# STATUS OF THE HADRONIC LIGHT-BY-LIGHT CONTRIBUTION TO THE MUON ANOMALOUS MAGNETIC MOMENT*

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We review the present status of the hadronic light-by-light contribution to muon $g-2$ and critically compare recent calculations.

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## 1. Introduction

The muon anomalous magnetic moment $g-2$ $[a_\mu \equiv (g-2)/2]$ has been measured by the E821 experiment (Muon $g-2$ Collaboration) at BNL with an impressive accuracy of 0.72 ppm [1] yielding the present world average [1]

$$a_\mu^{\rm exp} = 11\,659\,208.0(6.3) \times 10^{-10}\,, \tag{1}$$

with an accuracy of 0.54 ppm. New experiments [2,3] are being designed to measure $a_\mu$ with an accuracy of at least 0.25 ppm.

On the theory side, a lot of work has been devoted to reduce the uncertainty of the Standard Model prediction. For a recent updated discussion see [3], where an extensive list of references for both theoretical predictions and experimental results can be found.

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Here, we critically review the present status of the hadronic light-by-light contribution whose uncertainty will eventually become the largest theoretical error. This contribution is depicted Fig. 1. It consists of three photon legs coming from the muon line connected to the external electromagnetic field by hadronic processes.

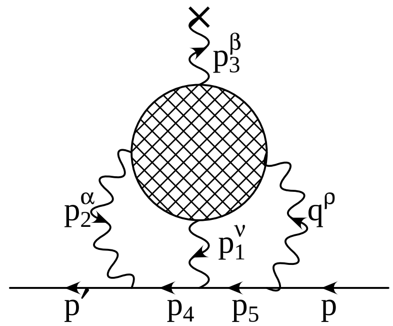


Fig. 1. The hadronic light-by-light contribution to the muon $g-2$.

Its contribution can be written as

$$\mathcal{M} = |e|^7 V_\beta \int \frac{d^4 p_1}{(2\pi)^4} \int \frac{d^4 p_2}{(2\pi)^4} \frac{1}{q^2\, p_1^2\, p_2^2 (p_4^2 - m^2)\,(p_5^2 - m^2)} \times \Pi^{\rho\nu\alpha\beta}(p_1,p_2,p_3)\, \bar{u}(p')\gamma_\alpha(\not{p}_4 + m)\gamma_\nu(\not{p}_5 + m)\gamma_\rho u(p)\,, \quad (2)$$

where $q = p_1 + p_2 + p_3$. To get the amplitude $\mathcal{M}$ in (2), one needs the full correlator $\Pi^{\rho\nu\alpha\beta}(p_1,p_2,p_3 \to 0)$, with

$$\Pi^{\rho\nu\alpha\beta}(p_1,p_2,p_3) = i^3 \int d^4x \int d^4y \int d^4z\; e^{i(p_1\cdot x + p_2\cdot y + p_3\cdot z)} \times \langle 0|T\left[V^\rho(0)V^\nu(x)V^\alpha(y)V^\beta(z)\right]|0\rangle\,, \quad (3)$$

with $V^\mu(x) = \left[\overline{q}\hat{Q}\gamma^\mu q\right](x)$ and $\hat{Q} = \mathrm{diag}(2,-1,-1)/3$ the quark charges. The photon leg with momentum $p_3 \to 0$ couples to the magnetic field.

Clearly, because we have two fully independent momenta many different energy scales are involved in the calculation of the hadronic light-by-light contribution to muon $g-2$. This makes it difficult to obtain the full needed behaviour of the correlator (3) from known constraints. Therefore no full first principles calculation exists at present. The needed results cannot be directly related to measurable quantities either. A first exploratory lattice QCD calculation has been attempted in [4].

Using $1/N_c$ and chiral perturbation theory (ChPT) expansion counting, one can organize the different contributions [5]:

- Goldstone boson exchange contributions are order $N_c$ and start contributing at order $p^6$ in ChPT.
- (Constituent) quark-loop and non-Goldstone boson exchange contributions are order $N_c$ and start contributing at order $p^8$ in ChPT.
- Goldstone boson loop contributions are order one in $1/N_c$ and start contributing at order $p^4$ in ChPT.
- Non-Goldstone boson loop contributions are order one in $1/N_c$ and start to contribute at order $p^8$ in ChPT.

The two existing *full* calculations, [6] and [7], are based on this classification. The Goldstone boson exchange contribution (GBE) was shown in [6,7] to be numerically dominant after strong cancellations between the other contributions. Ref. [8] showed that the leading double logarithm comes from the GBE and was positive. In [8,9] it was found that a global sign mistake occurred in the GBE of the earlier work [6,7], which was confirmed by their authors and by [10,11].

Recently, Melnikov and Vainshtein pointed out new short-distance constraints on the correlator (3) [12], studied and extended in [13]. The authors of [12] constructed a model which satisfies their main new short-distance constraints and in which the full hadronic light-by-light contribution is given by GBE and axial-vector exchange contributions. Here we explicitly compare and comment on the various contributions in the different calculations.

## 2. "Old" calculations: 1995–2001

With "old" we refer to the period 1995–2001. These calculations were organized according to the large $N_c$ and ChPT countings discussed above [5]. Notice that the ChPT counting was used just as a classification tool. We want to emphasize once more that the calculations in [6,7] showed that after several large cancellations in the rest of the contributions, the numerically dominant one is the Goldstone boson exchange. Here, we discuss mainly the results given in [6], with some comments and results from [7] and [9].

### *2.1. Pseudo-scalar exchange*

The pseudo-scalar exchange was saturated in [6,7] by the Goldstone boson exchange. This contribution is depicted in Fig. 2 with $M = \pi^0, \eta, \eta'$.

Ref. [6] used a variety of $\pi^0\gamma^*\gamma^*$ form factors

$$\mathcal{F}^{\mu\nu}(p_1,p_2) \equiv N_c/(6\pi)\,(\alpha/f_\pi)\,i\,\varepsilon^{\mu\nu\alpha\beta}p_{1\alpha}p_{2\beta}\,\mathcal{F}(p_1^2,p_2^2) \tag{4}$$

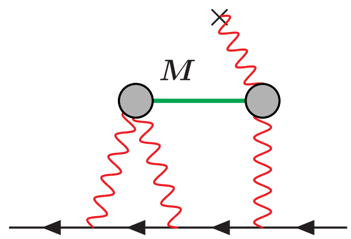


Fig. 2. A generic meson exchange contribution to the hadronic light-by-light part of the muon $g-2$.

fulfilling as many as possible QCD constraints. A more extensive analysis of this form factor was done, [14], finding very similar numerical results. In particular, the three-point form factors $\mathcal{F}(p_1^2, p_2^2)$ used in [6] had the correct QCD short-distance behavior[1]

$$\mathcal{F}(Q^2, Q^2) \;\to\; A/Q^2\,, \qquad \mathcal{F}(Q^2, 0) \to B/Q^2\,, \tag{5}$$

when $Q^2$ is Euclidean. These form factors were in agreement with available data including the slope at the origin as well as treating the $\pi^0$, $\eta$ and $\eta'$ mixing. All form factors converged for a cutoff scale $\mu \sim$ (2–4) GeV and produced small numerical differences when plugged into the hadronic light-by-light contribution.

Somewhat different $\mathcal{F}(p_1^2, p_2^2)$ form factors where used in [7, 9] but the results agree very well (after correcting for the global sign). For comparison, one can find the results of [6, 7, 9] in Table I after adding $\eta$ and $\eta'$ exchange contributions to the dominant $\pi^0$ one.

TABLE I

Results for the $\pi^0$, $\eta$ and $\eta'$ exchange contributions.

| $\pi^0$, $\eta$ and $\eta'$ exchange contribution | $10^{10} \times a_\mu$ |
|---|---|
| Bijnens, Pallante, Prades [6] | $8.5 \pm 1.3$ |
| Hayakawa, Kinoshita [7] | $8.3 \pm 0.6$ |
| Knecht, Nyffeler [9] | $8.3 \pm 1.2$ |
| Melnikov, Vainshtein [12] | $11.4 \pm 1.0$ |

[1] For this one and several other contributions [6] studied thus the observance of QCD short-distance constraints, contrary to the often stated claim that [12] is the first calculation to take such constraints into account, see *e.g.* [15].

### *2.2. Axial-vector exchange*

This contribution is depicted in Fig. 2 with $M = A = a_1^0, f_1$ and possible other axial-vector resonances. For this contribution one needs the $A\gamma\gamma^*$ and $A\gamma^*\gamma^*$ form factors. Not much is known about these but there are anomalous Ward identities which relate them to the $P\gamma\gamma^*$ and $P\gamma^*\gamma^*$ form factors.

This contribution was not calculated in [9]. Refs. [6] and [7] used nonet symmetry, which is exact in the large $N_c$ limit, for the masses of the axial-vector resonances. Their results are shown in Table II for comparison.

TABLE II

Results for the axial-vector exchange contributions.

| Axial-vector exchange contributions | $10^{10} \times a_\mu$ |
|---|---|
| Bijnens, Pallante, Prades [6] | $0.25 \pm 0.10$ |
| Hayakawa, Kinoshita [7] | $0.17 \pm 0.10$ |
| Melnikov, Vainshtein [12] | $2.2 \pm 0.5$ |

### *2.3. Scalar exchange*

This contribution is shown in Fig. 2 with $M = S = a_0, f_0$ and possible other scalar resonances. For this contribution one needs the $S\gamma\gamma^*$ and $S\gamma^*\gamma^*$ form factors. Within the extended Nambu–Jona-Lasinio (ENJL) model used in [6], chiral Ward identities impose relations between the constituent quark loop and scalar exchanges. The needed scalar form factors are also constrained at low energies by ChPT. Ref. [6] used nonet symmetry for the masses. This contribution was not included in [7] and [12].

In [10] it was found that the leading logarithms of the scalar exchange are the same as those of the pion exchange but with opposite sign. Ref. [6] finds that sign for the full scalar exchange contribution, obtaining

$$a_\mu(\text{Scalar}) = -(0.7 \pm 0.2) \cdot 10^{-10} \,. \tag{6}$$

### *2.4. Other contributions at leading order in $1/N_c$*

This includes any other contributions that are not exchange contributions. At short-distance, the main one is the quark-loop. At long distances they are often modeled as a constituent quark-loop with form-factors in the couplings to photons. This corresponds to the contribution shown in Fig. 3. Ref. [6] split up the quark momentum integration into two pieces by introducing an Euclidean matching scale $\Lambda$. At energies below $\Lambda$, the ENJL model was used to compute the quark-loop contribution while above

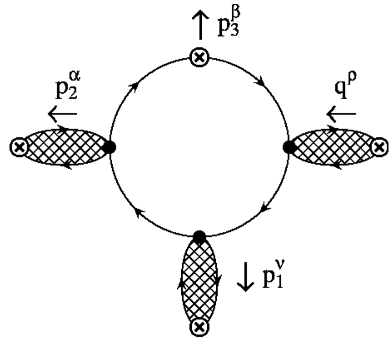


Fig. 3. Quark-loop contribution as modeled in ENJL.

$\Lambda$ a bare (partonic) heavy quark loop of mass $\Lambda$ was used. The latter part scales as $1/\Lambda^2$ and mimics the high energy behavior of QCD for a massless quark with an IR cut-off around $\Lambda$. Adding these two contributions yields a stable result as can be seen in Table III.

TABLE III

Sum of the short- and long-distance quark loop contributions as a function of the matching scale $\Lambda$.

| $\Lambda$ [GeV] | 0.7 | 1.0 | 2.0 | 4.0 |
|---|---|---|---|---|
| $10^{10} \times a_\mu$ | 2.2 | 2.0 | 1.9 | 2.0 |

### *2.5. NLO in $1/N_c$: Goldstone boson loops*

At next-to-leading order (NLO) in $1/N_c$, the leading contribution in the chiral counting to the correlator in (2), corresponds to charged pion and Kaon loops which can be depicted analogously to the quark-loop in Fig. 3 but with pions and Kaons running inside the loop instead.

In [6] the needed form-factors in the $\gamma^* P^+ P^-$ and $\gamma^*\gamma^* P^+ P^-$ vertices were studied extensively. In particular which forms were fully compatible with chiral Ward identities were studied. Full vector meson dominance (VMD) is one model fulfilling the known constraints[2]. The conclusion reached there was that there is a large ambiguity in the momentum dependence starting at order $p^6$ in ChPT. Both the full VMD model of [6] and the hidden gauge symmetry (HGS) model of [7] satisfy the known constraints. Unfortunately, this ambiguity cannot be resolved since there is no data for $\gamma^*\gamma^* \to \pi^+\pi^-$. Adding the charged pion and Kaon loops, the results obtained in [6] and [7] are listed in Table IV.

[2] Note that neither the ENJL model nor any fixed order in ChPT was used.

TABLE IV

Results for the charged and Kaon loop contributions to the hadronic light-by-light contribution to muon $g-2$.

| Charged pion and kaon loop contributions | $10^{10}\times a_\mu$ |
|---|---|
| Bijnens, Pallante, Prades (Full VMD) [6] | $-1.9\pm0.5$ |
| Hayakawa, Kinoshita (HGS) [7] | $-0.45\pm0.85$ |
| Melnikov, Vainshtein [12] | $0\pm1.0$ |

In view of the model dependence of this contribution, the difference between [6] and [7] for this contribution needs to be added *linearly* to the final uncertainty of the hadronic light-by-light contribution to $a_\mu$.

## 3. New short-distance constraints: 2003–2004

Melnikov and Vainshtein pointed out in [12] a new short-distance constraint on the correlator (3). This constraint is for

$$\langle T[V^\nu(p_1)V^\alpha(p_2)V^\rho(-q=-p_1-p_2)]|\gamma(p_3\to0)\rangle\,, \tag{7}$$

and follows from the OPE for two vector currents when $p_1^2\simeq p_2^2>>q^2$:

$$T[V^\nu(p_1)V^\alpha(p_2)]\sim\varepsilon^{\nu\alpha\mu\beta}\,(\hat{p}_\mu/\hat{p}^2)\,[\overline{q}\hat{Q}^2\gamma_\beta\gamma_5q](p_1+p_2)\,, \tag{8}$$

with $\hat{p}=(p_1-p_2)/2\simeq p_1\simeq-p_2$ and $\hat{Q}$ is the light quark electrical charge matrix (3). This constraint was afterwards generalized in [13].

The authors of [12] saturated the full correlator by exchanges. The new OPE constraint is satisfied by introducing a pseudo-scalar exchange with the point-like vertex on the $p_3$ side. This change strongly breaks the symmetry between the two ends of the exchanged particle in Fig. 2. Not all OPE constraints on the correlator are satisfied at the same time by this model, but in [12] they argued that this made only a small numerical difference.

To the pseudo-scalar exchange they added an axial-vector exchange contribution which was found to be extremely sensitive to the mixing of the resonances $f_1(1285)$ and $f_1(1420)$ as can be seen in Table V, taken from the results of [12]. The authors in [12] took the ideal mixing result for their final result for $a_\mu$.

TABLE V

Results quoted in [12] for the pseudo-vector exchange depending of the $f_1(1285)$ and $f_1(1420)$ resonances mass mixing.

| Mass mixing | $10^{10} \times a_\mu$ |
|---|---|
| No OPE and nonet symmetry with $M = 1.3$ GeV | 0.3 |
| New OPE and nonet symmetry with $M = 1.3$ GeV | 0.7 |
| New OPE and nonet symmetry with $M = M_\rho$ | 2.8 |
| New OPE and ideal mixing with experimental masses | $2.2 \pm 0.5$ |

## 4. Comparison

Let us now try to compare the three calculations [6, 7, 12]. In Table VI, the results of the leading order in $1/N_c$ are shown. The quark loop is of the same order and has to be *added* to get the full hadronic light-by-light while the model used in [12] is saturated just by exchanges. In the GBE the effect of the new OPE in [12] is a little larger than the quark loop contributions of [6]. It also increases the axial-vector exchange with nonet symmetry from 0.3 $\times 10^{-10}$ to 0.7 $\times 10^{-10}$ One thus sees a reasonable agreement in the comparison of the $\mathcal{O}(N_c)$ results of [6, 7, 12] when using the same mass mixing for the axial-vectors, namely, (10.9$\pm$ 1.9, 9.4 $\pm$ 1.6, 12.1 $\pm$1.0).

The final differences are due to the additional increase of $1.5\times 10^{-10}$ from the ideal mixing in the axial vector exchange in [12] and the scalar exchange of $-0.7\times 10^{-10}$ in [6].

TABLE VI

Full hadronic light-by-light contribution to $a_\mu$ at $\mathcal{O}(N_c)$. The difference between the two results of [6] is the contribution of the scalar exchange $-(0.7\pm 0.1)\times 10^{-10}$. This contribution was not included in [7] and [12].

| Hadronic light-by-light at $\mathcal{O}(N_c)$ | $10^{10} \times a_\mu$ |
|---|---|
| Nonet symmetry + scalar exchange [6] | $10.2 \pm 1.9$ |
| Nonet symmetry [6] | $10.9 \pm 1.9$ |
| Nonet symmetry [7] | $9.4 \pm 1.6$ |
| New OPE and nonet symmetry [12] | $12.1 \pm 1.0$ |
| New OPE and ideal mixing [12] | $13.6 \pm 1.5$ |

Let us now see what are the different predictions at NLO in $1/N_c$. In [12], the authors studied the chiral expansion of the charged pion loop using the HGS model used in [7]. This model is known not to give the correct QCD high energy behavior in some two-point functions, in particular it does not fulfill Weinberg Sum Rules, see *e.g.* [6]. Within this model [12] showed that there is a large cancellation between the first three terms of an expansion of the charged pion loop contribution in powers of $(m_\pi/M_\rho)^2$. It is not clear how one should interpret this. In [6] some studies of the cut-off dependence of this contribution were done and the bulk of their final number came from fairly low energies which should be less model dependent. However, it is clear that there is a large model dependence in the NLO in $1/N_c$ contributions. But simply taking it to be $(0 \pm 1) \times 10^{-10}$ as in [12] is rather drastic and certainly has an underestimated error.

Let us now compare the results for the full hadronic light-by-light contribution to $a_\mu$ when summing all contributions. The final result quoted in [6], [7] and [12] can be found in Table VII. The apparent agreement between the results of [6] and [7] is hiding non-negligible differences which numerically almost compensate. There are differences in the quark loop and charged pion and Kaon loops and [7] does not include the scalar exchange.

TABLE VII

Results for the full hadronic light-by-light contribution to $a_\mu$.

| Full hadronic light-by-light | $10^{10} \times a_\mu$ |
|---|---|
| Bijnens, Pallante, Prades [6] | $8.3 \pm 3.2$ |
| Hayakawa, Kinoshita [7] | $8.9 \pm 1.7$ |
| Melnikov, Vainshtein [12] | $13.6 \pm 2.5$ |

Comparing the results of [6] and [12], we have seen several differences of order $1.5 \times 10^{-10}$, differences which are not related to the one induced by the new short-distance constraint introduced in [12]. These differences are numerically of the same order or smaller than the uncertainty quoted in [6] but add up as follows. The different axial-vector mass mixing account for $-1.5 \times 10^{-10}$, the absence of scalar exchange in [12] accounts for $-0.7 \times 10^{-10}$ and the absence of the NLO in $1/N_c$ contribution in [12] accounts for $-1.9 \times 10^{-10}$. These model dependent differences add up to $-4.1 \times 10^{-10}$ out of the final $-5.3 \times 10^{-10}$ difference between the results of [6] and [12]. Clearly, the new OPE constraint found in [12] alone does not account for the large final difference as a reading of [12] seems to suggest.

## 5. Conclusions and prospects

At present, the only possible conclusion is that the situation of the hadronic light-by-light contribution to $a_\mu$ is unsatisfactory. However, one finds a *numerical* agreement within roughly one sigma when comparing the

$\mathcal{O}(N_c)$ results found in [6,7,12], see Table VI. A new full $\mathcal{O}(N_c)$ calculation studying the full correlator with the large $N_c$ techniques developed in [16,17] and references therein, seems feasible and desirable.

At NLO in $1/N_c$ one needs to control both Goldstone and non-Goldstone boson loop contributions. The high model dependence of the Goldstone boson loop is clearly visible in the different results of [6] and [7] and discussed in [6] and [12]. For non-Goldstone boson loops, little is known on how to consistently treat them, a recent attempt in another context is given in [18].

In the meanwhile, we propose as an educated guess[3]

$$a_\mu = (11 \pm 4) \times 10^{-10} \tag{9}$$

for the hadronic light-by-light contribution. We believe that this number summarizes our present understanding of the hadronic light-by-light contribution to $a_\mu$. One can arrive at this number in several different ways: the short-distance constraint and the ideal mixing for the axial-vector exchange should lead to some increase of the results of [6,7]; the scalar exchange and the pion and kaon loops are expected to lead to some decrease of the result of [12]; one can also average the leading in $1/N_c$ results (three middle results of Table VI). The final error remains a guess but the error in (9) is chosen to include all the known uncertainties.

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