MIGDAL'S MODEL AND HOLOGRAPHY*

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Migdal's model on the spectrum of vector mesons is reassessed. We discuss how its departure from a Padé approximant is closely linked to the issue of quark–hadron duality breakdown. We also show that Migdal's model is not truly a model of large-N_c QCD.

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1. Introduction

More than 30 years ago Migdal proposed a model for hadronic resonances based on rational approximants [1]. The final aim was to determine the spectrum of vector mesons in the large-N_c limit which best fulfills quark–hadron duality. The strategy was to start from the short distance behavior of the vector vacuum polarization function $\langle VV \rangle$ and build the Padé approximant thereof. Migdal further suggested to modify the continuum limit to better ensure quark–hadron duality. The final result was a spectrum of single poles located at zeroes of the Bessel J_0 function.

Migdal's model of resonances was recently revived [2] in the context of the AdS–QCD holographic models. The so called *hard wall model* [3] was shown to be the holographic dual of Migdal's model. Interestingly, another holographic model, the so called *soft wall model* [4], was recently identified as the holographic dual of the large-N_c Regge model of Ref. [5].

We will reassess Migdal's model with the help of both the 4-dimensional large-N_c Regge model and the holographic duals. We will see that quark–hadron duality breakdown plays a prominent role and look for its geometrical

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interpretation in holographic models. We will conclude that the modeling of duality breakdown in Migdal's model is not compatible with generic features of large-N_c QCD.

2. Migdal's model

The $[N,M]$ Padé approximant to a function $\Pi_V(q^2)$ is defined as the quotient of two polynomials $\mathcal{P}_M$ and $\mathcal{Q}_N$

$$\Pi_V(q^2) = \Pi_V^{[N,M]}(q^2) + \mathcal{R}_{[N,M]}(q^2)\,, \qquad \Pi_V^{[N,M]}(q^2) \equiv \frac{\mathcal{P}_M(q^2)}{\mathcal{Q}_N(q^2)}\,, \tag{1}$$

such that the first $N+M+1$ derivatives of the function at a point match those of the approximant. $\mathcal{R}_{[N,M]}(q^2)$ is the residue, and convergence of the Padé approximant means that the residue has to vanish as $N, M \to \infty$. Migdal chose to build the symmetric $[N,N]$ Padé approximant for the parton-model logarithm around a point $q^2 = -\mu^2$ in the far Euclidean axis. It is easy to check that the Padé approximant satisfies the following $2N+1$ equations

$$\frac{d^n}{d(q^2)^n}\left[\Pi_V(q^2)\mathcal{Q}_N(q^2) - \mathcal{P}_N(q^2)\right]\Big|_{q^2=-\mu^2} = 0, \qquad n = 0,\ldots,2N\,. \tag{2}$$

When $n \geq N+1$, $\mathcal{P}_N(q^2)$ cancels and the denominator $\mathcal{Q}_N(q^2)$ can be determined from the last N equations. Using that $\Pi_V(q^2)$ obeys a once-substracted dispersion relation, one can show that

$$\int_0^\infty \frac{dt}{(t+\mu^2)^{n+1}}\mathcal{Q}_N(t) = 0, \qquad n = N+1,\ldots,2N\,, \tag{3}$$

which can be shown to lead to Legendre polynomials $P_N^{(0,0)}$,

$$\mathcal{Q}_N(q^2) = {}_2F_1\left(-N,-N;1;-\frac{q^2}{\mu^2}\right) = (q^2+\mu^2)^N\, P_N^{(0,0)}\left(\frac{\mu^2-q^2}{\mu^2+q^2}\right)\,. \tag{4}$$

Eq. (4) can now be plugged back in Eq. (2) to yield [6]

$$\begin{aligned}\Pi_V^N(q^2) \;&\simeq\; \frac{2}{(q^2+\mu^2)^N}\sum_{k=0}^{N}\left(\begin{array}{c} k \\ j \end{array}\right)^2\left[\frac{H_{N-k}-H_k}{P_N^{(0,0)}(\chi)}\right]\left(-\frac{q^2}{\mu^2}\right)^k,\\ \chi \;&=\; \frac{\mu^2-q^2}{\mu^2+q^2}\,,\end{aligned} \tag{5}$$

which is the Padé approximant to the logarithm, *i.e.*, in the continuum limit $N \to \infty$ we recover the logarithm we started from.

2.1. Migdal's limit and quark–hadron duality breakdown

However, in the original approach of Migdal, the continuum limit was taken in a more sophisticated way, namely $q^2 \ll \mu^2$, $N \to \infty$ with N/μ fixed. Under this correlated limit, the Legendre polynomials can be expressed as a Bessel J_0 function

$$\lim_{N\to\infty} P_N^{(0,b)}\left(1-\frac{\xi^2}{2N^2}\right) = J_0(\xi)\,, \tag{6}$$

and the $\langle VV\rangle$ Green function takes the form

$$\begin{aligned} \Pi_V^N(q^2) &= -\frac{4}{3}\frac{N_c}{(4\pi)^2}\left[\log\frac{q^2}{\mu^2} - \pi\frac{Y_0\,(q\Lambda\,)}{J_0\,(q\Lambda\,)}\right], \\ \Lambda &= \frac{2N}{\mu}\,. \end{aligned} \tag{7}$$

It is important to stress that the equation above is no longer the Padé approximant, due to the presence of the Bessel functions. The Padé approximant has the form

$$\Pi_V^N(q^2) = -\frac{4}{3}\frac{N_c}{(4\pi)^2}\log\frac{-q^2}{\mu^2} - \frac{\mathcal{R}_N(q^2)}{\mathcal{Q}_N(q^2)}\,, \tag{8}$$

where the residue is a Meijer G function [6] that vanishes in the continuum limit, thus ensuring convergence. The correlated limits taken by Migdal freeze the residue, thereby spoiling the convergence of the rational approximant. The non-vanishing residue also explains why the resulting spectrum has distinct resonances *after* $N \to \infty$. Migdal's original motivation behind the correlated limits was to enforce quark–hadron duality. However, the correlated limits actually *break* duality. It can be shown that in the far Euclidean region the correlator of Eq. (7) reduces to ($Q^2 = -q^2$)

$$\Pi_V(Q^2) = -\frac{4}{3}\frac{N_c}{(4\pi)^2}\log\frac{Q^2}{\mu^2} + \mathcal{O}(e^{-2Q\Lambda})\,, \qquad (Q^2 \gg 0). \tag{9}$$

The last term is not part of the OPE. It is generically referred to as quark–hadron duality violations, defined as

$$\Pi_V(q^2) \simeq \Pi_V^{\rm OPE}(q^2) + \Delta(q^2)\,, \qquad (q^2 > 0). \tag{10}$$

In Migdal's model, it can be shown that

$$\Delta(q^2) = \frac{N_c}{12\pi}\frac{Y_0\,(q\Lambda\,)}{J_0\,(q\Lambda\,)}\,. \tag{11}$$

In other words, duality is broken and the duality violating pieces collect the singularities of the spectrum. This is exactly the role that duality violations have to play, since the OPE, being a regular expansion, is unable to capture the resonance poles.

There have been some studies devoted to the issue of quark–hadron duality violation [5]. The particular form of Eq. (9) is known to be induced by finite-size singularities. It is important to stress however that this duality breakdown is not what is expected from a large-N_c model of resonances, where infinite-size singularities are present.

3. A large-N_c toy model in 4 dimensions

In the large-N_c limit, the spectral function of the $\langle VV \rangle$ correlator takes the form

$$\frac{1}{\pi}\text{Im}\ \Pi_V(t) = \sum_{n=0}^{\infty} F_n^2 \delta(t - M_V^2(n))\,. \tag{12}$$

We will consider the simplest model in the large-N_c limit with asymptotic Regge behavior, namely [5]

$$F_n^2 = F^2, \qquad M_V^2(n) = m_\rho^2 + an\,. \tag{13}$$

Using dispersion relations, the set of Dirac deltas can be resummed to give

$$\Pi_V(q^2) = \frac{F^2}{a}\left[\psi\left(\frac{m_\rho^2}{a}\right) - \psi\left(\frac{-q^2+m_\rho^2}{a}\right)\right]. \tag{14}$$

The free parameters F and a can be determined by matching to the parton-model logarithm and requiring a vanishing dimension-two condensate

$$c_2 = -F^2\left(\frac{m_\rho^2}{a} - \frac{1}{2}\right). \tag{15}$$

Duality violations can also be determined analytically. Using Eq. (10),

$$\Delta(q^2) = \frac{N_c}{12\pi}\cot\left[\pi\frac{-q^2+m_\rho^2}{a}\right]. \tag{16}$$

which in the Euclidean region decays exponentially as $e^{-|q^2|/\mu^2}$. Notice the difference with Eq. (9).

4. Holographic duals

Both Migdal's model and the toy model introduced in the previous sections have 5-dimensional duals through the AdS–CFT correspondence[1]. Their action can be written as

$$S = -\int d^4x\, dz\, e^{-\Phi(z)}\sqrt{g}\frac{1}{4g_5^2}\mathrm{Tr}\left[(F_{\hat{\mu}\hat{\nu}}F^{\hat{\mu}\hat{\nu}})_{\mathrm{L}} + (F_{\hat{\mu}\hat{\nu}}F^{\hat{\mu}\hat{\nu}})_{\mathrm{R}}\right], \qquad (17)$$

where g is the AdS metric

$$g_{\hat{\mu}\hat{\nu}}\, dx^{\hat{\mu}}dx^{\hat{\nu}} = \frac{1}{z^2}(\eta_{\mu\nu}dx^{\mu}dx^{\nu} + dz^2)\,, \qquad \eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)\,, \qquad (18)$$

and $\Phi(z)$ is the dilaton field. In the *hard wall model*, $\Phi(z) = \phi$, $\epsilon \leq z \leq \Lambda$, where $z = \epsilon$ and $z = \Lambda$ are the four-dimensional boundary branes. In this model, the solution to $\langle VV\rangle$ is given by Eq. (7). Holography therefore provides a nice geometrical interpretation of the infrared scale Λ of Migdal's model.

The so called *soft wall model* was originally proposed as a AdS–QCD model with built-in Regge behavior. The theory does not have an infrared brane, and instead the dilaton field has a quadratic profile, $\Phi(z) \sim cz^2$. The *soft wall model* is the holographic dual to our toy model with $m_\rho^2 = 4c = a$ [4]. Notice that modifications on the infrared of 5-dimensional holographic models correspond to different spectrums and therefore to different violations of quark–hadron duality. In the language of holography a realistic large-N_c spectrum is achieved, as illustrated in the *soft wall model*, by the removal of the infrared brane.

4.1. Some phenomenology of the soft-wall model

The *soft wall model* has two independent parameters, g_5 and c. c is determined by m_ρ and g_5 can be fixed by imposing matching of the model to the perturbation theory logarithm. Thus,

$$g_5^2 = \frac{12\pi^2}{N_c}\,, \qquad c = \frac{m_\rho^2}{4} \simeq 150\,\mathrm{MeV}^2\,. \qquad (19)$$

By matching to our toy model one immediately finds that duality violations take the form [8]

$$\Delta(q^2) = \frac{\pi}{2g_5^2}\cot\left[\pi\frac{-q^2+4c}{4c}\right]. \qquad (20)$$

[1] For a detailed analysis of the relationship between Migdal's model, Padé approximants and holography we refer the reader to [7].

Unfortunately, $m_\rho^2 = a$ predicts a Regge slope a and decay constants F at about half their experimental values. Furthermore, using Eq. (15), the *soft wall model* yields a negative dimension-two condensate. Interestingly, $2m_\rho^2 = a$ would bring both a and F closer to experimental values as well as making the dimension-two condensate vanish.

5. Discussion

We have shown that Migdal's model for the spectrum of vector mesons in QCD is *not* a Padé approximant. Padé approximants are discretizations of functions and, if convergence is to be fulfilled, in the continuum limit $N \to \infty$ they should yield the function one has started with. In Migdal's model, one starts with the parton model logarithm and ends in a theory of discrete resonances, even after $N \to \infty$.

This departure from the Padé approximant is due to the modified continuum limit, which also generates an infrared scale Λ. This infrared scale has a nice geometrical interpretation as the infrared boundary brane of 5-dimensional holographic models. The scale breaks the convergence of the Padé approximant and is actually a modeling of quark–hadron duality breakdown. The existence of duality breakdown is a necessary ingredient in any model of resonances: from the operator product expansion one cannot reach the spectrum. Notice however that Migdal's modeling of duality breakdown is based on a manipulation of the asymptotic behavior of $\langle VV \rangle$. Therefore, it is *by construction* unable to distinguish models with the same asymptotics.

Migdal's spectrum (and hence the spectrum in the *hard wall model*) tends to the Padé approximant at large space-like momenta and hence reproduces the perturbative logarithm. However, on the physical axis, due to the presence of duality breakdown, the picture is very different: the Padé tries to reconstruct the logarithmic branch cut by piling simple poles arbitrarily close, thus invalidating any possible interpretation of them as resonances in a large-N_c model. Inclusion of duality breakdown is thus, contrary to what Migdal claimed, essential.

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