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STRUCTURE OF QUARK–LEPTON GENERATION AND GENERALISED CANONICAL COMMUTATION RELATIONS

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Dirac-like linearisation of $\boldsymbol{x}^2+\boldsymbol{p}^2$ with noncommuting position and momentum variables leads to the representation of the standard $\mathrm{U}(1)\otimes\mathrm{SU}(3)$ symmetry of the three-dimensional harmonic oscillator in the relevant Clifford algebra and the emergence of a formula which we previously proposed to identify with the Gell-Mann–Nishijima–Glashow relation between charge, third component of weak isospin and weak hypercharge. This matrix representation exhibits features not present in the standard treatment of harmonic oscillator. We show that these features, strictly corresponding to the structure of a single quark–lepton generation in the Standard Model, may be understood from the point of view of specific O(6) phase-space transformations, which go beyond $\mathrm{U}(1)\otimes\mathrm{SU}(3)$, and modify standard canonical commutation relations. It is demonstrated that the whole structure of a single quark–lepton generation corresponds to assuming that the imaginary unit appearing in the canonical commutation relations may acquire an additional "+" or "−" sign separately for each of the three directions.

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1. Introduction

A general idea is often advocated that there should exist a connection between the discrete quantum attributes of elementary particles and the properties of the continuous arena used for the description of classical macroscopic processes [1]. Recently, this idea was discussed in Ref. [2] from a somewhat different perspective. In particular, it was proposed there that, as a starting point in such a discussion, it should be acceptable to adopt a nonrelativistic framework. Later it was pointed out in [3] that in fact all those discrete quantum numbers of elementary particles for which a connection with spacetime is known can be inferred via a nonrelativistic reasoning. As stressed in [3], this claim refers also to the existence of the particle–antiparticle degree of freedom, as observed in [4].

The essence of the conceptual argument of [2] was that instead of identifying the arena of nonrelativistic physics with the observable three-dimensional position space (with physical processes occurring in Newtonian time), one should adopt the description given by the nonrelativistic Hamiltonian formalism, in which position and momentum coordinates are treated as *independent* variables. In this language it is the concept of phase space which provides the relevant arena to be used for the description of classical macroscopic processes. With such a shift in the meaning of the concept of arena, the issue of a possible symmetry between the position and momentum coordinates could be discussed in a more adequate language. In fact, an argument in favour of introducing more symmetry between momentum and position was put forward by Max Born in his reciprocity theory of elementary particles [5] already in 1949. His approach stemmed from the observation that various laws of physics exhibit symmetry under reciprocity transformations: $\boldsymbol{x} \to \boldsymbol{p}$, $\boldsymbol{p} \to -\boldsymbol{x}$. Consideration of these transformations requires the introduction of one additional physical constant which permits the expression of momenta and positions in the same dimensional units. When the Planck constant is added, a natural mass scale is then set.

Accordingly, Ref. [2] was mainly concerned with the problem of mass. First, it was argued that the assignment of a standard concept of mass to quarks leads to conceptual problems related to the issue of quark confinement. Then, it was pointed out that the phase-space approach admits such a generalisation of the concept of mass that the original conceptual problems might hopefully dissappear. The generalisation in question consisted in noting that:

(1) the standard concept of mass may be said to be associated with the concept of momentum $\boldsymbol{p}$ only (*i.e.* the energy of a standard free particle is given in terms of its mass and momentum, whether in a relativistic or a nonrelativistic approach), with $\boldsymbol{x}$ conspicuously absent, and

(2) the phase-space formulation admits not just one but four different ways of dividing the six-dimensional phase-space vector $\boldsymbol{x} \oplus \boldsymbol{p}$ into a pair of 3-dimensional objects. Thus, the original division of $\boldsymbol{x} \oplus \boldsymbol{p}$ into two canonically conjugated sets of positions and momenta

$$\{(x_1, x_2, x_3), (p_1, p_2, p_3)\} \tag{1}$$

may be generalised by admitting three additional alternatives, namely:

$$\{(x_1, p_2, p_3), (p_1, x_2, x_3)\},$$
$$\{(p_1, x_2, p_3), (x_1, p_2, x_3)\},$$
$$\{(p_1, p_2, x_3), (x_1, x_2, p_3)\}. \tag{2}$$

As argued in [2], with (x_1, p_2, p_3), (p_1, x_2, p_3) and (p_1, p_2, x_3) playing the role of generalised position coordinates, it seems then natural to assign the concept of mass not only to (p_1, p_2, p_3) but also to the three sets (p_1, x_2, x_3), (x_1, p_2, x_3), and (x_1, x_2, p_3), and conjecture that the three new alternatives correspond to quarks. The apparent lack of O(3) invariance was considered not a drawback, but a virtue of the approach: an argument against the existence of standard quark propagators.

In the next step Ref. [2] combined $\boldsymbol{x}^2$ and $\boldsymbol{p}^2$, both being invariants of the O(3) group (rotations and reflections in three dimensions), into a single form $\boldsymbol{x}^2 + \boldsymbol{p}^2$, so that full $\boldsymbol{x} \leftrightarrow \boldsymbol{p}$ symmetry was introduced. Then, Ref. [2] considered all transformations that keep invariant both $\boldsymbol{x}^2 + \boldsymbol{p}^2$ and the position-momentum Poisson brackets (or canonical commutation relations). The resulting symmetry group, conjectured to be fundamental, is obviously $\mathrm{U}(1)\otimes\mathrm{SU}(3)$, as is well known from the case of the three-dimensional harmonic oscillator.

The philosophical and physical ideas of [2] provided some guidance on how to construct an appropriate mathematical structure enabling their description and endowed with physical interpretation. Accordingly, Ref. [2] contained some "toy" attempts in that general direction. While from the present paper it is clear that these attempts were technically deficient, in no way have these deficiencies invalidated the general conceptual idea of [2].

Paper [2] was followed by a more technically oriented paper of [3], in which $\boldsymbol{x}^2 + \boldsymbol{p}^2$ was linearised *á la* Dirac. The $\mathrm{U}(1)\otimes\mathrm{SU}(3)$ algebra in question was then represented in the relevant Clifford algebra. Within the latter algebra, the eigenvalues of the U(1) generator were shown to be $(+1/3, +1/3, +1/3, -1)$, exactly as needed for a description of a weak hypercharge Y for three coloured quarks and one lepton (with U(1) eigenvalues for their antiparticles being opposite in sign). The total U(1) generator received contributions from the phase space and the Clifford algebra, yielding a relation which we proposed to identify with the Gell-Mann–Nishijima–Glashow formula [6] for lepton and quark charges:

$$Q = I_3 + Y/2, \tag{3}$$

where I_3 is the third component of weak isospin, and Y is weak hypercharge. Thus, a connection going beyond the $\mathrm{U}(1)\otimes\mathrm{SU}(3)\otimes\mathrm{SU}(2)_{\mathrm{L}}$ group structure of the Standard Model was achieved.

As in Ref. [3] in this paper we shall not be concerned with the other main ingredient of the Standard Model, *i.e.* with the gauge principle, although ways of introducing it in the phase-space language have been discussed in the literature (see *e.g.* [7]). Rather, we shall continue studying the very idea of connecting properties of phase space with quantum numbers of elementary particles. In particular, we shall derive a precise relationship between the conceptual arguments of [2], which lead to Eqs. (1), (2), and the mathematical framework of [3], in which the Dirac linearisation prescription leads to the Gell-Mann–Nishijima–Glashow formula of Eq. (3). The SU(4) transformations between quarks and leptons (with leptons being the "fourth colour" [8]) are then related to phase space SO(6) transformations. Likewise, the transformations between the $I_3 = \pm 1/2$ eigenstates of the third component of isospin are also interpreted as corresponding to specific phase-space transformations (*i.e.* to reflection applied to only one of the two sets of canonically conjugated variables).

2. Linearisation

Dirac-like linearisation of the $\boldsymbol{x}^2 + \boldsymbol{p}^2$ form was achieved in [2] with the help of enlarged Dirac matrices A_k, B_k, and B:

$$\begin{aligned} A_k &= \sigma_k \otimes \sigma_0 \otimes \sigma_1 \,, \\ B_k &= \sigma_0 \otimes \sigma_k \otimes \sigma_2 \,, \\ B &= \sigma_0 \otimes \sigma_0 \otimes \sigma_3 \,, \end{aligned} \tag{4}$$

satisfying the anticommutativity conditions:

$$\begin{aligned} A_k A_l + A_l A_k &= 2\delta_{kl} \,, \\ A_k B_l + B_l A_k &= 0 \,, \\ B_k B_l + B_l B_k &= 2\delta_{kl} \,, \\ A_k B + B A_k &= 0 \,, \\ B_k B + B B_k &= 0 \,, \\ BB &= 1 \,, \end{aligned} \tag{5}$$

with matrices A_k,B (or B_k,B) behaving like Dirac matrices α_k, β. Matrix $B = iA_1A_2A_3B_1B_2B_3$ constitutes the seventh anticommuting matrix of the relevant Clifford algebra.

With noncommuting $\boldsymbol{x}$ and $\boldsymbol{p}$ we have:

$$(\boldsymbol{A}\cdot\boldsymbol{p}+\boldsymbol{B}\cdot\boldsymbol{x})(\boldsymbol{A}\cdot\boldsymbol{p}+\boldsymbol{B}\cdot\boldsymbol{x}) = (\boldsymbol{p}^2+\boldsymbol{x}^2)\,\sigma_0\otimes\sigma_0\otimes\sigma_0+R^\sigma \equiv R^z + R^\sigma, \tag{6}$$

where the superscripts z and σ are used to label the phase-space and the matrix-space terms, and (unless otherwise stated, we use summation convention for repeated indices)

$$\begin{aligned} R^\sigma &= \sum_1^3 R_k^\sigma = -\frac{i}{2}[A_k, B_k] = \sigma_k \otimes \sigma_k \otimes \sigma_3 \,, \\ R_k^\sigma &= \sigma_k \otimes \sigma_k \otimes \sigma_3 \qquad \text{(no sum over } k) \,. \end{aligned} \tag{7}$$

Furthermore, we define $\mathcal{Y}$, $\mathcal{Y}_k$, y, and y_k as follows

$$\mathcal{Y} = R^\sigma B = B R^\sigma = \sum_{k=1}^3 \mathcal{Y}_k = \sum_{k=1}^3 y_k \otimes \sigma_0 = y \otimes \sigma_0 \,, \tag{8}$$

with

$$\begin{aligned} \mathcal{Y}_k &= y_k \otimes \sigma_0 = \sigma_k \otimes \sigma_k \otimes \sigma_0 \qquad \text{(no sum over } k) \,, \\ y &= \sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_3 \,. \end{aligned} \tag{9}$$

Alternatively, instead of A_k and B_k, one may consider matrix analogs of operators $a_k \equiv (x_k + ip_k)/\sqrt{2}$ and $a_k^\dagger \equiv (x_k - ip_k)/\sqrt{2}$, namely:

$$\begin{aligned} C_k &= \frac{1}{\sqrt{2}}(B_k + iA_k) \,, \\ C_k^\dagger &= \frac{1}{\sqrt{2}}(B_k - iA_k) \,. \end{aligned} \tag{10}$$

In terms of C_k and $C_k^\dagger$, the relevant anticommutation relations of equations (5) read:

$$\begin{aligned} \{C_k, C_l\} &= \{C_k^\dagger, C_l^\dagger\} = 0 \,, \\ \{C_k, C_l^\dagger\} &= \{C_k^\dagger, C_l\} = 2\delta_{kl} \,, \\ \{C_k, B\} &= \{C_k^\dagger, B\} = 0 \,. \end{aligned} \tag{11}$$

3. Charge, weak hypercharge and weak isospin

In Ref. [3] it was proposed to identify the operator of Eq. (6), multiplied by B, with six times the operator of charge for fundamental particles (for the origin of the factor of six, see [3]):

$$Q = \frac{1}{6}(R^z + R^\sigma)B \overset{R^z=+3}{\equiv} I_3 + \frac{Y}{2} \,, \tag{12}$$

where the lowest eigenvalue of $R^z = +3$ was adopted, as one would expect appropriate for fundamental objects.

The operator of the third component of weak isospin I_3 was identified as related to B through

$$I_3 = \frac{1}{2}B\,, \tag{13}$$

and the weak hypercharge Y as related to $\mathcal{Y}$ through:

$$Y = \frac{1}{3}\,\mathcal{Y}\,. \tag{14}$$

It was further observed in [3] that y satisfies the following equation:

$$y^2 + 2\,y - 3\,\sigma_0 \otimes \sigma_0 = 0\,. \tag{15}$$

Consequently, the eigenvalues of y are $+1$ and -3 (for Y, respectively: $+1/3$ and -1). Furthermore, the eigenvalue of $+1$ is triple degenerate. Note that the value of $y = +1$ ($Y = +1/3$) corresponds to an eigenvalue of R^σ which (in absolute magnitude) is three times smaller than the minimal value of R^z. The appearance of these small eigenvalues, absent in the spectrum of the three-dimensional harmonic oscillator requires explanation and understanding in the phase-space language. This is what the present paper is about.

It is of great interest to see how the eigenvalues of Y are built out of the eigenvalues of y_k. Since all commutators $[y_k, y_l]$, and $[y, y_k]$ vanish (for any k, l), it follows that one can diagonalise all y_k ($\mathcal{Y}_k$) and y ($\mathcal{Y}$) simultaneously. Performing such a diagonalisation we obtain

$$\begin{aligned}
y_1 = \sigma_1 \otimes \sigma_1 &= \begin{bmatrix} & & & +1 \\ & & +1 & \\ & +1 & & \\ +1 & & & \end{bmatrix} \rightarrow \begin{bmatrix} -1 & & & \\ & +1 & & \\ & & -1 & \\ & & & +1 \end{bmatrix}, \\
y_2 = \sigma_2 \otimes \sigma_2 &= \begin{bmatrix} & & & -1 \\ & & +1 & \\ & +1 & & \\ -1 & & & \end{bmatrix} \rightarrow \begin{bmatrix} +1 & & & \\ & +1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}, \\
y_3 = \sigma_3 \otimes \sigma_3 &= \begin{bmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & +1 \end{bmatrix} \rightarrow \begin{bmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & +1 \end{bmatrix},
\end{aligned} \tag{16}$$

so that

$$Y \rightarrow \begin{bmatrix} +\frac{1}{3} & & & \\ & +\frac{1}{3} & & \\ & & -1 & \\ & & & +\frac{1}{3} \end{bmatrix} \otimes \sigma_0, \qquad \leftarrow \text{colour } \#'s \left\{ \begin{array}{l} 1 \\ 3 \\ 0 \text{ (lepton)} \\ 2 \end{array} \right. , \tag{17}$$

with a clear division into the triplet and singlet subspaces. The obtained eigenvalues of Y, appropriate for the description of three ("coloured") quarks and one lepton, lead via Eq. (12) to fractional quark and integer lepton charges of the Standard Model. In Eq. (17) we labelled the three directions of the $y = +1$ ($Y = +1/3$) subspace by assigning "colour" number k to this direction for which y_k takes on the value of -1. As discussed in [3], the way in which the eigenvalues of Y are built out of the eigenvalues of y_k corresponds exactly to the rishon model of leptons and quarks proposed by Harari [9].

For the sake of this discussion and Section 6 we recall from [3] that for any element X the operation of charge conjugation is effected as follows:

$$X \to \overline{X} = CX^*C^{-1}, \tag{18}$$

with

$$C = -i\sigma_2 \otimes \sigma_2 \otimes \sigma_2 = -C^{-1}. \tag{19}$$

This means in particular that

$$\begin{aligned} \overline{A}_k &= A_k\,, & \overline{B}_k &= B_k\,, & \overline{B} &= -B\,,\\ \overline{p}_k &= -p_k\,, & \overline{x}_k &= x_k\,, & \overline{i} &= -i\,,\\ \overline{C}_k &= C_k^\dagger\,, & \overline{C^\dagger}_k &= C_k\,. \end{aligned} \tag{20}$$

The operation of charge conjugation is not equivalent to Hermitian conjugation, as can be seen either from $\overline{p}_k = -p_k$, or from the application of relation (18) to a product of matrices. It follows that

$$\overline{R}^z = R^z\,, \qquad \overline{R}^\sigma = -R^\sigma\,, \tag{21}$$

$$\overline{I}_3 = -I_3\,, \qquad \overline{Y} = Y\,. \tag{22}$$

Consequently

$$\overline{Q} = CQ^*C^{-1} = -\frac{1}{6}(R^z - R^\sigma)B \overset{R^z \equiv +3}{\equiv} -I_3 + \frac{Y}{2}\,, \tag{23}$$

and the observed charge of the antiparticle is

$$\overline{Q}_{\mathrm{obs}} = -\overline{Q} = I_3 - \frac{Y}{2}\,, \tag{24}$$

as seen from the comparison of $\exp(i\phi Q)$ with its charge conjugate version:

$$\exp(-i\phi\,\overline{Q}) = \exp(i\phi\,\overline{Q}_{\mathrm{obs}})\,. \tag{25}$$

4. SU(4) transformations

4.1. U(1)$\otimes$ *SU(3) generators*

From the case of the three-dimensional harmonic oscillator it is well known that the O(6) symmetry group of the $\boldsymbol{x}^2 + \boldsymbol{p}^2$ form is reduced to U(1)$\otimes$SU(3) when the condition of invariance of the canonical position and momentum commutation relations is imposed. In standard treatments of the three-dimensional harmonic oscillator one introduces nine shift operators:

$$H^z_{kl} = \tfrac{1}{2}\left\{a_k, a^\dagger_l\right\}, \tag{26}$$

(see for example [3]). The nine generators R^z and F^z_b $(b = 1, 2, \ldots 8)$ of the U(1)$\otimes$ SU(3) symmetry group are then expressed as linear combinations of H^z_{kl}'s. Explicit formulas may be found in [3].

In Ref. [3] we defined the following nine shift operators acting in matrix space, the counterparts of H^z_{kl}'s:

$$H^\sigma_{kl} = -\tfrac{1}{4}\left[C_k, C^\dagger_l\right] = (H^\sigma_{lk})^\dagger\ , \tag{27}$$

which act on C_n and $C^\dagger_n$ as follows:

$$\begin{aligned}
[H^\sigma_{kl}, C_n] &= -\delta_{ln} C_k\,, \\
[H^\sigma_{kl}, C^\dagger_n] &= +\delta_{kn} C^\dagger_l\,.
\end{aligned} \tag{28}$$

Matrix counterparts of the relevant generators, denoted as R^σ and F^σ_b $(b = 1, 2, \ldots 8)$, are built from H^σ_{kl}'s in a way completely analogous to that for R^z and F^z_b. In matrix space the U(1) generator is just R^σ of Eq. (7) (*i.e.* the trace-only part of H^σ_{kl}: $R^\sigma = 2H^\sigma_{kk} = -\frac{1}{2}[C_k, C^\dagger_k]$). Explicit formulas for the SU(3) generators in matrix space are given in Ref. [3].

4.2. Genuine SU(4) shift operators

In the following we shall study in detail the origin of the appearance of the fractional values of Y, which go beyond what is expected from the case of the three-dimensional harmonic oscillator and correspond to the triplet representation of SU(3). With this in mind, let us introduce six additional operators, in addition to H^σ_{kl}:

$$\begin{aligned}
\epsilon_{mkl} H^\sigma_{m0} &= -\tfrac{1}{4}[C_k, C_l]\,, \\
\epsilon_{mkl} H^\sigma_{0m} &= +\tfrac{1}{4}[C^\dagger_k, C^\dagger_l]\,,
\end{aligned} \tag{29}$$

satisfying

$$H^{\sigma}_{0m} = (H^{\sigma}_{m0})^{\dagger}\,. \tag{30}$$

The nine H^{σ}_{kl}'s and the six H^{σ}_{m0}, H^{σ}_{0m} operators together correspond to fifteen generators of SU(4).

Under the action of H^{σ}_{m0} and H^{σ}_{0m} matrices C_n and $C^{\dagger}_n$ transform as follows:

$$\begin{aligned}
\epsilon_{mkl}[H^{\sigma}_{m0}, C_n] &= 0\,,\\
\epsilon_{mkl}[H^{\sigma}_{m0}, C^{\dagger}_n] &= +\delta_{kn}C_l - \delta_{ln}C_k\,,\\
\epsilon_{mkl}[H^{\sigma}_{0m}, C^{\dagger}_n] &= 0\,,\\
\epsilon_{mkl}[H^{\sigma}_{0m}, C_n] &= -\delta_{kn}C^{\dagger}_l + \delta_{ln}C^{\dagger}_k\,.
\end{aligned} \tag{31}$$

Shift operators may be decomposed into terms acting between subspaces with different eigenvalues of Y and I_3 as:

$$\begin{aligned}
H^{\sigma}_{m0} &= Y_{-1}I_{+\frac{1}{2}}H^{\sigma}_{m0}I_{+\frac{1}{2}}Y_{+\frac{1}{3}} + Y_{+\frac{1}{3}}I_{-\frac{1}{2}}H^{\sigma}_{m0}I_{-\frac{1}{2}}Y_{-1}\,,\\
H^{\sigma}_{0m} &= Y_{+\frac{1}{3}}I_{+\frac{1}{2}}H^{\sigma}_{0m}I_{+\frac{1}{2}}Y_{-1} + Y_{-1}I_{-\frac{1}{2}}H^{\sigma}_{0m}I_{-\frac{1}{2}}Y_{+\frac{1}{3}}\,,
\end{aligned} \tag{32}$$

where we defined projection operators onto subspaces of definite hypercharge:

$$Y_{-1} = \frac{1-\mathcal{Y}}{4}\,, \qquad Y_{+\frac{1}{3}} = \frac{3+\mathcal{Y}}{4}\,, \tag{33}$$

and isospin:

$$I_{\pm\frac{1}{2}} = \frac{1 \pm B}{2}\,. \tag{34}$$

Thus, H^{σ}_{m0} and H^{σ}_{0m} are off-diagonal in Y: they connect triplet and singlet SU(3) subspaces with each other, as indicated by subscripts "$m0$" and "$0m$" of our notation.

4.3. Genuine SU(4) generators

In analogy to F^{σ}_b's being built out of H^{σ}_{kl}, we now form out of H^{σ}_{m0} and H^{σ}_{0m} the six "genuine" Hermitian generators of SU(4) (the remaining operators R^{σ} and F^{σ}_b will obviously be referred to as the U(1)$\otimes$SU(3) generators):

$$\begin{aligned}
F^{\sigma}_{+1} &= H^{\sigma}_{10} + H^{\sigma}_{01} = -\frac{i}{4}([A_2, B_3] - [A_3, B_2])\,,\\
F^{\sigma}_{+2} &= H^{\sigma}_{20} + H^{\sigma}_{02} = -\frac{i}{4}([A_3, B_1] - [A_1, B_3])\,,\\
F^{\sigma}_{+3} &= H^{\sigma}_{30} + H^{\sigma}_{03} = -\frac{i}{4}([A_1, B_2] - [A_2, B_1])\,,
\end{aligned} \tag{35}$$

and

$$
\begin{aligned}
F^{\sigma}_{-1} &= i(H^{\sigma}_{10} - H^{\sigma}_{01}) = -\frac{i}{4}([B_2, B_3] - [A_2, A_3])\,,\\
F^{\sigma}_{-2} &= i(H^{\sigma}_{20} - H^{\sigma}_{02}) = -\frac{i}{4}([B_3, B_1] - [A_3, A_1])\,,\\
F^{\sigma}_{-3} &= i(H^{\sigma}_{30} - H^{\sigma}_{03}) = -\frac{i}{4}([B_1, B_2] - [A_1, A_2])\,. \qquad (36)
\end{aligned}
$$

Explicit expressions for F^{σ}_{+n} and F^{σ}_{-n} are:

$$F^{\sigma}_{+n} = \tfrac{1}{2}\,\epsilon_{nkl}\,\sigma_k \otimes \sigma_l \otimes \sigma_3\,, \qquad (37)$$

and

$$F^{\sigma}_{-n} = \tfrac{1}{2}\,(\sigma_0 \otimes \sigma_n - \sigma_n \otimes \sigma_0) \otimes \sigma_0\,. \qquad (38)$$

The F^{σ}_{-n} describe simultaneous rotations in mutually opposite senses in $\boldsymbol{x}$ and $\boldsymbol{p}$ spaces. They constitute counterparts to "ordinary" simultaneous rotations in likewise senses generated by $S_n = \frac{1}{2}(\sigma_0 \otimes \sigma_n + \sigma_n \otimes \sigma_0) \otimes \sigma_0$ for which the two terms are added rather than subtracted.

By explicit calculation we find that

$$(F^{\sigma}_{+n})^2 = (F^{\sigma}_{-n})^2 = \tfrac{1}{2}\,(\sigma_0 \otimes \sigma_0 - \sigma_n \otimes \sigma_n) \otimes \sigma_0\,, \qquad (39)$$

and

$$
\begin{aligned}
(F^{\sigma}_{+n})^3 &= F^{\sigma}_{+n}\,,\\
(F^{\sigma}_{-n})^3 &= F^{\sigma}_{-n}\,. \qquad (40)
\end{aligned}
$$

The eight generators of SU(3) and the six genuine SU(4) generators together make fourteen generators. The fifteenth generator of SU(4) is proportional to the U(1) generator R^{σ}, and in the same normalisation (we adopted here the convention of a positive relative sign) it is:

$$F^{\sigma}_{15} \equiv \frac{1}{\sqrt{6}}\,R^{\sigma} = \frac{1}{\sqrt{6}}\,y \otimes \sigma_3 = \sqrt{6}\,Y I_3\,, \qquad (41)$$

Using Eqs. (16), the SU(4) generator F^{σ}_{15} assumes the form

$$F^{\sigma}_{15} \to \sqrt{\frac{3}{2}} \begin{bmatrix} +1/3 & & & \\ & +1/3 & & \\ & & -1 & \\ & & & +1/3 \end{bmatrix} \otimes \sigma_3\,, \qquad (42)$$

i.e. it distinguishes between the SU(3) singlet and SU(3) triplet spaces.

By an explicit calculation we find that under the action of $F^\sigma_{\pm n}$ matrices C_k and $C^\dagger_k$ transform as follows:

$$\begin{aligned}
[F^\sigma_{+n}, C_k] &= -\,\epsilon_{nkl} C^\dagger_l\,,\\
[F^\sigma_{+n}, C^\dagger_k] &= +\,\epsilon_{nkl} C_l\,,
\end{aligned} \tag{43}$$

$$\begin{aligned}
[F^\sigma_{-k}, C_l] &= +i\,\epsilon_{klm} C^\dagger_m\,,\\
[F^\sigma_{-k}, C^\dagger_l] &= +i\,\epsilon_{klm} C_m\,,
\end{aligned} \tag{44}$$

which should be compared with their transformation properties as triplets (antitriplets) under SU(3), as given in Eqs. (36), (37) of [3].

For R^σ_k we have

$$\begin{aligned}
[F^\sigma_{\pm k}, R^\sigma_k\,] &= 0 && (\text{no sum})\,,\\
[F^\sigma_{+k}, R^\sigma_m\,] &= -2i\,F^\sigma_{-k} && (m \neq k)\,,\\
[F^\sigma_{-k}, R^\sigma_m\,] &= +2i\,F^\sigma_{+k} && (m \neq k)\,,\\
[(F^\sigma_{\pm k})^2, R^\sigma_m\,] &= 0 && (\text{any } k, m)\,.
\end{aligned} \tag{45}$$

4.4. Finite genuine SU(4) transformations

Since H^σ_{m0} and H^σ_{0m} connect the SU(3) triplet and singlet spaces, in order to analyse transformations between quarks and leptons we have to look at appropriate transformations generated by genuine SU(4) generators $F^\sigma_{\pm n}$. Consider *e.g.* finite rotations generated by $F^\sigma_{\pm 2}$, as applied to arbitrary element X, *i.e.*:

$$\tilde{X} = e^{+i\phi F^\sigma_{-2}} X\, e^{-i\phi F^\sigma_{-2}}\,, \tag{46}$$

or

$$\tilde{X} = e^{+i\phi F^\sigma_{+2}} X\, e^{-i\phi F^\sigma_{+2}}\,. \tag{47}$$

4.4.1. Transformations of A_k and B_k

In order to see how A_k and B_k transform under finite rotations generated by F^σ_{-2} let us write:

$$F^\sigma_{-2} = \frac{1}{2}\left(F^B_2 - F^A_2\right)\,, \tag{48}$$

with

$$\begin{aligned}
F^A_2 &\equiv S_2 - F^\sigma_{-2} = \sigma_2 \otimes \sigma_0 \otimes \sigma_0\,,\\
F^B_2 &\equiv S_2 + F^\sigma_{-2} = \sigma_0 \otimes \sigma_2 \otimes \sigma_0\,,
\end{aligned} \tag{49}$$

which act separately on A_k and B_k:

$$\left[F_2^A, B_k\right] = \left[F_2^B, A_k\right] = \left[F_2^A, F_2^B\right] = 0\,. \tag{50}$$

Therefore,

$$\begin{aligned}\tilde{A}_k &= e^{-i\frac{\phi}{2}F_2^A}\,A_k\,e^{+i\frac{\phi}{2}F_2^A}\,,\\ \tilde{B}_k &= e^{+i\frac{\phi}{2}F_2^B}\,B_k\,e^{-i\frac{\phi}{2}F_2^B}\,,\end{aligned} \tag{51}$$

i.e. $\boldsymbol{A}$ and $\boldsymbol{B}$ rotate in opposite senses:

$$\begin{aligned}\tilde{A}_1 &= A_1\cos\phi - A_3\sin\phi\,,\\ \tilde{A}_2 &= A_2\,,\\ \tilde{A}_3 &= A_3\cos\phi + A_1\sin\phi\,,\\ \tilde{B}_1 &= B_1\cos\phi + B_3\sin\phi\,,\\ \tilde{B}_2 &= B_2\,,\\ \tilde{B}_3 &= B_3\cos\phi - B_1\sin\phi\,.\end{aligned} \tag{52}$$

Similarly, for rotations (47) generated by F^{σ}_{+2} we obtain:

$$\begin{aligned}\tilde{A}_1 &= A_1\cos\phi + B_3\sin\phi\,,\\ \tilde{A}_2 &= A_2\,,\\ \tilde{A}_3 &= A_3\cos\phi - B_1\sin\phi\,,\\ \tilde{B}_1 &= B_1\cos\phi + A_3\sin\phi\,,\\ \tilde{B}_2 &= B_2\,,\\ \tilde{B}_3 &= B_3\cos\phi - A_1\sin\phi\,.\end{aligned} \tag{53}$$

4.4.2. Transformations of $\boldsymbol{R^{\sigma}_m}$

Let us now see how R^{σ}_m transform under finite genuine SU(4) rotations generated by $F^{\sigma}_{\pm 2}$. For transformations (46) generated by F^{σ}_{-2}, we find from Eq. (40):

$$e^{+i\phi F^{\sigma}_{-2}} = 1 + (\cos\phi - 1)(F^{\sigma}_{-2})^2 + i\sin\phi\,F^{\sigma}_{-2}\,. \tag{54}$$

Therefore

$$\begin{aligned}e^{+i\phi F^{\sigma}_{-2}}R^{\sigma}_m &= R^{\sigma}_m\,e^{+i\phi F^{\sigma}_{-2}} + (\cos\phi - 1)[(F^{\sigma}_{-2})^2, R^{\sigma}_m]\\ &\quad + i\sin\phi\,[F^{\sigma}_{-2}, R^{\sigma}_m]\,.\end{aligned} \tag{55}$$

With the commutators on the r.h.s. above vanishing for $m = 2$ (see Eq. (45)), we find that

$$\tilde{R}^{\sigma}_2 = R^{\sigma}_2\,. \tag{56}$$

On the other hand, for $m \neq 2$ we obtain

$$e^{+i\phi F^\sigma_{-2}} R^\sigma_m e^{-i\phi F^\sigma_{-2}} = R^\sigma_m - 2\sin\phi F^\sigma_{+2}(1+(\cos\phi-1)(F^\sigma_{-2})^2 - i\sin\phi F^\sigma_{-2})\,. \quad (57)$$

Since

$$\begin{aligned} F^\sigma_{+2}F^\sigma_{-2} &= \frac{i}{2}(R^\sigma_1 + R^\sigma_3)\,, \\ F^\sigma_{+2}(F^\sigma_{-2})^2 &= F^\sigma_{+2}(F^\sigma_{+2})^2 = F^\sigma_{+2}\,, \end{aligned} \quad (58)$$

we finally obtain for $m \neq 2$:

$$\tilde{R}^\sigma_m = R^\sigma_m - \sin 2\phi\, F^\sigma_{+2} - \sin^2\phi\,(R^\sigma_1 + R^\sigma_3)\,. \quad (59)$$

For transformations (47) generated by F^σ_{+2} we find in a similar way that

$$\tilde{R}^\sigma_2 = R^\sigma_2\,, \quad (60)$$

and, for $m \neq 2$:

$$\tilde{R}^\sigma_m = R^\sigma_m + \sin 2\phi\, F^\sigma_{-2} - \sin^2\phi\,(R^\sigma_1 + R^\sigma_3)\,. \quad (61)$$

4.4.3. Quark–lepton transformations: $\phi = \pm\pi/2$

As we shall shortly see, transformations (46,47) are of particular interest for $\phi = \pm\pi/2$. For these values of ϕ one obtains either from Eqs. (56), (59), (60), (61), or directly from Eq. (7) and Eqs. (52), (53) that for both $F^\sigma_{-2}-$ and $F^\sigma_{+2}-$generated transformations one has the same result:

$$\begin{aligned} \tilde{R}^\sigma_1 = \tilde{y}_1 \otimes \sigma_3 &= -y_3 \otimes \sigma_3 = -R^\sigma_3\,, \\ \tilde{R}^\sigma_2 = \tilde{y}_2 \otimes \sigma_3 &= +y_2 \otimes \sigma_3 = +R^\sigma_2\,, \\ \tilde{R}^\sigma_3 = \tilde{y}_3 \otimes \sigma_3 &= -y_1 \otimes \sigma_3 = -R^\sigma_1\,. \end{aligned} \quad (62)$$

Applying these transformations to Eqs. (16) we find (after the same diagonalisation procedure) that

$$\begin{aligned} \tilde{y}_1 &\to \begin{bmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & -1 \end{bmatrix}, \\ \tilde{y}_2 &\to \begin{bmatrix} +1 & & & \\ & +1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}, \\ \tilde{y}_3 &\to \begin{bmatrix} +1 & & & \\ & -1 & & \\ & & +1 & \\ & & & -1 \end{bmatrix}, \end{aligned} \quad (63)$$

and

$$\tilde{Y} \rightarrow \begin{bmatrix} +\frac{1}{3} & & & \\ & +\frac{1}{3} & & \\ & & +\frac{1}{3} & \\ & & & -1 \end{bmatrix} \otimes \sigma_0 \,, \qquad \leftarrow \text{colour } \#'s \; \left\{ \begin{array}{l} 1 \\ 3 \\ 2 \\ 0 \text{ (lepton)} \end{array} \right. . \quad (64)$$

Comparing this with Eq. (17) we see that

$$Y = \frac{1}{3}(y_1 + y_2 + y_3) \otimes \sigma_0 \rightarrow \tilde{Y} = \frac{1}{3}(-y_1 + y_2 - y_3) \otimes \sigma_0 \,, \qquad (65)$$

i.e. (1) the signs in front of y_1 and y_3 are changed,
(2) the lepton subspace ($Y = -1$), and that of quark #2 are interchanged, and
(3) the subspaces #1 and #3 are unchanged.

Thus, finite SU(4) transformations generated by $F^{\sigma}_{\pm n}$ (and defined in analogy to Eqs. (46), (47)) interchange quark $\#n$ with lepton for $\phi = \pm\pi/2$.

On the other hand, the U(1)$\otimes$SU(3) transformations obviously cannot change the sign in front of y_k. Indeed, take *e.g.* $k = 1$. Then, as may be checked by looking at explicit formulas for F^{σ}_a gathered in [3], transformations generated by F^{σ}_a with $a = 3, 7, 8$ (and also R^{σ}) leave y_1 invariant as for these values of a one has $[F^{\sigma}_a, y_1] = 0$. In order to see what happens for other values of a, consider $a = 2$, *i.e.* $F^{\sigma}_2 = S_3$ (see [3]). Then, one calculates that

$$e^{i\phi S_3}\, y_1\, e^{-i\phi S_3} = y_1 \cos^2\phi + y_2 \sin^2\phi - \sin\phi\cos\phi\,(\sigma_1 \otimes \sigma_2 + \sigma_2 \otimes \sigma_1), \quad (66)$$

and for no value of ϕ one can obtain $-y_1$ or $-y_2$ on the r.h.s.. The action of the remaining SU(3) generators may be analysed analogously.

5. SO(6) transformations

5.1. General case

The F^{σ}_{-2}-generated transformations of equations (52) will leave the linearised form

$$A_k p_k + B_k x_k \qquad (67)$$

invariant, provided x_k and p_k transform accordingly (with $\boldsymbol{p}$ and $\boldsymbol{x}$ rotating in opposite senses):

$$\begin{aligned} \tilde{p}_1 &= p_1 \cos\phi - p_3 \sin\phi \,, \\ \tilde{p}_2 &= p_2 \,, \\ \tilde{p}_3 &= p_3 \cos\phi + p_1 \sin\phi \,, \end{aligned}$$

$$\begin{aligned}\tilde{x}_1 &= x_1 \cos\phi + x_3 \sin\phi\,,\\ \tilde{x}_2 &= x_2\,,\\ \tilde{x}_3 &= x_3 \cos\phi - x_1 \sin\phi\,.\end{aligned}\tag{68}$$

We now calculate the form of position and momentum commutation relations in new variables. For the position–position and momentum–momentum relations we find:

$$[\tilde{x}_k, \tilde{x}_l] = [\tilde{p}_k, \tilde{p}_l] = 0\tag{69}$$

for any k, l. Thus, the new positions (new momenta) commute among themselves.

The diagonal position–momentum commutation relations are:

$$\begin{aligned}[\tilde{x}_1, \tilde{p}_1] = [\tilde{x}_3, \tilde{p}_3] &= i\cos 2\phi\,,\\ [\tilde{x}_2, \tilde{p}_2] &= i\,,\end{aligned}\tag{70}$$

while the nondiagonal ones are:

$$\begin{aligned}[\tilde{x}_1, \tilde{p}_2] = [\tilde{x}_2, \tilde{p}_1] = [\tilde{x}_2, \tilde{p}_3] = [\tilde{x}_3, \tilde{p}_2] &= 0\,,\\ [\tilde{x}_1, \tilde{p}_3] = -[\tilde{x}_3, \tilde{p}_1] &= i\sin 2\phi\,,\end{aligned}\tag{71}$$

or, in matrix form:

$$[\tilde{x}_k, \tilde{p}_l] = i\Delta_{kl}\,,\tag{72}$$

with

$$\Delta = \begin{bmatrix} \cos 2\phi & 0 & \sin 2\phi \\ 0 & 1 & 0 \\ -\sin 2\phi & 0 & \cos 2\phi \end{bmatrix}.\tag{73}$$

For the F^{σ}_{+2}−generated transformations the analogs of Eqs. (68) are:

$$\begin{aligned}\tilde{p}_1 &= p_1 \cos\phi + x_3 \sin\phi\,,\\ \tilde{p}_2 &= p_2\,,\\ \tilde{p}_3 &= p_3 \cos\phi - x_1 \sin\phi\,,\\ \tilde{x}_1 &= x_1 \cos\phi + p_3 \sin\phi\,,\\ \tilde{x}_2 &= x_2\,,\\ \tilde{x}_3 &= x_3 \cos\phi - p_1 \sin\phi\,.\end{aligned}\tag{74}$$

The new position–position and momentum–momentum commutation relations are:

$$[\tilde{x}_1, \tilde{x}_2] = [\tilde{x}_2, \tilde{x}_3] = [\tilde{p}_1, \tilde{p}_2] = [\tilde{p}_2, \tilde{p}_3] = 0\,,\tag{75}$$

$$[\tilde{x}_3, \tilde{x}_1] = [\tilde{p}_1, \tilde{p}_3] = i\sin 2\phi\,,\tag{76}$$

while for the position-momentum commutation relations one obtains Eq. (72) with

$$\Delta = \begin{bmatrix} \cos 2\phi & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cos 2\phi \end{bmatrix}. \tag{77}$$

5.2. The diagonality condition

If, instead of requiring that $\Delta = 1$ (which is the case of the three-dimensional harmonic oscillator) one imposes a weaker (*i.e.* more general) condition that the new positions (new momenta) commute among themselves, and that the position-momentum commutation relations stay diagonal, for both F^{σ}_{-2}- and F^{σ}_{+2}-generated transformations one needs that $\sin 2\phi = 0$, *i.e.*

$$2\phi = 0\,,\ \pm\pi\,, \ldots\,, \tag{78}$$

or

$$\phi = \pm\frac{\pi}{2}\,,\ \pm\frac{3\pi}{2}\,. \tag{79}$$

The cases with $\phi = 0$ or with $\phi = \pm\pi$ are trivial (the latter is equivalent to an S_2-generated "ordinary" rotation by $\pm\pi$, see Eqs. (68), or Eqs. (74)). Since $\pm(3\pi)/2 = \pm\pi/2 \pm \pi$, the case $\phi = \pm(3\pi)/2$ is easily brought to the case $\phi = \pm\pi/2$. The latter is precisely the case of the interchange of lepton and quark # 2, as discussed in the previous section.

For $\phi = +\pi/2$ (the case $\phi = -\pi/2$ is related to $\phi = +\pi/2$ by an S_2-generated ordinary rotation by π) the SO(6) transformation generated by F^{σ}_{-2} reads:

$$\begin{aligned} &\tilde{p}_1 = -p_3\,, && \tilde{p}_2 = p_2\,, && \tilde{p}_3 = +p_1\,, \\ &\tilde{x}_1 = +x_3\,, && \tilde{x}_2 = x_2\,, && \tilde{x}_3 = -x_1\,, \end{aligned} \tag{80}$$

while that generated by F^{σ}_{+2} is:

$$\begin{aligned} &\tilde{p}_1 = +x_3\,, && \tilde{p}_2 = p_2\,, && \tilde{p}_3 = -x_1\,, \\ &\tilde{x}_1 = +p_3\,, && \tilde{x}_2 = x_2\,, && \tilde{x}_3 = -p_1\,. \end{aligned} \tag{81}$$

In both cases

$$\Delta = \begin{bmatrix} -1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{82}$$

In order to bring the above sets of expressions (80,81) to somewhat better-ordered forms we shall use specific U(1)⊗SU(3) transformations. First, we perform an S_2-generated ordinary rotation by $\pi/2$, which commutes with

$F^\sigma_{\pm 2}$-generated transformations and does not affect Δ. For F^σ_{-2} this leads to (keeping the same notation for transformed coordinates):

$$\begin{aligned} \tilde{p}_1 &= +p_1\,, & \tilde{p}_2 &= p_2\,, & \tilde{p}_3 &= +p_3\,, \\ \tilde{x}_1 &= -x_1\,, & \tilde{x}_2 &= x_2\,, & \tilde{x}_3 &= -x_3\,, \end{aligned} \tag{83}$$

while for F^σ_{+2} one gets:

$$\begin{aligned} \tilde{p}_1 &= -x_1\,, & \tilde{p}_2 &= p_2\,, & \tilde{p}_3 &= -x_3\,, \\ \tilde{x}_1 &= -p_1\,, & \tilde{x}_2 &= x_2\,, & \tilde{x}_3 &= -p_3\,, \end{aligned} \tag{84}$$

still satisfying Eq. (72) with Δ as in Eq. (82).

Performing appropriate rotations in p_1–x_1 and p_3–x_3 planes (generated by $R^z_{\bar{1}}$ and $R^z_{\bar{3}}$, *i.e.* by certain linear combinations of U(1)$\otimes$SU(3) generators R^z, F^z_3, and F^z_8, see [3]), both (83) and (84) may be brought to the same form with "+" signs everywhere:

$$\begin{aligned} \tilde{p}_1 &= x_1\,, & \tilde{p}_2 &= p_2\,, & \tilde{p}_3 &= x_3\,, \\ \tilde{x}_1 &= p_1\,, & \tilde{x}_2 &= x_2\,, & \tilde{x}_3 &= p_3\,, \end{aligned} \tag{85}$$

which still satisfy Eq. (72) with Δ as before.

In paper [2] it was pointed out that there exist four (*i.e.* 3+1) ways in which six coordinates of phase space may be divided into two sets of three "position" and three "momentum" coordinates. Furthermore, Ref. [2] speculated that these four possibilities may be related to the issue of mass and the existence of three coloured quarks and one lepton (as present in a single generation of the Standard Model when isospin is neglected). The above considerations show the precise way in which the original argument of [2] is connected with the quark–lepton hypercharge structure derived in Ref. [3] from the analysis of the Dirac-like linearisation of phase-space invariant $\boldsymbol{x}^2+\boldsymbol{p}^2$. In brief, for a given value of I_3, the transformations between a quark and a lepton correspond to those genuine SO(6) rotations in phase space which keep momentum–position commutation relations diagonal (but not identical).

More specifically, the four possible ways of dividing phase-space coordinates into two sets — one of canonical momenta $\boldsymbol{p}$ and one of canonical positions $\boldsymbol{x}$, correspond to four possibilities for matrix Δ in the commutation relations of standard momenta $\boldsymbol{p}$ and positions $\boldsymbol{x}$:

$$[x_k, p_l] = i\Delta_{kl}\,. \tag{86}$$

These four possibilities for Δ are as follows:

$$\begin{bmatrix} +1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & +1 \end{bmatrix}, \begin{bmatrix} +1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & +1 \end{bmatrix}. \tag{87}$$

Following an example given in Eq. (85), an appropriate redefinition of x_k's and p_k's brings Eqs. (86), (87) to the form when the canonical momenta and positions — as defined in Eqs. (1), (2) by the division of the set of $(x_1, x_2, x_3, p_1, p_2, p_3)$ into two groups — satisfy standard commutation relations with $+i\delta_{kl}$ everywhere on the r.h.s of position–momentum commutation relations. This is the precise connection between the general ideas of [2] and the more technical results of [3].

6. Isospin and reflections in phase space

In the previous section we discussed transformations between the subspaces corresponding to different ways of obtaining the eigenvalues for Y, *i.e.*

$$\begin{aligned}
(-1-1-1)/3 &= -1\,, \\
(-1+1+1)/3 &= 1/3\,, \\
(+1-1+1)/3 &= 1/3\,, \\
(+1+1-1)/3 &= 1/3\,.
\end{aligned} \tag{88}$$

In a similar spirit, in this subsection we shall discuss transformations between the subspaces corresponding to different eigenvalues of $I_3 = \pm 1/2$. These transformations are effected *e.g.* by

$$I_{\pm} \equiv \sigma_0 \otimes \sigma_0 \otimes \sigma_1\,, \tag{89}$$

which interchanges $I_3 = \pm 1/2$:

$$I_3 \rightarrow \tilde{I}_3 = I_{\pm}\; I_3\; I_{\pm} = -I_3\,. \tag{90}$$

The corresponding transformations of A_k and B_k are:

$$A_k \rightarrow \tilde{A}_k = A_k, \qquad B_k \rightarrow \tilde{B}_k = -B_k\,. \tag{91}$$

If the linearised form $A_k p_k + B_k x_k$ is to be invariant, the following are the corresponding transformations in phase space:

$$\tilde{\boldsymbol{p}} = \boldsymbol{p}, \qquad \tilde{\boldsymbol{x}} = -\boldsymbol{x}\,. \tag{92}$$

Had we used $\sigma_0 \otimes \sigma_0 \otimes \sigma_2$ in place of $\sigma_0 \otimes \sigma_0 \otimes \sigma_1$ we would have ended up with

$$\tilde{\boldsymbol{p}} = -\boldsymbol{p}, \qquad \tilde{\boldsymbol{x}} = \boldsymbol{x}\,. \tag{93}$$

In both cases we get overall reflections in either momentum or position space. In both cases we also have:

$$i \rightarrow \tilde{i} = i\,, \tag{94}$$

since no complex conjugation is performed. Therefore, in both cases we obtain that if

$$[x_k, p_l] = i\delta_{kl}\,, \tag{95}$$

corresponds to sector with $I_3 = +1/2$, then

$$[x_k, p_l] = -i\delta_{kl} \tag{96}$$

corresponds to sector with $I_3 = -1/2$ (compare Eqs. (86), (87). Consequently, we see that the isospin degree of freedom corresponds to the freedom of choosing either $+i$ or $-i$ on the r.h.s. of position–momentum commutation relations. It should be strongly stressed that this freedom of choice of the sign in front of i on the r.h.s of commutation relations is *not* related to complex conjugation as used in the operation of charge conjugation. Namely, when one performs the operation of charge conjugation (*c.f.* Eq. (20)) one takes complex conjugation $i \to i^* = -i$ on *both* sides of the original commutation relations, *i.e.*

$$[x_k, p_l] = [i\frac{d}{dx_l}, x_k] = i\delta_{kl}\,, \tag{97}$$

goes into

$$[x_k, -p_l] = [-i\frac{d}{dx_l}, x_k] = -i\delta_{kl}\,, \tag{98}$$

with *both* the relative sign between $\boldsymbol{p}$ and $\boldsymbol{x}$, *and* the sign in front of i on the r.h.s. changed. In this way one goes from $\boldsymbol{p} - e\boldsymbol{\mathcal{A}}$ to $-\boldsymbol{p} - e\boldsymbol{\mathcal{A}}$ (for real vector potential $\mathcal{A}$), and, at the same time, from a given representation to its complex conjugate. Thus, in charge conjugation operation, the canonical commutation relations are in fact unchanged, as may be seen by multiplying both sides of (98) by -1.

On the other hand, in the transformation between the $I_3 = \pm\frac{1}{2}$ sectors the commutation relations are changed. In fact, it is the very definition of what we mean by i that in some sense seems to be related to the possibility of the existence of the isospin degree of freedom. Namely, the only definition of i that we have is that $i^2 = -1$. This, however, is satisfied both by i and by $-i$, and, therefore, my "i" may be somebody's "$-i$". As long as one neglects weak interactions (which change I_3), the sector in which the imaginary unit $\mathcal{I}$ used on the right hand side of commutations relations is defined as $+i$ is fully disjoint from the other sector in which $\mathcal{I}$ is taken as $-i$. Then, in the other sector, one may freely replace $-i$ by $+i$ and obtain the familiar formalism with $\mathcal{I} = +i$ everywhere. In the real world, however, the replacement of i by $-i$ is obviously not allowed as weak interactions cannot be switched off.

The analogous substitution of i in place of $-i$ in the quark sector could be admitted only if one "switches off" strong interactions, which, contrary to the weak case, is obviously not an acceptable approximation to Nature.

When the isospin degree of freedom is taken into account, the Δ matrix present on the r.h.s. of Eq. (86), in addition to the possibilities given in Eq. (87), may acquire the following forms as well:

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & +1 \end{bmatrix}, \begin{bmatrix} +1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & +1 \end{bmatrix}, \begin{bmatrix} +1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \quad (99)$$

The eight possibilities of Eqs. (87), (99) correspond to the possibility of freely choosing $\mathcal{I}$ to be $+i$ or $-i$ "separately in each of the three directions", and to the existence, in a given Standard Model generation, of eight different quarks and leptons.

7. Summary

The original paper [2] addressed the problem of mass, arguing for such a treatment of positions and momenta, in which mass could be associated not only with momentum (as in the standard relativistic or nonrelativistic expressions for the energy of a free particle), but also with position. The arguments formulated therein suggested that there should be three counterparts to the standard case in which mass is associated with the division of the six canonical variables $(p_1, p_2, p_3, x_1, x_2, x_3)$ into momentum (p_1, p_2, p_3) and position (x_1, x_2, x_3). The three cases in question correspond to three additional possible divisions of the set of phase-space variables into pairs of canonically conjugated momenta and positions, *i.e.*: $\{(p_1, x_2, x_3), (x_1, p_2, p_3)\}$, $\{(x_1, p_2, x_3), (p_1, x_2, p_3)\}$, and $\{(x_1, x_2, p_3), (p_1, p_2, x_3)\}$.

In Ref. [3] a linearization procedure *á la* Dirac was applied to $\boldsymbol{x}^2 + \boldsymbol{p}^2$ with $\boldsymbol{x}$ and $\boldsymbol{p}$ satisfying standard commutation relations. The U(1)$\otimes$SU(3) phase-space symmetry was thus represented in the relevant Clifford algebra. The eigenvalues of the U(1) generator Y were shown to be $(+1/3, +1/3, +1/3, -1)$, as needed for the description of a weak hypercharge for three coloured quarks and one lepton. Furthermore, a formula identified with the Gell-Mann–Nishijima–Glashow formula $Q = I_3 + Y/2$ was derived.

In the present paper we established a connection between the approaches of [2] and [3]. Symmetries of the nonrelativistic quantum phase space were correlated with SU(4) and weak isospin quark–lepton transformations. We have shown how the commutation relations should be imposed — within each of the four possible divisions of the set of six phase-space variables into canonical momentum and position — in order that a proper correspondence with the charge, weak hypercharge and isospin quantum number structure of Ref. [3] is achieved.

In summary there are two ways of looking at the results obtained. The first one is that we keep standard meaning for momenta and positions, but allow the i's appearing on the r.h.s. of commutation relations to be $\pm i$, independently for each of the three directions. The other is that we keep the standard $+i$ on the r.h.s. of all commutation relations, but write them in terms of canonical momenta and positions which are obtained from the standard ones by appropriate redefinitions, realizing in this way the ideas of [2].

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