CHIRAL LOGARITHMS*

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The structure of leading logarithms in chiral perturbation theory was already studied some time ago by S. Weinberg, *Physica* **A96**, 327 (1979) and recently by M. Buchler, G. Colangelo, *Eur. Phys. J.* **C32**, 427 (2003). Because the leading logarithms may generate sizable numerical contributions to observables, it would be very interesting to know them or even sum them up to every order in the chiral expansion. We investigate these possibilities for two specific Green functions in chiral perturbation theory with two flavours, in the chiral limit.

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1. Introduction

We discuss the structure of chiral logarithms in the effective low-energy theory of QCD, chiral perturbation theory (ChPT). To simplify the discussion, we consider the case of two flavours u and d. At a given order in the chiral expansion, the logarithm with the highest power is called *leading logarithm* (LL). We want to address two questions: Is it possible

(1) to calculate the leading logarithm to every order in the chiral expansion?

(2) to sum up the leading logarithms to every order, similarly to summing up leading logarithmic singularities in renormalizable theories?

In the following, we consider the two-point function of two scalar quark currents in the chiral limit $m_u = m_d = 0$,

$$H(s) = i\int dx e^{ipx}\langle 0|TS^0(x)S^0(0)|0\rangle; \quad S^0 = \bar{u}u + \bar{d}d; \quad s = p^2\,,$$

and the scalar form factor $F(s)$,

$$\langle 0|S^0(0)|\pi^i(p)\pi^k(p')\rangle \;=\; \delta^{ik}F(s)\,, \quad s = (p+p')^2\,.$$

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2. Chiral logarithms from unitarity, analyticity and the Roy equations

To answer the first question, we rely on unitarity, analyticity and the Roy equations. Once the LL of the partial wave t_0^0 of $\pi\pi$ scattering is known at chiral order $p^{2(N-2)}$, unitarity of the S-matrix determines the LL of the two-point function $H(s)$ and of the scalar form factor $F(s)$ at chiral order p^{2N} and p^{2N-1}, respectively [3]. For illustration, we note the pertinent relation between $H(s)$ and $F(s)$,

$$\mathrm{disc}\, H(s) = i(2\pi)^4 \sum_n \delta^{(4)}(P_n - p) |\langle 0|S^0(0)|n\rangle|^2 = \frac{3i}{16\pi}|F(s)|^2 + \cdots .$$

In the sum over intermediate states, only two pion intermediate states are relevant for the LL [3]. Furthermore, once the partial waves t_ℓ^I are known at tree-level, invoking unitarity and the Roy equations [4] allows one to calculate the pertinent combination of LLs of t_0^0 used in the unitarity relation for $\mathrm{disc}\, F(s)$. The reason that the Roy equations are used is the following: The partial waves $t_\ell^I(s)$ are analytic in the complex s-plane cut along the positive and the negative real axis and unitarity does not provide sufficient information about the contribution generated by the left-hand cut in t_ℓ^I. We write the low-energy expansion of $H(s)$ and $F(s)$ as

$$\begin{aligned} H(s) &= \frac{B^2}{16\pi^2}\left\{P_0 + P_1 L + P_2 L^2 + \cdots\right\}, \quad L = \ln\left(-\frac{s}{\mu^2}\right), \\ F(s) &= 2B\{T_0 + T_1 L + T_2 L^2 + \cdots\}, \end{aligned}$$

where the coefficients P_i and T_i are polynomials in $N = s/(16\pi^2 F^2)$. Up to five loops, the leading contributions to these polynomials are given by [3]

$$\begin{aligned} P_1 &= -6, & P_2 &= 6N, & P_3 &= -\frac{61}{9}N^2, \\ P_4 &= \frac{68}{9}N^3, & P_5 &= -\frac{140347}{16200}N^4, & T_1 &= -N, \\ T_2 &= \frac{43}{36}N^2, & T_3 &= -\frac{143}{108}N^3, & T_4 &= \frac{15283}{9720}N^4. \end{aligned}$$

Higher-order terms in T_i, P_i are omitted here, because they are suppressed in the chiral counting. Available one- and two-loop results [5–8] together with the renormalization group equation [2] provide a direct check on the polynomials P_i, T_i, with $i \leq 3$.

In the chiral limit, a straightforward use of the Roy equations produces infrared divergences, because $\mathrm{Im}\, t_\ell^I(s)$ behave like s^4 as $s \to 0^+$ for $\ell \geq 2$.

There appear logarithmic and arbitrary power-like divergences in $1/M_\pi$. In the calculation of P_5, the emerging divergence is of the form $s^4 \ln(M_\pi)$. However, this divergence does not affect the five-loop LL, as we explicitly checked using the renormalization group equations for the partial wave t_0^0 at the order p^8. Therefore we expect the method presented here to work to all orders.

3. Renormalizable effective theory for the LLs of ChPT

Thinking about the second question, one runs into the problem that ChPT is a nonrenormalizable theory and there exists no method at present to sum leading logarithmic singularities.

We take the linear sigma model — which is renormalizable — as effective theory for the LLs of ChPT and show why summation still escapes.

We rely on the fact that the generating functionals of the linear sigma model (equipped with additional external fields) in the heavy mass limit and ChPT agree at first nonleading order provided the low-energy constants of ChPT are pertinent functions of the parameters of the linear sigma model [5]. In a first step, we check whether also the two-loop leading logarithm in the linear sigma model agrees with ChPT. To that end, we have to introduce our notation of the linear sigma model. The Lagrangian of the O(4) linear sigma model coupled to external scalar sources reads

$$\mathcal{L} = \frac{1}{2}\partial_\mu \varphi^a \partial^\mu \varphi^a + \frac{m^2}{2}\varphi^a\varphi^a - \frac{g}{4}(\varphi^a\varphi^a)^2 + j^a\varphi^a, \quad a = 0, ..., 3\,.$$

If $m^2 > 0$, the O(4) symmetry is spontaneously broken down to O(3), leading to three Goldstone bosons. In order to expand around the ground state $\varphi_{\mathrm{G}} = (v, \mathbf{0})$ of the spontaneously broken theory, one rewrites the Lagrangian with the shifted fields $\varphi = (\phi + v, \boldsymbol{\pi})$. To every order of the calculation, one has to determine v such that the vacuum expectation value vanishes, $\langle 0|\phi(x)|0\rangle = 0$. To one loop, the parameters have to be renormalized in the following way:

$$g = \mu^{4-d} g_r \Big[1 - 24 g_r \lambda\Big], \quad m^2 = m_r^2\Big[1 - 12 g_r\lambda\Big], \quad \varphi = Z^{\frac{1}{2}}\varphi_R, \quad Z = 1\,,$$

$$d = 4 - 2\varepsilon, \quad \lambda = -\frac{1}{32\pi^2}\left(\frac{1}{\varepsilon} + \Gamma'(1) + \ln(4\pi) + 1\right)\,.$$

We identify the renormalized scalar two-point function

$$G_R^{(2,\mathbf{0})}(s) = iZ\int d^4x e^{ipx}\langle 0|T\phi(x)\phi(0)|0\rangle, \quad s = p^2\,,$$

at small external momenta s as the corresponding quantity of the scalar two-point function $H(s)$ in ChPT. Calculating the leading logarithms to one and two loops in the quantity $G_R^{(2,\mathbf{0})}(s)$ yields [9]

$$G_R^{(2,\mathbf{0})}(s,g_r,m_r^2,\mu) = \frac{1}{2m_r^2}\left[c^{(0)} + c^{(1)}g_r + c^{(2)}g_r^2 + O(g_r^3)\right],$$
$$c^{(1)} = -\frac{3}{16\pi^2}\ln\left(-\frac{s}{\mu^2}\right) + \cdots, \qquad c^{(2)} = \frac{3}{256\pi^4}\frac{s}{m_r^2}\ln^2\left(-\frac{s}{\mu^2}\right) + \cdots,$$

where the ellipsis denote terms with a higher power of s/m_r^2 as well as subleading logarithms. Note that the coefficients $c^{(i)}$ depend on s, m_r and μ and also contain mass logarithms. Comparing with ChPT, one finds that the one- and two-loop leading logarithms agree exactly with ChPT [9]. Given this strong evidence, we assume that the linear sigma model reproduces the LLs of ChPT to every order.

Based on this assumption, we use the renormalization group equation (RGE) for renormalized, Fourier transformed Green functions $G_R^{(k,\boldsymbol{j})}$ with $k\,(j)$ sigma (pion) fields,

$$\left(\mathcal{D} + (k + \sum_{t=1}^{3} j_t)\gamma\right) G_R^{(k,\boldsymbol{j})}(p_i,g_r,m_r^2,\mu) = 0; \quad \boldsymbol{j} = (j_1,j_2,j_3),$$
$$\mathcal{D} = \mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial g_r} - m_r^2\gamma_m\frac{\partial}{\partial m_r^2}, \quad \beta = \mu\frac{\partial}{\partial\mu}g_r = \sum_{k=2}^{\infty}\beta^{(k)}g_r^k,$$
$$\gamma_m = -\frac{1}{m_r^2}\mu\frac{\partial}{\partial\mu}m_r^2 = \sum_{k=1}^{\infty}\gamma_m^{(k)}g_r^k, \quad \gamma = \frac{1}{2}\beta\frac{\partial}{\partial g_r}\log Z = O(g_r^2),$$

to establish recursion relations between the coefficients of the LLs. It is convenient to decompose the coefficients $c^{(k)}$ as

$$c^{(k)} = \sum_{n=0}^{k}\sum_{l=0}^{k-n} a_{l,k-n-l}^{(k)}\left(\frac{s}{m_r^2}\right) L_s^l L_m^{k-n-l}, \quad L_s = \ln\left(-\frac{s}{\mu^2}\right), \quad L_m = \ln\left(\frac{2m_r^2}{\mu^2}\right).$$

As the coefficients $a^{(k)}$ are analytic functions in s/m_r^2, we expand them in a power series

$$a_{k,l}^{(k)} = \sum_{t=0}^{\infty} a_{k,l}^{(k,t)}\left(\frac{s}{m_r^2}\right)^t.$$

The recursion relation for the LLs following from the RGE reads [9]

$$-2Na_{N,0}^{(N,t)} - 2a_{N-1,1}^{(N,t)} + \left((N-1)\beta^{(2)} + (1+t)\gamma_m^{(1)}\right)a_{N-1,0}^{(N-1,t)} = 0. \qquad (1)$$

In this relation, only the leading order results for the β- and γ_m-functions appear. Still, given the LL at order g_r^{N-1}, $a_{N-1,0}^{(N-1,t)}$, it is not possible to calculate the LL at order g_r^N, $a_{N,0}^{(N,t)}$ as there appears a coefficient of a subleading logarithm, $a_{N-1,1}^{(N,t)}$. This troublesome coefficient $a_{N-1,1}^{(N,t)}$ is not accessible by the renormalization group. Recursion relations which connect the coefficients of subleading logarithms do not provide enough information, as can be seen in Fig. 1. Furthermore, these relations only connect the coefficients at the same order in s/m_r^2, whereas the leading logarithm at order g_r^N is proportional to $(s/m_r^2)^{N-1}$.

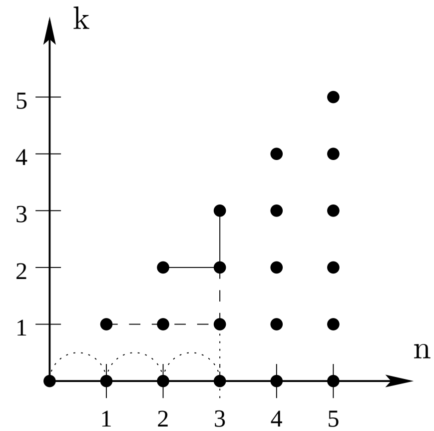

Fig. 1. Illustration of the connections between the coefficients of the scalar two-point function in the spontaneously broken phase of the linear sigma model at order g_r^3. The quantity n represents the order in g_r and k stands for the exponent of the logarithm L_s. Every type of line indicates recursion relations which contain the connected coefficients. The solid line corresponds to Eq. (1). There is one such picture for every order in $\frac{s}{m_r^2}$.

However, a summation of logarithms is still possible, but it is infeasible to single out the contribution from one type of logarithm, like L_s. The situation becomes clear by introducing a new scale ρ and splitting up all mass- and momentum logarithms as

$$L_s = \ln\left(-\frac{s}{\rho^2}\right) + L_\mu, \quad L_m = \ln\left(\frac{2m_r^2}{\rho^2}\right) + L_\mu, \quad L_\mu = \ln\left(\frac{\rho^2}{\mu^2}\right) .$$

Therefore, all terms of the form $g_r^N L_s^k L_m^l$ with $k + l = N$ in the scalar two-point function generate a logarithm L_μ^N. At a given order g_r^N, one is left with one explicitly scale dependent logarithm L_μ with power N. These leading logarithms L_μ can be summed to all orders. The coefficient of the leading L_μ-logarithm however is the sum of all coefficients of terms $g_r^N L_s^k L_m^l$ with $k+l=N$.

4. Summary and conclusion

(i) Once the LLs of the partial wave t_0^0 are given at chiral order p^{2N}, unitarity and analyticity determine the LLs of the scalar form factor $F(s)$ and the scalar two-point function $H(s)$ at chiral order $p^{2(N+1)}$ and $p^{2(N+2)}$, respectively.

(ii) Given the tree-level partial waves t_ℓ^I, unitarity and the Roy equations allow to successively determine the pertinent combination of logarithms in the partial wave t_0^0 used in the unitarity relation mentioned in point *(i)*. Because the infrared singularities appearing in the Roy equations do not alter the five-loop LL, we strongly believe that this procedure works to all orders.

(iii) The linear sigma model reproduces the LLs of the scalar two-point function in ChPT up to and including two loop order.

(iv) Assuming that the identity *(iii)* holds to all orders, we apply the RGE to work out recursion relations for the LLs. These recursion relations, however, also contain subleading terms and cannot be used to calculate the LL at order g_r^N if the LL at order g_r^{N-1} is given.

(v) A summation of logarithms is possible. However, as the scalar two-point function depends on three scales, s, m_r and μ, a separation between the leading momentum logarithms L_s^N and other logarithms to the power N like $L_s^k L_m^l$ with $k+l=N$ is not accessible.

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