LARGE-N_C ESTIMATE OF THE CHIRAL LOW-ENERGY CONSTANTS*

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Chiral low-energy constants incorporate short-distance information from the dynamics involving heavier degrees of freedom not present in the chiral Lagrangian. We have studied the contribution of the lightest resonances to the chiral low-energy constants, up to $\mathcal{O}(p^6)$, within a systematic procedure guided by the large-N_C limit of QCD and also including short-distance asymptotic constraints.

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1. Introduction

Chiral symmetry of massless Quantum Chromodynamics has become the key tool in the study of the very low-energy domain of strong interactions (typically $E \sim M_\pi$), where QCD turns non-perturbative. It is indeed the guiding principle in the construction of Chiral Perturbation Theory (χPT), the effective field theory of QCD in this energy region [1–3]. The χPT Lagrangian has a perturbative structure guided by powers of external momenta and light quark masses. It involves the multiplet of pseudoGoldstone bosons, *i.e.* pseudoscalar mesons (π, K, η), and classical auxiliary fields. The theory, up to a fixed order in the expansion $\mathcal{O}(p^n)$, can be obtained by a construction guided by chiral symmetry:

$$\mathcal{L}_{\chi\mathrm{PT}} = \mathcal{L}_2^{\chi\mathrm{PT}} + \mathcal{L}_4^{\chi\mathrm{PT}} + \mathcal{L}_6^{\chi\mathrm{PT}} + \ldots + \mathcal{L}_n^{\chi\mathrm{PT}} \,. \tag{1}$$

$\mathcal{L}_2^{\chi\mathrm{PT}}$ embodies the spontaneous breaking of the chiral symmetry and depends only on two parameters: F, the decay constant of the pion, and $B_0 F^2 = -\langle 0|\overline{\psi}\psi|0\rangle$, the vacuum expectation value of the light quarks; both

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of them in the chiral limit. Higher orders in the expansion bring in the information from short-distance contributions that have been integrated out, for instance resonance states. As in any effective field theory (EFT) this information is incorporated into the low-energy constants (LECs) that weight the operators of the theory:

$$\mathcal{L}_4^{\chi\mathrm{PT}} = \sum_{i=1}^{10} L_i \mathcal{O}_i^{(4)} \,, \qquad \mathcal{L}_6^{\chi\mathrm{PT}} = \textstyle\sum_{i=1}^{90} C_i \mathcal{O}_i^{(6)} \,, \tag{2}$$

for SU(3). Explicit expressions for the operators can be read from Refs. [3,4]. The scale that specifies the chiral expansion, $\Lambda_\chi \sim M_V$ (being M_V the mass of the $\rho(770)$, the lightest hadron not included in the theory), indicates that LECs in χPT should receive contributions from the energy regime at or above that scale [5]. The determination of the contributions of the lightest multiplets of resonances to the $\mathcal{O}(p^4)$ LECs in $\mathcal{L}_4^{\chi\mathrm{PT}}$ [6] has shown that they indeed saturate the values extracted from the phenomenological analyses. As a consequence it is reasonable to think that the most important contribution to the LECs is provided by the energy region immediately above the integrated scale ($E \sim \Lambda_\chi$).

2. The role of resonance chiral theory

As illustrated in the $\mathcal{O}(p^4)$ case [6,7] a procedure to systematically determine the resonance contributions to the LECs in χPT is available. Essentially the idea is to construct a Lagrangian theory in terms of resonances, pseudoscalar mesons and auxiliary fields respecting the underlying chiral symmetry. Then, upon integration of the heavier states, the χPT Lagrangian is recovered. The outcome of this first step is that LECs are traded by the equally unknown couplings of the resonance Lagrangian though, at this point, one may also notice relations between LECs. In a second stage information on the resonance couplings is obtained, whether from phenomenology or, more interestingly, by imposing theoretical constraints from the QCD asymptotic behaviour of form factors or Green functions.

Contrarily to χPT, the lack of a mass gap between the spectrum of light-flavoured resonances and the perturbative continuum prevents the construction of an appropriate EFT to describe the interaction of resonances and pseudoscalar mesons. However there are several tools that allow us to grasp relevant features of QCD and to implement them in an EFT-like Lagrangian model. The two relevant basis are the following:

(i) A theorem put forward by Weinberg [1] and worked out by Leutwyler [8] states that if one writes down the most general possible Lagrangian, including all terms consistent with assumed symmetry principles, and then calculates matrix elements with this Lagrangian to any given order of perturbation theory, the result will be the most general possible S-matrix amplitude consistent with analyticity, perturbative unitarity, cluster decomposition and the principles of symmetry that have been required.

(*ii*) The inverse of the number of colours of the SU(N_C) gauge group can be taken as a perturbative expansion parameter [9]. Large-N_C QCD shows features that resemble, both qualitatively and quantitatively, the $N_C = 3$ case [10]. In practice, the consequences of this approach are that meson dynamics in the large-N_C limit is described by tree diagrams of an effective local Lagrangian involving an infinite spectrum of zero-width mesons.

Both statements can be combined by constructing a Lagrangian theory in terms of SU(3) (pseudoGoldstone mesons) and U(3) (heavier resonances) flavour multiplets as active degrees of freedom. This has been established [6,7,11] systematically and devises what is known as Resonance Chiral Theory (RχT) that shows the following main features:

1. The construction of the operators in the Lagrangian is guided by chiral symmetry for the pseudoGoldstone mesons and by unitary symmetry for the resonances. The general structure of these couplings is:

$$\mathcal{O} = \langle\, R_1 R_2 \dots R_m\, \chi(p^n)\, \rangle \qquad \in \quad \mathcal{L}_{(n)}^{\overbrace{RR\ldots R}^{m}}\,, \tag{3}$$

   where R_j indicates a resonance field and $\chi(p^n)$ is a chiral structured tensor, involving the pseudoGoldstone mesons and auxiliary fields only. Then, the usual chiral counting in χPT [1] represented by the power of momenta can straightforwardly be applied to $\chi(p^n)$. With these settings chiral symmetry is preserved upon integration of the resonance fields and, at the same time, the low-energy behaviour of the amplitudes is guaranteed.

2. Symmetries do not provide information on the coupling constants as these incorporate short-distance dynamics not included explicitly in the Lagrangian. The latter is supposed to bridge between the energy region below resonances ($E \ll M_V$) and the parton regime ($E \gg M_V$). This hypothesis indicates that it should match both regions and it satisfies, by construction, the chiral constraints. To suit the high-energy behaviour one can match, for instance, the OPE of Green functions (that are order parameters of chiral symmetry) with the corresponding expressions evaluated within our theory. In addition the asymptotic

trend of form factors of QCD currents is estimated from the spectral structure of two-point functions or the partonic make-up and it is enforced on the couplings. This heuristic strategy is well supported by the phenomenology [10–14].

With this pattern the content of the theory is, schematically, given by:

$$\mathcal{L}_{\mathrm{R}\chi\mathrm{T}} = \mathcal{L}_2^{\chi\mathrm{PT}} + \sum_n \mathcal{L}_{n>2}^{\mathrm{GB}} + \mathcal{L}_{\mathrm{R}} \, , \tag{4}$$

where $\mathcal{L}_{n>2}^{\mathrm{GB}}$ has the same structure than $\mathcal{L}_4^{\chi\mathrm{PT}}$, $\mathcal{L}_6^{\chi\mathrm{PT}}$, ...in Eq. (2) though with different coupling constants, and $\mathcal{L}_{\mathrm{R}}$ involves terms with resonances and their couplings to pseudoGoldstone modes.

RχT lacks an expansion parameter. There is of course the guide provided by $1/N_C$ that translates into the loop expansion, however there is no counting that limits the number of operators with resonances that have to be included in the initial Lagrangian. However the number of resonance fields to be kept relies fundamentally in the physical system that we are interested in and the maximum order of the chiral tensor $\chi(p^n)$ in Eq. (3) is very much constrained by the required high-energy behaviour.

As commented above large-N_C requires, already at $N_C \to \infty$, an infinite spectrum in order to match the leading QCD logarithms, though we do not know how to implement this in a model-independent way. The usual approach in RχT is to include the lightest resonances because of their phenomenological relevance, though there is no conceptual problem that prevents the addition of a finite number of multiplets. This cut in the spectrum may produce inconsistencies in the matching procedure outlined above [15]. To deal with this one can include more states that may delay the appearance of that problem.

3. Resonance contributions to the $\mathcal{O}(p^6)$ LECs

In Ref. [11] we have constructed the RχT Lagrangian needed to evaluate the resonance contributions to the $\mathcal{O}(p^6)$ LECs in Eq. (2). It has the following structure:

$$\mathcal{L}_{\mathrm{R}\chi\mathrm{T}} = \mathcal{L}_2^{\chi\mathrm{PT}} + \mathcal{L}_4^{\mathrm{GB}} + \mathcal{L}_6^{\mathrm{GB}} + \mathcal{L}_{\mathrm{kin}}^{\mathrm{R}} + \mathcal{L}_{(2)}^{\mathrm{R}} + \mathcal{L}_{(4)}^{\mathrm{R}} + \mathcal{L}_{(2)}^{\mathrm{RR}} + \mathcal{L}_{(0)}^{\mathrm{RRR}} + \mathcal{L}_{\varepsilon} \, . \tag{5}$$

The inclusion of spin-1 resonances has been performed within the antisymmetric tensor formalism. However, there are independent contributions to the $\mathcal{O}(p^6)$ LECs coming from odd-intrinsic-parity couplings involving spin-1 resonances in the Proca formalism. These are included in $\mathcal{L}_{\varepsilon}$. In the above representation it can be shown [7] that, at $\mathcal{O}(p^4)$, all local terms in $\mathcal{L}_4^{\mathrm{GB}}$

have to vanish in order not to upset the asymptotic behaviour of QCD correlators. A corresponding result at $\mathcal{O}(p^6)$ is still lacking but we have also assumed that all the couplings in $\mathcal{L}_6^{\mathrm{GB}}$ are set to zero.

$\mathcal{L}_{\mathrm{R}\chi\mathrm{T}}$ in Eq. (5) involves 6 a priori unknown couplings in $\mathcal{L}_{(2)}^{\mathrm{R}}$, 70 in $\mathcal{L}_{(4)}^{\mathrm{R}}$, 38 in $\mathcal{L}_{(2)}^{\mathrm{RR}}$, 7 in $\mathcal{L}_{(0)}^{\mathrm{RRR}}$ and 3 in $\mathcal{L}_{\varepsilon}$. Some additional work provides an enormous simplification:

(i) Upon integration of resonances not all couplings appear independently in the LECs. In general only several combinations of couplings intervene and to take into account this case one can perform suitable redefinitions of the fields. This procedure spoils in general the high-energy behaviour of the theory but it is correct for the evaluation of the LECs. Indeed the 70 couplings in $\mathcal{L}_{(4)}^{\mathrm{R}}$ reduce to 23.

(ii) The next step is to enforce additional short-distance information, *i.e.* the leading behaviour at large momenta, for two and three-point functions and form factors. This procedure, set in Ref. [7], relies in well-known features of partonic scattering or asymptotic QCD [16]. Two-current correlators and associated form-factors provide 19 new constraints on couplings, while the three-point Green functions studied till now: $\langle$VAP$\rangle$ [13] and $\langle$SPP$\rangle$ [11, 14], give 6 and 5 independent restrictions, respectively.

Hence we can already determine fully the resonance contribution to the $\mathcal{O}(p^6)$ couplings C_{78} and C_{89} (that appear in $\pi \to \ell\nu_\ell\gamma$ and $\pi \to \ell\nu_\ell\gamma^*$, respectively), C_{87} (in $\langle A_\mu A_\nu\rangle$), C_{88} and C_{90} (in $F_V^\pi(q^2)$ and the q^2 dependence of the form factors in $K_{\ell 3}$), C_{38} (in $\langle SS\rangle$) and C_{12} and C_{34} (in $F_V^{\pi,K}(q^2)$ and $f_+^{K^0\pi^-}(0)$).

It is significant to observe that, as shown in the analysis of the $\langle$SPP$\rangle$ Green function, the use of a Lagrangian theory like RχT though involved [11] brings more information (encoded in the symmetries of the Lagrangian) than the use of a parametric ansatz [14].

4. $K_{\ell 3}$ decays: determination of $f_+^{K^0\pi^-}(0)$

$K_{\ell 3}$ decays have the potential to provide one of the most accurate determinations of the V_{us} CKM element. The main uncertainty in extracting this parameter comes from theoretical calculations of the vector form factor $f_+^{K^0\pi^-}(0)$ defined by:

$$\langle\, \pi^-(p)\,|\,\overline{s}\gamma_\mu u\,|\,K^0(q)\,\rangle = f_+^{K^0\pi^-}(t)(q+p)_\mu + f_-^{K^0\pi^-}(t)\,(q-p)_\mu\,, \qquad (6)$$

with $t=(q-p)^2$. Deviations of $f_+^{K^0\pi^-}(0)$ from unity (the octet symmetry limit) are of second order in the SU(3) breaking:

$$f_+^{K^0\pi^-}(0) = 1 + f_{p^4} + f_{p^6} + \ldots \tag{7}$$

The first correction is $\mathcal{O}(p^4)$ in χPT, through one loop calculation and no local terms, and it gives $f_{p^4} = -0.0227$, essentially without uncertainty [17]. At $\mathcal{O}(p^6)$ two-loop, one-loop and local terms contribute and the latter make the determination more uncertain. Loops give $f_{p^6}^{\rm loops}(M_\rho) = 0.0093(5)$ [14,18]. The explicit form for the tree-level contribution is:

$$f_{p^6}^{\rm tree}(M_\rho) = 8\frac{\left(M_K^2-M_\pi^2\right)^2}{F_\pi^2}\left[\frac{\left(L_5^r(M_\rho^2)\right)^2}{F_\pi^2} - C_{12}^r(M_\rho^2) - C_{34}^r(M_\rho^2)\right], \tag{8}$$

that involves LECs of χPT both at $\mathcal{O}(p^4)$ and $\mathcal{O}(p^6)$. In Ref. [14], and using the method outlined in these proceedings, we have determined the contribution of scalar and pseudoscalar resonances to the LECs present in Eq. (8), and we get:

$$f_{p^6}^{\rm tree}(M_\rho) = -\frac{\left(M_K^2-M_\pi^2\right)^2}{2M_S^4}\left(1-\frac{M_S^2}{M_P^2}\right)^2 . \tag{9}$$

As can be seen in Fig. 1, it produces a tiny result: $f_{p^6}^{\rm tree}(M_\rho) = -0.002(12)$

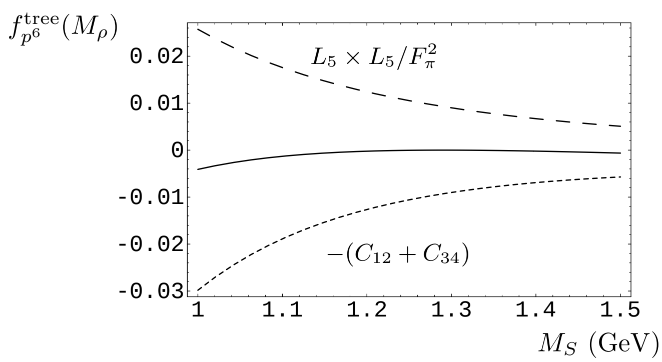

Fig. 1. We display $f_{p^6}^{\rm tree}(M_\rho)$ as a function of M_S for $M_P = 1.3$ GeV (solid line). We also plot the two components: the dashed line represents the term proportional to $L_5 \times L_5$, while the dotted line represents the term proportional to $-(C_{12}+C_{34})$. The cancellation between both contributions is very large.

due to a strong cancellation between both terms. We end up with the final result:

$$f_+^{K^0\pi^-}(0) = 0.984(12) . \tag{10}$$

In Table I we compare our result with other determinations coming from different sources. This comparison shows a clear pattern: both quenched and unquenched lattice results are in very good agreement with the quark model prediction by Leutwyler and Roos [17], while our analytic determination of $f_+^{K^0\pi^-}(0)$ shows a clear tension. It is also interesting to notice the tiny modification that unquenching produces in the lattice results though, as shown in Ref. [18], chiral logarithms are very much important in the determination of $f_+^{K^0\pi^-}(0)$. A better understanding is still required on this issue.

TABLE I

Comparison of different predictions for $f_+^{K^0\pi^-}(0)$. The value quoted for Ref. [18] has been modified as explained in Ref. [14].

Reference	$f_+^{K^0\pi^-}(0)$
Quark model [17]	0.961(8)
Lattice (quenched) [19]	0.960(9)
Lattice (unquenched) [20]	0.968(11)
Lattice (unquenched) [21]	0.961(5)
(Chiral + [17]) [18]	0.971(10)
$K\pi$ scalar f.f. [22]	0.974(11)
Ours [14]	0.984(12)

5. Perspective

The study of the resonance energy region is essential for the understanding of hadron phenomenology driven by non-perturbative QCD: hadronic tau decays, final-state interactions, kaon decays, *etc.*

RχT is a systematic setting that allows to implement known features of QCD into a Lagrangian framework. We have shown that it provides sensible results but much more work is still needed to reach a better understanding both on the phenomenology and on the underlying ideas that join together in this formulation.

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