SHELL MODEL OF THE BIG BANG IN THE SPECIAL-RELATIVITY FRAMEWORK

A. Czachor

Institute of Atomic Energy, Świerk
05-400 Otwock, Poland
a.czachor@cyf.gov.pl

(Received March 6, 2007; revised version received April 25, 2007)

The equation of motion of massive spherical shell expanding in the field of its own gravitational potential has been solved within the special relativity mechanics, assuming fixed total energy of such system. The initial velocity of such "shell-universe" is always finite and equal to the velocity of light. When the total energy is less than the rest mass energy of the shell, the expansion terminates in time and the shell collapses, while otherwise it expands indefinitely; at long times it resembles the Friedman model of universe. For zero total energy the shell radius goes in time t as $\sin(\Omega t)$, where the "frequency" Ω is proportional to the rest mass of the shell. A given "lifetime" of the expansion-terminated shell universe can be achieved in two ways: "grand" or "small" expansion scenarios. Another version of the model, relying explicitly on the energy–gravitational-mass equivalence, leads to similar (but not identical) predictions. The predictions of the model are compared with the predictions of the GRT "dust shell" model. Possible impact of this special relativity model of expanding universe on its general relativity counterpart is suggested.

PACS numbers: 98.80.Bp, 98.80.Jk, 98.80.Es

1. Introduction

The theory of expanding universe, as given in the textbooks [1, 2] on the Einstein's general relativity theory (GRT), generally suffers an evident shortcoming originating yet from the famous Friedman paper [3, 4] — the initial velocity of the expansion is infinite. It seems, this feature is the result of the convention, that the GRT formulation should in proper limit converge to the expansion pattern following from the classic Newtonian mechanics, also showing such feature.

The dynamics of the spherical "dust sphere" has been first treated by Israel [5], within the GRT. The initial expansion of such a sphere also tends to occur with infinite velocity, although the reason for it is in this case less obvious and somewhat hidden in the derivation procedures. There have been afterwards many developments in the field of the general-relativity spherical-shell-models — they show several merits for astrophysics and cosmology. We do not enter here the GR part of the story and propose the readers instead a comprehensive revue by Kijowski, Magli and Malafarina [9].

We would like to note, that the problem of singularity of initial expansion, appearing in the GR theories of expanding universe, does not seem to have been clearly resolved till now. It may be an artifact due to approximations involved in solving the Einstein equations of motion, or it is the property inherent to the basic assumption of the universe being isotropic, uniform and spherical. There is also the possibility that this initial singularity cannot be avoided as long as the gravitation field and the matter are in the theory two separate notions [4].

One should add here a comment on the inflation process as the beginning of the Universe expansion, in times of the order of 10^{-33} s. Being exponential, the expansion velocity in this model soon exceeds the velocity of light c. One should therefore ask how it compares to the initial velocity of expansion in the present model, which is just c.

The answer is that they cannot be compared at all — the models belong to different logical categories. In our SRT model one considers the motion of a material sphere expanding in absolute time t, and there at the beginning we have the expansion linear in time, $R = ct$.

The inflation phenomenon has now a rich literature, but it is present already in the de Sitter GRT model of isotropic uniform static empty universe [10], derived at the beginning of the General Relativity era. According to Gribbin [11], in the inflation process "it is the spacetime itself that is expanding, carrying matter along for the ride". Following Möller [1], in the de Sitter model the particles initially at rest remain at rest during expansion, although distances between them exponentially increase, $R \approx \exp(t/t_0), t_0$ being of the order of 10^{-33} s. Gribbin emphasizes, that the inflation theory predicts just such small fluctuations in the otherwise uniform background radiation and in the distribution of galaxies, as have been found in recent astronomic observations of most early Universe using the COBE telescope.

In this paper we carry on the exact calculation of the expansion of material shell within the special relativity theory, relying on the energy conservation law. Although this approach is simple-minded, its predictions are reasonable in the whole time and size domain, except perhaps the very early inflation period.

2. The spherical shell model of special relativity — the rest mass version (I)

Using the special-relativity theory (SRT) one can set up an interesting and simple analogue of the general-relativity big-bang model of universe, showing a finite initial velocity of expansion. We consider here such SRT model: the expanding spherical material shell, similar to the above mentioned dust-sphere GRT model of Israel. Assuming the rest mass M_0 and the radius r of the sphere, its gravitational potential energy is $-kM_0^2/r$, where k is the gravitational constant.

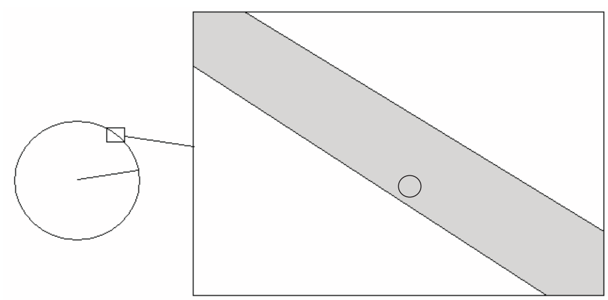

Fig. 1. Circle — the expanding shell of radius r. Rectangular — a magnified piece of the shell, showing its finite thickness. The shade represents the mass distribution within the shell, whereas the small circle immersed in the shade might be the visible universe.

Let us consider radial movement of such spherical shell (see Fig. 1) of the radius $r(t)$ and radial velocity $v(t) = dr(t)/dt$, where t is time. Within the SRT the total energy of such a system is

$$E = \frac{M_0 c^2}{\sqrt{1-(v/c)^2}} - k\frac{M_0^2}{r}\,, \tag{1}$$

where c is the velocity of light. We can introduce dimensionless variables

$$\varepsilon = \frac{E}{M_0 c^2}\,, \quad \beta = \frac{v}{c}\,, \quad x = \frac{r}{R}\,, \quad R = \frac{kM_0}{c^2}\,. \tag{2}$$

One can solve Eq. (1) for the dimensionless radial velocity, to obtain the equation of motion of the shell

$$\beta = \left(1 - \frac{x^2}{(\varepsilon x + 1)^2}\right)^{1/2}\,. \tag{3}$$

One sees immediately that the maximal shell radius corresponding to radial velocity $\beta = 0$ is

$$x_{\max} = \frac{1}{1-\varepsilon}\,, \tag{4}$$

i.e. the expansion terminates if $\varepsilon < 1$. Fig. 2 shows the behavior of the velocity for several values of the energy ε.

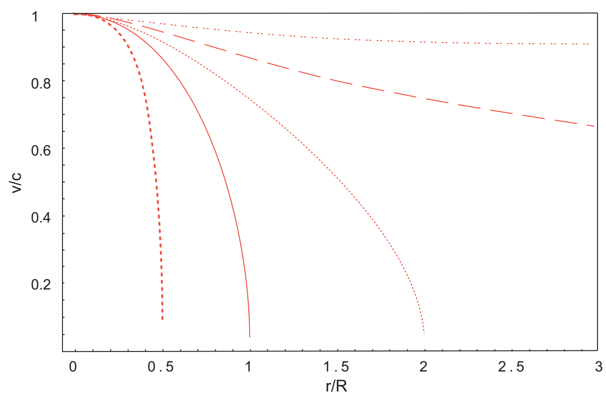

Fig. 2. Radial velocity of the shell, $\beta = v/c$, *versus* the dimensionless radius $x = r/R$, where $R = kM_0/c^2$, according to Eq. (3), for $\varepsilon = -1, 0, 0.5, 1, 1.5$, subsequently from left to right. All pictures below are given for the version I of the model, corresponding to Eq. (1).

To pursue the expansion in time we integrate Eq. (3). Assuming $x = 0$ at $t = 0$, it can be written as

$$\int_0^x \frac{\varepsilon x + 1}{(Ax^2 + Bx + C)^{1/2}}\,dx = \frac{c}{R}\int_0^t dt\,, \tag{5}$$

where

$$A = \varepsilon^2 - 1\,, \quad B = 2\varepsilon\,, \quad C = 1\,.$$

Let us introduce the dimensionless time $\tau = ct/R$. Three cases have to be distinguished:

I. $A < 0\,,$ *i.e.* $\varepsilon < 1$

$$\tau = \frac{\varepsilon}{A}\left[\left(Ax^2 + 2\varepsilon x + 1\right)^{1/2} - 1\right] + (-A)^{-3/2}[\arcsin\varepsilon - \arcsin(Ax+\varepsilon)]\,. \tag{6}$$

II. $A > 0\,,\ i.e.\ \varepsilon > 1$

$$\tau = \frac{\varepsilon}{A}\left[\left(Ax^2 + 2\varepsilon x + 1\right)^{1/2} - 1\right]$$
$$+A^{-3/2}\left[\ln(\sqrt{A} + \varepsilon) - \ln\left(\sqrt{A}\sqrt{Ax^2 + 2\varepsilon x + 1} + Ax + \varepsilon\right)\right] \underset{x\to\infty}{\longrightarrow} x\,. \tag{7}$$

III. $A = 0\,,\ i.e.\ \varepsilon = 1$

$$\tau = \frac{1}{3}\left[(2x+1)^{1/2}(x+2) - 2\right] \underset{x\to\infty}{\longrightarrow} x^{3/2}\,. \tag{8}$$

Fig. 3 shows the expansion of the shell for different values of the energy. One can see that in this model one obtains 3 analogues of the universe history of the big bang type: II — hyperbolic expansion, III — "parabolic" expansion, I — terminated expansion and possible return.

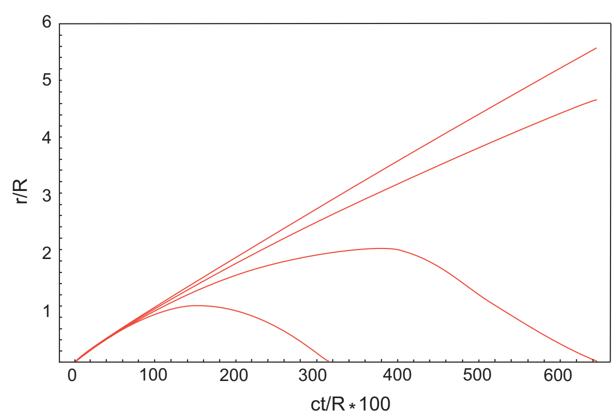

Fig. 3. Dimensionless shell radius $x = r/R$ *versus* the dimensionless time $\tau = ct/R$ plots, for the total energy $\varepsilon \equiv E/(M_0c^2) = 0, 0.5, 1, 1.5$, subsequently, going up from the lowest curve, see Eqs. (6), (7), (8). For each curve the initial velocity $v = c$. The case $\varepsilon = 0$ corresponds to the sinusoidal history given by Eq. (9). Note that the absolute value of the shell velocity $v(t)$ within this model does not exceed c.

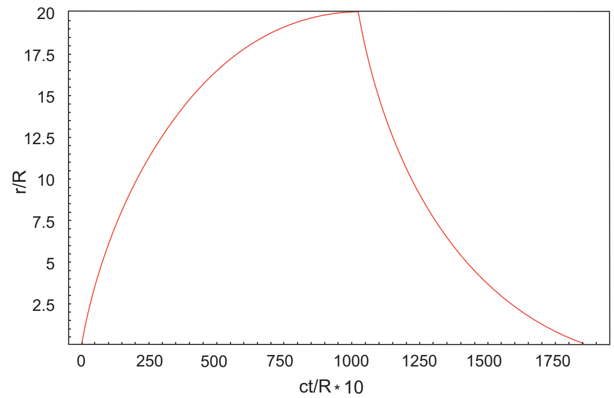

Fig. 4. Shell expansion, notation as in Fig. 3, at $\varepsilon = 0.95$, *i.e.* close to the critical value of the energy parameter $\varepsilon = 1$. The curve is continuous and the pictorial sharp step at maximum is round in proper scale.

3. Sinusoidal history of the Shell Universe

In the class I we have a very interesting case — the Shell Universe with zero total energy, $\varepsilon = 0$, (say, the "cheap" universe). Eq. (6) gives $x = \sin(\tau)$, or:

$$r = R\sin(\Omega t) \tag{9}$$

i.e. the pendulum-like single-swing behavior, with the "frequency"

$$\Omega = \frac{c^3}{kM_0}\,. \tag{10}$$

At $\Omega t = \pi$ such an universe suffers a total collapse, $r = 0$, *i.e.* its age cannot exceed

$$T_0 = \pi\frac{kM_0}{c^3}\,, \tag{11}$$

where the subscript "zero" in T_0 is related to the value $\varepsilon = 0$. One can call this quantity the lifetime of such universe. The history of the energy-less universe depends on its rest mass only.

Let us evaluate the Hubble constant for this special case

$$H \equiv \frac{v(t)}{r(t)} = \Omega\cot(\Omega t)\,. \tag{12}$$

Interestingly, at small t it is the rest-mass independent quantity — just the inverse of the elapsed time.

Of course one can use Eqs. (6)–(8) to write it for general cases. In particular, one easily finds from Eq. (6) for $\varepsilon < 1$ the general formula for the lifetime of such shell-model universe

$$T_\varepsilon = T_0(1 - \varepsilon^2)^{-3/2}(1 + 2\arcsin\varepsilon/\pi)\,. \tag{13}$$

It is interesting to note that the shortest lifetime occurs at the negative energy $\varepsilon_s \cong -0.24411$, see Fig. 5. There follows from it an interesting observation: for a given rest mass M_0 of the shell one can arrive at the same lifetime in the low-energy "small expansion scenario" $-1 < \varepsilon < \varepsilon_s$, and in the "grand expansion scenario" $\varepsilon_s < \varepsilon < 1$, see Eq. (4) and Fig. 6.

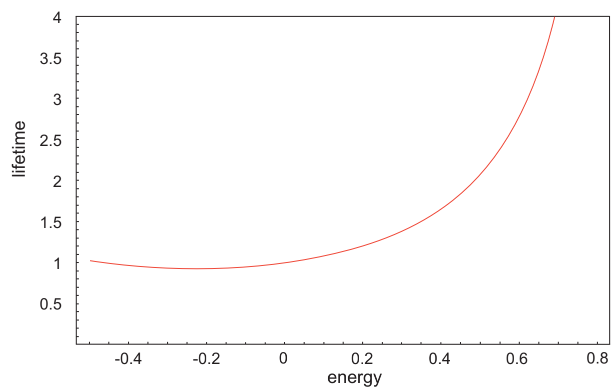

Fig. 5. The lifetime of the shell-model universe, T_ε/T_0, *versus* its total energy $\varepsilon = E/(M_0c^2)$, in the energy range $\varepsilon < 1$, see Eq. (13).

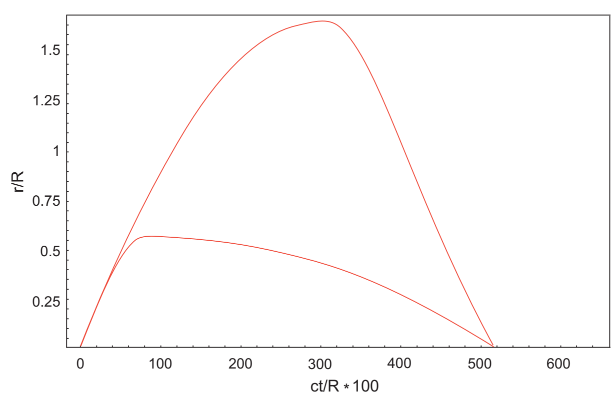

Fig. 6. Two different expansions characterized by the identical lifetime: the "small expansion scenario", $\varepsilon = -0.76$, lower plot, and the "grand expansion scenario", $\varepsilon = 0.4$, upper plot. Notation as in Fig. 3.

4. Shell model of special relativity — the total mass-energy version (II)

Having in mind the Einsten's equivalence of the energy and the gravitational mass, proven *e.g.* for photons by Pound and Rebka [6], one may apply it to formulate another version (say, version II) of the shell model of universe: to put $(E/c^2)^2$ instead of M_0^2 in the gravitation term of Eq. (1). The equation of motion (3) takes on the form

$$\beta = \left[1 - \frac{x^2}{(\varepsilon x + \varepsilon^2)^2}\right]^{1/2} . \tag{14}$$

Looking for $\beta = 0$ one finds that the maximal radius for the closed shell-universe ($\varepsilon < 1$) is now shorter than in the first version

$$x_m = \frac{\varepsilon^2}{1-\varepsilon} . \tag{15}$$

In particular, in this version II at $\varepsilon = 0$ such a system does not exist at all; in a way it is consistent with elementary intuition. Note that both versions coincide at $\varepsilon = 1$. The integration is formally such as before, see Eq. (7), now with $A = \varepsilon^2 - 1$, $B = 2\varepsilon^2$, $C = \varepsilon^4$ and nominator $\varepsilon x + \varepsilon^2$. It follows that, for open universe, $\varepsilon \geq 1$, the indices s in the time dependence of $x \approx t^s$ at $t \to \infty$ are such as in the first version of the model: $s = 2/3$ for $\varepsilon = 1$, $s = 1$ for $\varepsilon > 1$. The lifetime is such as given by Eq. (13), the RHS being multiplied by ε^2. Also in this model the expansion velocity at small times is c.

All above results follow directly from the energy conservation law. With some extra work one can arrive at them via the special-relativity Newton force law, too.

5. A comparison of the SRT and GRT spherical shell models

Within the General Relativity Theory the problem of radial motion of a thin material spherical shell (shell of dust) has been treated by Israel [5], assuming the Schwarzschild metric form for the exterior vacuum region V^+

$$\left(ds^2\right)^+ = f^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) - f dt^2 ,$$

where $f(r) = 1 - 2m/r$ and m is the gravitational mass of the shell, equivalent of the total energy, (*i.e.* E in our notation). For the interior region V^- one obtains the metric by formally putting $m = 0$ in the above formula.

With s being the proper time along the streamlines one deals here with the sphere radius $r = R(s)$ and $\dot{R} = dR/ds$. After meticulous derivations the GRT equation for the shell motion has the form [5]

$$\ddot{R}\left[(1+\dot{R}^2)^{1/2} + \left(1+\dot{R}^2 - \frac{2m}{R}\right)^{1/2}\right] = -\frac{m(1+\dot{R}^2)^{1/2}}{R^2}\,.$$

The first integral of this equation is

$$(1+\dot{R}^2)^{1/2} = a + \frac{m}{2aR}\,,$$

where a is the constant of integration [5]. Its physical content is found to be such that the rest-mass of the shell is m/a, thus a is equivalent to our dimensionless energy ε. Let us just mention here that the final integrations of this GRT equation, "the career of collapsing shell", the hypothesis of elastic reversible rebounding of the sphere from $R = 0$ and related logical problems, are treated in another paper of Israel [7] and many subsequent papers (see [9]).

We shall compare now the essentials of the sphere motion within the SRT and the GRT. One should emphasize, however, that such a comparison can be fair only to a certain degree, because the very notions of time, mass and radius have in the GRT subtler and more diversified meaning than in the SRT — see *e.g.* the comment in the book of Synge on spherical stars [8]. As our comparison is mainly qualitative, we have decided to give it below in a simplistic way.

In the units given by Eqs. (2) we can solve the above equation for $\dot{R} \equiv \beta$ to obtain the GRT analogue of Eqs. (3), (14)

$$\beta = \left[\left(\varepsilon + \frac{1}{2x}\right)^2 - 1\right]^{1/2}\,. \tag{16}$$

One can see that for $x \to \infty$ and $\varepsilon > 1$ the expansion is qualitatively such as in the SRT versions I and II, while for $\varepsilon < 1$ in all cases the expansion terminates and in the GRT case the final radius is finite

$$x_{\max} = \frac{1}{2}\frac{1}{1-\varepsilon} \tag{17}$$

to be compared with Eqs. (4), (15). We can see that this formula is similar to the corresponding SGT formula (4) — they differ only by the constant factor 1/2, which may be related to a different scaling of the sphere radius in both cases.

The predictions are dramatically different at $x \to 0$, because in this limit both SRT versions of the present model lead to reasonable finite value of the velocity of initial expansion, $v \to c$, while in the GRT cases the initial velocity of expansion tends to be infinite. This prediction is basically unphysical.

6. Discussion and conclusions

The shell introduced here need not perhaps be conceived as a mathematical sphere, but rather as a material sphere of the finite thickness, which is small in the scale of the radius r. The observed universe could be hidden within this finite thickness, see Fig. 1. Note, that within the present approach there is no limit on the rest mass M_0 of the shell universe — such big bang explosion may occur for any rest mass.

Besides the demonstration merit, which is of a considerable value, the present model of the universe history leads to some M_0-related numbers: R, T, H, providing scale for cosmic events. As it is the energy conservation law, that makes ground for our considerations, we believe this shell model gives at least a right intuition of the true behavior of the universe expansion.

The Friedman and the Israel approximations and assumptions involved in the derivation of the GRT equations of motion lead to physically acceptable results only for late periods of expansion, when velocities are relatively low. As mentioned in the Introduction, if the gravitation field and matter are treated as independent notions, the problem of singular initial velocity of expansion may be immanent of the General Relativity Theory [4]. It is hoped, that present approach will help formulating the theory free from the initial infinity. The present Author leaves such a task to the GRT specialists.

Finally, let us permit ourselves for a bit of imagination. Due to its simplicity and transparency, the present Special Relativity Newtonian model of expanding universe opens door for some trial extrapolations. Its important extension could be the sequence in time of the big bang events. *E.g.* one can figure out and analyze an appearance of the "new" expanding shell of the rest mass M_1 at the time $t = t_1$ after the "old" one of the rest mass M_0 had started at $t = 0$. The presence of such "inner" mass M_1 slows down the expansion of the M_0 shell, until possibly (if its total energy is high enough) the shell M_1 overpasses the shell M_0, having at this time radial velocity higher than the M_0. The interplay between the initial energies and masses can result in quite a complicated non-monotonic behavior in time of such two-shell system, such as an increase of the Hubble constant at a certain stage of the universe expansion. One should also note that for such a complex system one can still figure out the tempting variant of the "cheap" universe of zero total energy, $\varepsilon = \varepsilon_0 + \varepsilon_1 = 0$, and in such case the shell of the mass M_1 could "borrow" the energy from that of the mass M_0, and vice versa. Such suggestions deserve scrutiny within the general relativity theory, too.

REFERENCES

[1] C. Möller, *The Theory of Relativity*, II edition, Clarendon Press Oxford, 1972.

[2] M. Demiański, *Relativistic Astrophysics*, Polish Scientific Publishers, Warszawa; Pergamon Press, Oxford, New York, Toronto, Sydney, Paris, Frankfurt 1985.

[3] A. Friedman, *Z. Phys.* **10**, 377 (1922).

[4] A. Einstein, *The Meaning of Relativity*, V edition, Princeton University Press, Princeton 1955.

[5] W. Israel, *Nuovo Cimento* **44B**, 1 (1966).

[6] R.V. Pound, G.A. Jr Rebka, *Phys. Rev. Lett.* **4**, 357 (1960).

[7] W. Israel, *Phys. Rev.* **153**, 1388 (1967).

[8] J.L. Synge, *Relativity: the General Theory*, North-Holland Publishing Company, Amsterdam 1960.

[9] J. Kijowski, G. Magli, D. Malafarina, *Gen. Relativ. Gravitation* **38**, 1697 (2006).

[10] W. de Sitter, *Mon. Not. Roy. Astr. Soc.* **76**, 699 (1916); *Mon. Not. Roy. Astr. Soc.* **77**, 155 (1916); *Mon. Not. Roy. Astr. Soc.* **78**, 3 (1917).

[11] J. Gribbin, Cosmology for Beginners, 2007,
`http://www.lifesci.sussex.ac.uk/home/John_Gribbin/cosmo.htm`