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INFELD–VAN DER WAERDEN WAVE FUNCTIONS FOR GRAVITONS AND PHOTONS

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A concise description of the curvature structures borne by the Infeld-van der Waerden $\gamma\varepsilon$-formalisms is provided. The derivation of the wave equations that control the propagation of gravitons and geometric photons in generally relativistic space-times is then carried out explicitly.

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1. Introduction

The most striking physical feature of the Infeld-van der Waerden $\gamma\varepsilon$-formalisms [1] is related to the possible occurrence of geometric wave functions for photons in the curvature structures of generally relativistic space-times [2]. It appears that the presence of intrinsically geometric electromagnetic fields is ultimately bound up with the imposition of a single condition upon the metric spinors for the γ-formalism, which bears invariance under the action of the generalized Weyl gauge group [1–5]. In the case of a spacetime which admits nowhere-vanishing Weyl spinor fields, background photons eventually interact with underlying gravitons. Nevertheless, the corresponding coupling configurations turn out to be in both formalisms exclusively borne by the wave equations that control the electromagnetic propagation. Indeed, the same graviton-photon interaction prescriptions had been obtained before in connection with a presentation of the wave equations for the ε-formalism [6, 7]. As regards such earlier works, the main procedure seems to have been based upon the implementation of a set of differential operators whose definitions take up suitably contracted two-spinor covariant commutators. However, the metric inner structure of the γ-formalism and the intrinsic spin-density character of the fundamental objects of the ε-formalism, were both left out to the extent that no geometric specification was assigned to the electromagnetic wave functions allowed for. A fairly complete elementary description of the formalisms, which has really filled in this gap, is supplied by Ref. [2].

The present paper brings out the techniques which enable one to describe in a concise way the physical situation being entertained. Its relevance stems only from the significance of any works that deal adequately with algebraic descriptions of spin structures in the realm of general relativity. As far as our calculational procedures are concerned, the crucial point comes directly from the construction of a pair of algebraically independent computational rules that include utilizing a more geometric version of the covariant commutators referred to previously. We will see that the implementation of our commutators gives rise to a system of wave equations for gravitons and geometric photons, which possess in either formalism a gauge-invariance property associated with appropriate spinor-index configurations.

It will be expedient to take for granted at the outset all the conventions adopted in Ref. [2], but we will occasionally emphasize some of them. We will restrict ourselves to using only holonomic coordinates on a torsionless curved spacetime $\mathfrak{M}$ even though many of our expressions would still remain applicable if anholonomic coordinates were put into practice. Without any risk of confusion, we will utilize the same indexed symbol $\bigtriangledown_a$ upon spelling out covariant differentials in both formalisms. All wave functions shall be considered physically as classical objects. Throughout the work, no specific energy character will be explicitly attributed to them. Hence, we will not attempt herein to look upon any wave functions as quantum fields. The requirement that ensures the presence of background wave functions for photons in $\mathfrak{M}$, amounts to taking $\beta_a \neq 0$ everywhere along with the eigenvalue equations

$$\bigtriangledown_a \gamma_{BC} = i\beta_a \gamma_{BC} \,, \quad \bigtriangledown_a \gamma^{BC} = (-i\beta_a)\gamma^{BC} \,,$$

with γ_{BC} standing for one of the unprimed-index metric spinors for the γ-formalism. Consequently, γ_{BC} must not bear covariant constancy. Its independent component γ shows up as a world-invariant spin-scalar density of weight $+1$, which enters into the prescription

$$\gamma_{BC} = \gamma\varepsilon_{BC}, \, \gamma = \mid \gamma \mid \exp(i\Phi) \,.$$

Thus, the absolute value $\mid \gamma \mid$ behaves as a real-valued world-invariant spin-scalar density of absolute weight $+1$, whereas $\exp(i\Phi)$ comes into play as a composite spin-scalar density of weight $+1$ and absolute weight -1, which accordingly bears the same world character as $\mid \gamma \mid$. The spinor ε_{BC} is one of the metric spinors for the ε-formalism. It is effectively viewed as a gauge-invariant spin-tensor density of weight -1. For the eigenvalue $i\beta_a$, we have the purely imaginary gauge-invariant expression:

$$i\beta_a = i(\nabla_a \Phi + 2\Phi_a)\,,$$

where the quantity Φ_a denotes the (common) electromagnetic potential of some contracted $\gamma\varepsilon$-affine connexions on $\mathfrak{M}$. It will also be convenient to call for the natural system of units wherein $c = \hbar = 1$.

The paper has been outlined as follows. Section 2 exhibits the spin-curvature structure of $\mathfrak{M}$. There, the sets of geometric formulae for both formalisms will be built up in conjunction with one another. The relevant calculational techniques are given in Section 3. In respect of the derivation of our wave equations, we will work out the electromagnetic case in Section 4. The wave equations for gravitons will be derived afterwards in Section 5. We will make a few remarks on the paper in Section 6.

2. Spin curvature

Actually, the simplest procedure for bringing out the spin-curvature structure of $\mathfrak{M}$ consists first in considering some differentiable world-invariant spin vectors ζ^A and ξ_A along with the torsion-freeness property

$$[\nabla_a, \nabla_b](\zeta^C \xi_C) = 0\,, \tag{1}$$

and then writing the alternative covariant-commutator configurations

$$[\nabla_a, \nabla_b]\zeta^C = W_{abM}{}^C \zeta^M\,, \quad [\nabla_a, \nabla_b]\xi_C = -W_{abC}{}^M \xi_M\,. \tag{2}$$

The W-object involved in (2) is one of the conjugate Infeld-van der Waerden mixed curvature objects [1, 2] of $\mathfrak{M}$. In either formalism, it carries the whole information on the respective curvature spinors in conformity with the bivector scheme

$$\omega_{ABCD} = \omega_{(AB)CD} \doteqdot \frac{1}{2} S^a_{AA'} S^{bA'}_B W_{abCD}, \tag{3}$$

and

$$\omega_{A'B'CD} = \omega_{(A'B')CD} \doteqdot \frac{1}{2} S^a_{AA'} S^{bA}_{B'} W_{abCD}, \tag{4}$$

where the S-objects are the connecting objects of the formalism at issue[1]. The ω-spinors for the γ-formalism, and their complex conjugates, are all subject to gauge-tensor laws, whilst the former ω-spinors for the ε-formalism appear as gauge-invariant spin-tensor densities of weight -2 and absolute weight -2.

[1] In both formalisms, the S-objects are defined so as to involve closely the symmetries borne by Eqs. (3) and (4).

It can be shown [2] that the information on the gravitational and electromagnetic contributions to the geometric structure of $\mathfrak{M}$ is entirely encoded into the curvature spinors of either formalism. An easy way of extracting the individual patterns in both cases takes into consideration the splitting prescriptions

$$\omega_{ABCD} = \omega_{(AB)(CD)} + \frac{1}{2}\omega_{(AB)N}{}^{N}M_{CD}, \tag{5}$$

and

$$\omega_{A'B'CD} = \omega_{(A'B')(CD)} + \frac{1}{2}\omega_{(A'B')N}{}^{N}M_{CD}, \tag{6}$$

with the kernel letter M standing here as elsewhere for either γ or ε. In either formalism, the Riemann–Christoffel curvature tensor R_{abcd} of $\mathfrak{M}$ appears to be associated to the gauge-covariant spinor [2]

$$R_{AA'BB'CC'DD'} = (M_{A'B'}M_{C'D'}\omega_{AB(CD)} + M_{AB}M_{C'D'}\omega_{A'B'(CD)}) + \text{c.c.}\,, \tag{7}$$

where the symbol "c.c." denotes an overall complex-conjugate piece. The index-pair symmetry borne by R_{abcd} requires the implementation of the additional symmetries [7, 8]

$$\omega_{AB(CD)} = \omega_{(CD)AB}, \omega_{A'B'(CD)} = \omega_{(CD)A'B'}\,, \tag{8}$$

which imply that $\omega_{A'B'(CD)}$ must be taken as an Hermitian object in both formalisms. Hence, introducing the first-left dual configuration[2]

$$^{*}R_{AA'BB'CC'DD'} = [(-i)(M_{A'B'}M_{C'D'}\omega_{AB(CD)} - M_{AB}M_{C'D'}\omega_{A'B'(CD)})] + \text{c.c.}\,, \tag{9}$$

brings about the reality statement

$$M_{A'D'}M^{BC}\omega_{AB(CD)} = M_{AD}M^{B'C'}\omega_{A'B'(C'D')}\,, \tag{10}$$

which is equivalent to

$$\mathrm{Im}(M^{AD}M^{BC}\omega_{AB(CD)}) = 0,\ R = 4M^{AD}M^{BC}\omega_{AB(CD)}, \tag{11}$$

with R being the pertinent Ricci scalar. Furthermore, if use is made of the four-index reduction formula [7]

$$\begin{aligned}\varOmega_{ABCD} = \varOmega_{(ABCD)} &- \tfrac{1}{4}\left(M_{AB}\varOmega^{M}{}_{(MCD)} + M_{AC}\varOmega^{M}{}_{(MBD)} + M_{AD}\varOmega^{M}{}_{(MBC)}\right)\\ &- \tfrac{1}{3}\left(M_{BC}\varOmega^{M}{}_{A(MD)} + M_{BD}\varOmega^{M}{}_{A(MC)}\right) - \tfrac{1}{2}M_{CD}\varOmega_{AB}{}^{M}{}_{M}\,,\end{aligned} \tag{12}$$

[2] Eq. (9) corresponds in both formalisms to $^{*}R_{abcd} \doteqdot \frac{1}{2}e_{ab}{}^{mn}R_{mncd}$, with the e-object being one of the alternating Levi-Civita world *tensors* on $\mathfrak{M}$. It can be shown that $^{*}R_{ab}{}^{bc} = 0$ (see Refs. [5, 7]).

we will obtain the important gravitational equality

$$\omega_{AB(CD)} = \omega_{(ABCD)} - \frac{R}{12} M_{A(C} M_{D)B} , \tag{13}$$

which presumably absorbs the property $M_{A(C} M_{D)B} = M_{C(A} M_{B)D}$.

The electromagnetic contribution to the curvature spinors for either formalism amounts to the contracted pieces

$$\frac{i}{2} \omega_{ABC}{}^{C} \doteq \phi_{AB} , \qquad \frac{i}{2} \omega_{A'B'C}{}^{C} \doteq \phi_{A'B'} . \tag{14}$$

Such ϕ-quantities are locally thought of as wave functions for geometric photons, with each of which being inextricably rooted into the curvature structure of $\mathfrak{M}$. We have the unambiguous Maxwell-bivector decomposition

$$S^{a}_{AA'} S^{b}_{BB'} F_{ab} \doteq 2 S^{a}_{AA'} S^{b}_{BB'} \nabla_{[a} \Phi_{b]} = M_{AB} \phi_{A'B'} + M_{A'B'} \phi_{AB} , \tag{15}$$

along with the relationships

$$\omega_{ABC}{}^{C} = 2i \nabla^{C'}_{(A} \Phi_{B)C'} , \quad \omega_{A'B'C}{}^{C} = 2i \nabla^{C}_{(A'} \Phi_{B')C} . \tag{16}$$

It should be stressed that the wave function $\phi_{A}{}^{B}$ and its complex conjugate are gauge-invariant spin tensors in both formalisms.

3. Computational procedures

The key covariant-derivative-operator pattern is written out explicitly in either formalism as [2]

$$S^{a}_{AA'} S^{b}_{BB'} [\nabla_a , \nabla_b] = M_{A'B'} \Delta_{AB} + M_{AB} \Delta_{A'B'} . \tag{17}$$

Both the Δ-kernels carried by the right-hand side of (17) are symmetric second-order differential operators which bear linearity as well as the Leibniz-rule property. In the γ-formalism, they behave under gauge transformations as formal covariant spin tensors, with one of the respective conjugate defining expressions appearing as

$$\Delta_{AB} \doteq \nabla_{C'(A} \nabla^{C'}_{B)} - i \beta_{C'(A} \nabla^{C'}_{B)} = - \nabla^{C'}_{(A} \nabla_{B)C'} . \tag{18}$$

The ε-formalism version of (18) is given by the simple configurations

$$\Delta_{AB} = \nabla_{C'(A} \nabla^{C'}_{B)} , \qquad \Delta_{A'B'} = \nabla_{C(A'} \nabla^{C}_{B')} , \tag{19}$$

which correspondingly behave as gauge-invariant spin-tensor densities of weight -1 and antiweight -1, respectively. In both formalisms, the contravariant form of the Δ-operators is defined by

$$\Delta^{AB} \doteqdot M^{AC}M^{BD}\Delta_{CD}\,. \tag{20}$$

In particular, this definition produces the γ-formalism structure

$$\Delta^{AB} = -(\nabla^{C'(A}\nabla^{B)}_{C'} + i\beta^{C'(A}\nabla^{B)}_{C'}) = \nabla^{(A}_{C'}\nabla^{B)C'}\,. \tag{21}$$

It follows that the rules for computing covariant and contravariant Δ-derivatives in both formalisms are symbolically the same.

A glance at Eqs. (2) tells us that we can write down in each formalism the derivative patterns

$$\Delta_{AB}\zeta^{C} = \omega_{ABM}{}^{C}\zeta^{M}\,, \qquad \Delta_{A'B'}\zeta^{C} = \omega_{A'B'M}{}^{C}\zeta^{M}\,. \tag{22}$$

It should be clear that the prescriptions for computing any Δ-derivatives of ξ_A can right away be deduced from (22) by performing Leibniz expansions of the product $\zeta^C\xi_C$. We thus obtain

$$\Delta_{AB}\xi_{C} = -\omega_{ABC}{}^{M}\xi_{M}\,, \qquad \Delta_{A'B'}\xi_{C} = -\omega_{A'B'C}{}^{M}\xi_{M}\,, \tag{23}$$

along with the complex conjugates of (22) and (23). For a complex spin-scalar density η of weight w on $\mathfrak{M}$, one has the derivatives [2]

$$\Delta_{AB}\eta = -w\eta\omega_{ABC}{}^{C}\,, \qquad \Delta_{A'B'}\eta = -w\eta\omega_{A'B'C}{}^{C}\,. \tag{24}$$

The patterns of Δ-derivatives of spin objects of arbitrary valences can be likewise constructed by carrying out Leibniz expansions for outer products between spin vectors. For instance,

$$\Delta_{AB}(\eta T_{C\ldots D}) = (\Delta_{AB}\eta)T_{C\ldots D} + \eta\Delta_{AB}T_{C\ldots D}\,, \tag{25}$$

with $T_{C\ldots D}$ being some differentiable spin tensor.

It is observed in Ref. [2] that whenever Δ-derivatives of arbitrary Hermitian quantities are computed in both formalisms, there occurs a cancellation of the electromagnetic pieces of (5) and (6), independently of which admissible index configurations are ascribed to the Δ-operators. Such a cancellation also happens when we allow Δ-operators to act freely upon any unprimed or primed spin tensor that bears the same number of covariant and contravariant indices. For $w < 0$, it still occurs in the expansion (25) when $T_{C\ldots D}$ is taken to carry $-2w$ indices and $\mathrm{Im}\eta \neq 0$. A similar property obviously holds for cases which involve outer products between contravariant spin tensors and complex spin-scalar densities carrying suitable positive weights. In fact, these properties[3] can all be established in a transparent way by implementing the electromagnetic conjugacy scheme

$$\omega_{ABC}{}^{C} = -\omega_{ABC'}{}^{C'} , \qquad \omega_{A'B'C}{}^{C} = -\omega_{A'B'C'}{}^{C'} , \tag{26}$$

along with the gravitational definitions

$$\mathrm{X}_{ABCD} \doteqdot \omega_{AB(CD)} , \qquad \varXi_{CA'DB'} \doteqdot \omega_{A'B'(CD)} . \tag{27}$$

It will become manifest later that the completion of our computational procedures crucially involves making use of the algebraic rules

$$2\,\nabla^{A'}_{[C}\,\nabla_{A]A'} = M_{AC}\square = \nabla^{A'}_{D}(M_{CA}\nabla^{D}_{A'}) , \tag{28}$$

and

$$2\,\nabla^{[C}_{A'}\,\nabla^{A]A'} = M^{CA}\square = \nabla^{D}_{A'}(M^{AC}\nabla^{A'}_{D}) , \tag{29}$$

where $\square$ is the (gauge-invariant) d'Alembertian operator for ∇_a. We thus have the operator splittings

$$\nabla^{A'}_{C}\nabla_{AA'} = \frac{1}{2}M_{AC}\square - \varDelta_{AC} , \qquad \nabla^{C}_{A'}\nabla^{AA'} = \varDelta^{AC} - \frac{1}{2}M^{AC}\square . \tag{30}$$

4. Wave equations for geometric photons

In both formalisms, Maxwell's theory is written as

$$\nabla^{AA'}(M_{A'B'}\phi_{AB}) = 0 , \qquad \nabla^{AA'}(M_{AB}\phi_{A'B'}) = 0 . \tag{31}$$

The γ-formalism unprimed-field version of (31) amounts to the eigenvalue equations

$$\nabla^{AB'}\phi_{AB} = i\beta^{AB'}\phi_{AB} \Leftrightarrow \nabla_{AB'}\phi^{AB} = (-i)\beta_{AB'}\phi^{AB} . \tag{32}$$

In the ε-formalism, the above statements are reduced to the gauge-invariant massless-free-field equations

$$\nabla^{AB'}\phi_{AB} = 0 \Leftrightarrow \nabla_{AB'}\phi^{AB} = 0 . \tag{33}$$

Equations (32) and (33) can at once be recast into the homogeneous form

$$\nabla^{AB'}\phi_{A}{}^{B} = 0 , \tag{34}$$

[3] The properties under consideration will be used so many times in Secs. 4 and 5 that we shall no longer refer to them explicitly.

whence, invoking the contravariant splitting exhibited by (30), yields

$$\nabla^{C}_{A'} \nabla^{AA'} \phi_A{}^B = \Delta^{AC}\phi_A{}^B - \tfrac{1}{2} M^{AC}\Box\phi_A{}^B = 0\,. \tag{35}$$

Because of the invariant character of $\phi_A{}^B$, the Δ-expansion of (35) carries in either formalism only the (symmetric) configuration

$$\Delta^{AB}\phi_A{}^C = \frac{R}{6}\phi^{BC} - \omega^{(ABCD)}\phi_{AD} = \Delta^{AC}\phi_A{}^B\,. \tag{36}$$

Therefore, after some trivial index manipulations, we obtain the wave equation

$$\left(\Box + \frac{R}{3}\right)\phi_A{}^B = (-2)\Psi_{AD}{}^{BC}\phi_C{}^D\,, \tag{37}$$

with the definition

$$\Psi_{ABCD} \doteqdot \omega_{(ABCD)}\,. \tag{38}$$

The derivation of the γ-formalism equation that controls the propagation of ϕ_{AB} was carried out in Ref. [2] on the basis of the utilization of the differential structure

$$2\Delta^{AC}\phi_{AB} - \gamma^{AC}\Box\phi_{AB} = \nabla^{C}_{A'}(2i\beta^{AA'}\phi_{AB})\,. \tag{39}$$

In addition to having to account for the derivative

$$2\Delta^{AC}\phi_{AB} = \frac{R}{3}\phi_B{}^C - 2\Psi_B{}^{CMN}\phi_{MN} - 2\omega^{AC}{}_M{}^M\phi_{AB}\,, \tag{40}$$

one has to perform somewhat lengthy calculations towards accomplishing an irreducible form of the right-hand side of (39). A much simpler derivation procedure amounts to combining together the expansion

$$\Box(\phi_A{}^M\gamma_{MB}) = (\Box\phi_A{}^M)\gamma_{MB} + \phi_A{}^M\Box\gamma_{MB} + 2(\nabla^h\phi_A{}^M)\nabla_h\gamma_{MB}\,, \tag{41}$$

and the eigenvalue equation

$$\Box\gamma_{BC} = (-\overline{\Theta}_{(\boldsymbol{E})})\gamma_{BC}\,, \tag{42}$$

with $\Theta_{(\boldsymbol{E})} \doteqdot \beta^h\beta_h + i\nabla_h\beta^h$. We thus arrive at the spin-tensor statement

$$\left(\Box - 2i\beta^h\nabla_h - \Theta_{(\boldsymbol{E})} + \frac{R}{3}\right)\phi_{AB} = 2\Psi_{AB}{}^{CD}\phi_{CD}\,. \tag{43}$$

In a similar way, for ϕ^{AB}, we invoke the upper-index version of (42), namely

$$\Box\gamma^{BC} = (-\Theta_{(\boldsymbol{E})})\gamma^{BC}\,, \tag{44}$$

to obtain

$$\left(\Box + 2i\beta^h \nabla_h - \overline{\Theta}_{(\boldsymbol{E})} + \frac{R}{3}\right)\phi^{AB} = 2\Psi^{AB}{}_{CD}\phi^{CD}\,, \tag{45}$$

along with the complex conjugates of (43) and (45). For the ε-formalism, we have[4]

$$\left(\Box + \frac{R}{3}\right)\phi_{AB} = 2\Psi_{AB}{}^{CD}\phi_{CD}\,, \qquad \left(\Box + \frac{R}{3}\right)\phi^{AB} = 2\Psi^{AB}{}_{CD}\phi^{CD}\,. \tag{46}$$

5. Wave equations for gravitons

The totally symmetric spinor defined by Eq. (38) is the Weyl spinor field of either formalism. Usually, it enters together with its complex conjugate [6,7] into the spinor expression for the Weyl tensor C_{abcd} of $\mathfrak{M}$, that is to say

$$S^a_{AA'}S^b_{BB'}S^c_{CC'}S^d_{DD'}C_{abcd} = M_{A'B'}M_{C'D'}\Psi_{ABCD} + \text{c.c.} \tag{47}$$

In the γ-formalism, we have the gauge-covariant eigenvalue equations [2]

$$\nabla^{AB'}\Psi_{ABCD} = 2i\beta^{AB'}\Psi_{ABCD} \Leftrightarrow \nabla_{AB'}\Psi^{ABCD} = (-2i)\beta_{AB'}\Psi^{ABCD}\,, \tag{48}$$

which can be equivalently rewritten as the vacuum field equation

$$\nabla^{AA'}\Psi_{AB}{}^{CD} = 0\,. \tag{49}$$

In both formalisms, $\Psi_{AB}{}^{CD}$ appears as an invariant spin-tensor wave function whence the ε-formalism version of the first of Eqs. (48), for instance, is given by

$$\nabla^{AB'}\Psi_{ABCD} = 0\,. \tag{50}$$

The conjugate Ψ-spinors for both formalisms can be taken to represent locally the ten independent gravitational degrees of freedom. They appear as massless uncharged wave functions carrying spin ± 2, which constitute dynamical states for gravitons in $\mathfrak{M}$.

It has become evident that the basic procedures for deriving the wave equations for the Ψ-fields of either formalism, are essentially the same as the ones for the electromagnetic case. Thus, for the field borne by Eq. (49), the γ-formalism splitting is

$$\nabla^E_{A'}\nabla^{AA'}\Psi_{AB}{}^{CD} = \Delta^{AE}\Psi_{AB}{}^{CD} - \tfrac{1}{2}\gamma^{AE}\Box\Psi_{AB}{}^{CD} = 0\,. \tag{51}$$

[4] Needless to say, techniques similar to the ones we have employed lead to the traditional wave equation for the potential.

The explicit computation of the Δ-derivative of (51) leads us to [2]

$$\left(\Box + \frac{R}{2}\right) \Psi_{AB}{}^{CD} = 6\Psi_{MN}{}^{(CD}\Psi^{EL)MN}\gamma_{EA}\gamma_{LB} \,. \tag{52}$$

Owing to the common gauge character of $\Psi_{AB}{}^{CD}$, we can state that the calculation yielding (52) possesses the same form in both formalisms. An observation regarding the gauge specification of Ψ_{ABCD} as well as its four-index feature could indeed be made at this stage, whence one can promptly write out the ε-formalism statement

$$\left(\Box + \frac{R}{2}\right) \Psi_{ABCD} = 6\Psi_{MN(AB}\Psi_{CD)}{}^{MN} \,. \tag{53}$$

We can derive the γ-formalism counterpart of (53) by taking into account the combination of the differential prescriptions

$$\Box\Psi_{ABCD} \;=\; \Box(\Psi_{AB}{}^{LM}\gamma_{LC}\gamma_{MD}) \,, \tag{54a}$$

$$\Box(\gamma_{LC}\gamma_{MD}) \;=\; \left(-\overline{\Theta}_{(\boldsymbol{G})}\right)\gamma_{LC}\gamma_{MD} \,, \tag{54b}$$

and

$$2(\nabla_a \Psi_{AB}{}^{LM})\, \nabla^a\, (\gamma_{LC}\gamma_{MD}) = 4(2\beta^h\beta_h + i\beta^h\nabla_h)\Psi_{ABCD} \,, \tag{55}$$

with $\Theta_{(\boldsymbol{G})} \doteqdot 2(\beta^h\beta_h + \Theta_{(\boldsymbol{E})})$. The resulting statement amounts, in effect, to the wave equation

$$\left(\Box - 4i\beta^h\, \nabla_h - \Theta_{(\boldsymbol{G})} + \frac{R}{2}\right) \Psi_{ABCD} = 6\Psi_{MN(AB}\Psi_{CD)}{}^{MN} \,. \tag{56}$$

A useful metric property of the γ-formalism is that the wave equations for covariant and contravariant fields of the same physical kind can be immediately attained from one another by calling upon the simultaneous interchanges[5]

$$i\beta^h\nabla_h \leftrightarrow (-i)\beta^h\nabla_h,\; (\Theta_{(\boldsymbol{E})}, \Theta_{(\boldsymbol{G})}) \leftrightarrow (\overline{\Theta}_{(\boldsymbol{E})}, \overline{\Theta}_{(\boldsymbol{G})}) \,. \tag{57}$$

The wave equation for Ψ^{ABCD} is thus spelt out as

$$\left(\Box + 4i\beta^h\, \nabla_h - \overline{\Theta}_{(\boldsymbol{G})} + \frac{R}{2}\right) \Psi^{ABCD} = 6\Psi_{MN}{}^{(AB}\Psi^{CD)MN}. \tag{58}$$

[5] This property was established for the first time in Ref. [2]. It had already been clearly exhibited in Section 4 by the statements (42) through (45).

6. Concluding remarks and outlook

One of the most significant aspects of the work just presented is the fact that it has enhanced the elementary role played by geometric wave functions for photons within the framework of general relativity. All the couplings that should somehow be involved in any natural description of generally relativistic spinor structures have been effectively brought out.

We believe that our differential techniques for keeping track of spinor-index configurations can perhaps be utilized for describing at large scales some of the physical properties of the cosmic radiation background. An interesting point concerning the calculational features of this situation rests upon the implementation of conformally flat space-times, and thence also upon the strict use of identically vanishing wave functions for gravitons.

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