a topological effect and is in no contradiction with the fact that $d\star d\boldsymbol{\eta} = 0$ everywhere in smooth regions of $\mathcal{K}$ flow.

## 4. Conclusions

We examined cylindrically symmetric and stationary dust flow along geodesic world-lines of locally non-rotating observers of which space trajectories are rings about common axis of rotation. Such flow can be constructed globally with the exception of regions where singularities of the flow reside and that are additional sources of gravitational field. The flow is differential (the shear tensor is non-zero), non-expanding (the dilation scalar vanishes identically) and locally does not rotate (the vorticity tensor vanishes identically). Local mass density of the flow is necessarily negative-definite which makes the flow nonphysical as far as usual forms of matter are concerned.

There exist asymptotically flat spacetimes of the flow. Matter of the flow moves differentially on circular orbits with respect to asymptotic stationary observers. The spacetimes have vanishing total mass and contain internal singularities where distributional sources of positive (maybe infinite) mass and of angular momentum are located, and this should be understood in the sense that a surface integral over the sphere at infinity that reproduces total mass, vanishes for asymptotically flat $\mathcal{K}$ flow. For radii sufficiently large the resulting mass function is positive and attains zero monotonically in the limit of infinite radius. As local energy density is negative definite in the regions where spacetime is smooth, the internal singularities of the flow must be distributional sources of positive mass (maybe infinite) of which contribution to the total mass is screened by the regular regions, such that the mass function is zero at infinity. This phenomenon is quite analogous with the screening of singularities of negative mass by regions of positive energy density of asymptotically flat van Stockum–Bonnor flow. By construction of $\mathcal{K}$ flow the specific angular momentum per particle is zero, nevertheless, total angular momentum of a class of asymptotically flat spacetimes may be non-zero and we gave an example in (3.3). As the $\mathcal{K}$ flow has no specific angular momentum, the total angular momentum must be located in singularities of the spacetimes.

Among other solutions, the model contains an infinite sequence of smooth asymptotically flat multipolar solutions to which a class of other asymptotically flat solutions of $\mathcal{K}$ flow can be decomposed. There also exist an infinite sequence of the corresponding internal multipolar solutions that are not asymptotically flat. The solutions are given in general by formulas (3.1) and (3.2).

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