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RADIATION FROM CHIRAL SLABS AND CYLINDERS WITH ZERO PERMITTIVITY

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The possibility to simulate an effective medium with a permittivity close to zero in some frequency band with as consequence that such a medium suitably excited behaves for the outside world as an ultrarefractive antenna with a narrow radiation pattern was recently proved. We prove here that slabs and cylinders made of a Tellegen chiral metamaterial with zero permittivity excited with a time harmonic filamentous current respecting the symmetry of these structures constitute ultrarefractive antennas. We also analyze the equations satisfied by the electromagnetic field inside and outside a metaTellegen paraboloid of revolution excited with an electric current running along its axis.

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1. Introduction

Numerical evidence of ultrarefractive optics was recently proved with the conclusions that a dielectric photonic crystal can simulated an effective medium having a permittivity close to zero in some frequency band [1]. Then, introducing a radiation source in such a structure with an excitation frequency that lies within the specified pass band, builds up an antenna having a significantly narrow pattern in the far field outside the structure [2–5].

These results stimulated further investigations, for instance, metaslabs are considered in [6], as well as cylinders and spheres, made of a Drude material with zero index of refraction and matched to surrounded free space. These structures excited by a proper electric current are shown to behave as antennas with a narrow far field pattern.

A similar analysis is performed here in a somewhat different context with slabs and cylinders made of a lossy Tellegen chiral metamaterial [7] with zero permittivity in some frequency band so that the constitutive relations become

$$\boldsymbol{D} = -i\xi \boldsymbol{H}\,, \qquad \boldsymbol{B} = \mu \boldsymbol{H} + i\xi \boldsymbol{E}\,, \qquad i = \sqrt{-1}\,. \tag{1}$$

Permeability μ and chirality ξ are complex functions of the angular frequency ω. These structures are excited with a time harmonic current, of the filament type, respecting the slab and the cylinder symmetry. Thus, fields are not disturbed by reflections at boundaries, but this condition is impossible to satisfy for a sphere. These metaslabs and cylinders, excited in this way in a convenient frequency band are shown to be antennas with an ultranarrow radiation pattern. We also consider a meta-Tellegen paraboloid of revolution around the z-axis with its apex at the origin of coordinates while an excitation current runs along $0z$ and, we analyze the equations satisfied by the electromagnetic field inside and outside this structure but we do not discuss their solutions.

Maxwell's equations in presence of a charge e and of a current $\boldsymbol{j}$ have the general expressions for harmonic fields with the factor $\exp(i\omega t)$ implicit throughout and $c = 1$

$$\nabla\times\boldsymbol{H} - i\omega\boldsymbol{D} = \boldsymbol{j}\,, \qquad \nabla\times\boldsymbol{E} + i\omega\boldsymbol{B} = 0\,, \qquad \nabla\cdot\boldsymbol{D} = e, \qquad \nabla\cdot\boldsymbol{B} = 0\,, \tag{2}$$

with the conservation relation $\nabla \cdot \boldsymbol{j} + i\omega e = 0$.

In a Tellegen medium of zero permittivity with the constitutive relations (1) these equations become

$$\begin{aligned} \nabla \times \boldsymbol{H} - \omega\xi\boldsymbol{H} &= \boldsymbol{j}\,, \qquad & e + i\xi\nabla\cdot\boldsymbol{H} = 0\,, \\ \nabla \times \boldsymbol{E} - \omega\xi\boldsymbol{E} + i\omega\mu\boldsymbol{H} &= 0\,, \qquad & \mu\nabla\cdot\boldsymbol{H} + \xi\nabla\cdot\boldsymbol{E} = 0\,. \end{aligned} \tag{3}$$

We start with a discussion of 1D-slab antennas.

2. 1D-Tellegen metaslabs with zero permittivity

As a preliminary, we consider a situation in which the fields depend only on z, $e = 0$, and the current $\boldsymbol{j}$ has the components

$$j_x = j_0\ \delta(z)\,, \qquad j_y = j_z = 0\,, \tag{4}$$

in which j_0 is a constant and $\delta(z)$ the Dirac distribution. Then, $E_z = H_z = 0$ the divergence in Eqs. (3) are satisfied and the curl in Eqs. (3) become

$$\partial_z H_y + \omega\xi\, H_x + j_0\delta(z) \;=\; 0\,, \qquad \partial_z\, H_x - \omega\xi\, H_y = 0\,, \tag{5a}$$

$$\partial_z E_y + \omega\xi\, E_x - i\,\omega\mu\, H_x \;=\; 0\,, \qquad \partial_z E_x - \omega\xi\, E_y + i\,\omega\mu\, H_y = 0\,. \tag{5b}$$

Eliminating H_y from (5a), E_y from (5b) gives the 1D-inhomogeneous wave equations

$$\begin{aligned} \partial_z^2 H_x + \omega^2\xi^2 H_x &= -\omega\xi\, j_0\, \delta(z)\,, && (6a)\\ \partial_z^2 E_x + \omega^2\xi^2 E_x &= i\omega\mu\,[2\omega\xi\, H_x + j_0\delta(z)]\,. && (6b)\end{aligned}$$

Now, the solution of the differential equation $y'' + \omega^2\xi^2 y = f(z)$ is

$$y = c_1(z)\sin(\omega\xi z) + c_2(z)\cos(\omega\xi z)\,, \tag{7a}$$

in which the amplitudes $c_1(z), c_2(z)$ have the derivatives

$$c_1'(z) = (1/\omega\xi)\ f(z)\cos(\omega\xi z), \qquad c_2'(z) = -(1/\omega\xi)\ f(z)\sin(\omega\xi z)\,. \tag{7b}$$

For the right hand side of (6a), $f(z) = -\omega\xi\, j_0\, \delta(z)$ and the corresponding primitives $c_{1,2}(z)$ are given in Appendix A: $c_1(z) = U(z)$, $c_2(z) = 0$. Then, according to (7a), the solution of (6a) is

$$H_x = -j_0\ \sin(\omega\xi z)\, U(z)\,, \tag{8a}$$

while we get from (5a) and (8a)

$$H_y = -j_0\ \cos(\omega\xi z)\, U(z)\,. \tag{8b}$$

In these expressions, $U(z)$ is the Heaviside function $U(z) = 1$ for $z \geq 0, = 0$ for $z < 0$.

Let us now write the right hand side of Eq. (6b)

$$f(z) = f^0(z) + f^1(z), \quad f^0(z) = i\omega\mu\, j_0\, \delta(z)\,, \quad f^1(z) = 2i\,\omega^2\mu\xi\, H_x\,. \tag{9}$$

We get $E_x = E_x^0 + E_x^1$ and comparing $f^0(z)$ with the right hand side of (6a) gives at once according to (8a):

$$E_x^0 = -(i\mu/\xi)\, H_x = (i\mu\, j_0/\xi)\ \sin(\omega\xi\, z)\, U(z)\,, \tag{10}$$

while for $f^1(z)$ the derivatives of (7b) become

$$\begin{aligned} c_1' &= 2i\omega\mu\, H_x\ \cos(\omega\xi z) = -i\omega\mu\, j_0\ \sin(2\omega\xi z)\, U(z)\,,\\ c_2' &= -2i\omega\mu\, H_x\ \sin(\omega\xi\, z) = 2i\omega\mu\, j_0\ \sin^2(\omega\xi\, z)\, U(z)\,. && (11)\end{aligned}$$

The primitives $c_{1,2}(z)$ are also obtained in Appendix A and we get

$$c_1 = -(ij_0\mu/\xi)\ \sin^2(\omega\xi\, z)\, U(z)\,, \quad c_2 = (ij_0\mu/\xi[\omega\xi\, z - \sin(2\omega\xi\, z)/2] \tag{12}$$

so, according to (7a)

$$E_x^1 = (ij_0\mu/\xi)[\omega\xi\, z\ \cos(\omega\xi\, z) - \sin(\omega\xi\, z)]\, U(z)\,, \tag{13}$$

and taking into account (10)

$$E_x = E_x^0 + E_x^1 = i\mu j_0\,\omega z\,\cos(\omega\xi\, z)\,U(z) = -i\mu\omega\, z\, H_y\,. \tag{14}$$

Now, we get from (5b)

$$E_y = (1/\omega\xi)\,\partial_z E_x + i\mu/\xi\, H_y\,, \tag{15a}$$

and according to (8b) and (14)

$$E_y = -i\mu j_0\,\omega z\,\sin(\omega\xi\, z)\,U(z) = i\mu\omega z\, H_x\,. \tag{15b}$$

The average Poynting's vector at the angular frequency ω has only a non-null component in which the asterisk denotes the complex conjugation

$$S_z = {}^1\!/\!{}_2\mathrm{Re}(E_x H_y^* - E_y H_x^*)\,, \tag{16a}$$

and taking into account (14), (15b) we get according to (8a) and (8b)

$$\begin{aligned} S_z &= \mathrm{Im}[(\mu\omega\, z/2)(H_x H_x^* + H_y H_y^*)]\,U(z) \\ &= \mathrm{Im}[(\mu\omega z\, j_0^2/2)\,\cos\{\omega(\xi - \xi^*)\}]\,U(z)\,, \end{aligned} \tag{16b}$$

so that in a lossless medium $S_z = 0$ since μ is real.

We now consider a 1D-Tellegen slab with filamentous boundaries $-\infty < x < \infty$ at $z = -d$ and $z = d$. A time harmonic filamentous current exists along the axis $z = 0$ with the expression (4). Then, the electromagnetic field inside this 1D-slab has the components (8a), (8b) and (14), (15b) since by symmetry the reflected fields on the boundaries cancel each other.

To analyze the behavior of this Tellegen metaslab as an antenna, we need the solutions of Maxwell's equations in the outward free space. Now, in a medium with permittivity ε_0, permeability μ_0, Maxwell's equations for fields depending only on z reduce to $E_z^\dagger = H_z^\dagger = 0$ and to

$$\partial_z H_y^\dagger = -i\omega\varepsilon_0\, E_x^\dagger\,, \qquad \partial_z H_x^\dagger = i\omega\varepsilon_0\, E_y^\dagger\,, \tag{17a}$$

$$\partial_z E_y^\dagger = i\omega\mu_0\, H_x^\dagger\,, \qquad \partial_z E_x^\dagger = -i\omega\mu_0\, H_y^\dagger\,. \tag{17b}$$

Eliminating $E_{x,y}^\dagger$ gives the wave equations $(\partial_z^2 + n_0^2\omega^2)H_{x,y}^\dagger = 0$, $n_0^2 = \varepsilon_0\mu_0$, with the solutions

$$H_{x,y}^\dagger = A_{x,y}\cos(\omega n_0 z) + B_{x,y}\sin(\omega n_0 z)\,, \tag{18a}$$

and substituting (18a) into (17a), we get

$$\begin{aligned} i\varepsilon_0\, E_x^\dagger &= n_0 A_y \sin(\omega n_0 z) - n_0 B_y \cos(\omega n_0 z)\,, \\ i\varepsilon_0\, E_y^\dagger &= -n_0 A_x \sin(\omega n_0 z) + n_0 B_x \cos(\omega n_0 z)\,. \end{aligned} \tag{18b}$$

In (18a,b), $A_{x,y}$, $B_{x,y}$ are the four amplitudes of the fields in the half-space $z \geq d$ determined by the boundary conditions on the face $z = d$ of the Tellegen 1D-slab and we have

$$H^{\dagger}_{x,y}(d) = H_{x,y}(d)\,, \qquad E^{\dagger}_{x,y}(d) = E_{x,y}(d)\,, \tag{19}$$

in which $H_{x,y}(d)$, $E_{x,y}(d)$ are the expressions (8a), (8b), (14), (15b) for $z = d$. Then using (18a,b) we get

$$\begin{aligned} A_{x,y}\cos(\omega n_0 d) + B_{x,y}\sin(\omega n_0 d) \;&=\; H_{x,y}(d)\,, \\ -i\varepsilon_0 n_0[A_y \sin(\omega n_0 d) - B_y \cos(\omega n_0 d)] \;&=\; E_x(d)\,, \\ i\varepsilon_0 n_0[A_x \sin(\omega n_0 d) - B_x \cos(\omega n_0 d)] \;&=\; E_y(d)\,. \end{aligned} \tag{20a}$$

The solution of (20a) is

$$\begin{aligned} \{A_y,\, B_y\} \;&=\; \{\cos(\omega n_0 d),\, \sin(\omega n_0 d)\} H_y(d) \\ &\quad +i\varepsilon_0/n_0\{\sin(\omega n_0 d),\, -\cos(\omega n_0 d)\}\, E_x(d)\,, \\ \{A_x,\, B_x\} \;&=\; \{\cos(\omega n_0 d),\, \sin(\omega n_0 d)\}\, H_x(d) \\ &\quad -i\varepsilon_0/n_0\{\sin(\omega n_0 d),\, -\cos(\omega n_0 d)\}\, E_y(d)\,. \end{aligned} \tag{20b}$$

We are, of course, interested in the Poynting vector with the nonnull component

$$S^{\dagger}_z(z) = {}^1\!/\!_2\mathrm{Re}(E^{\dagger}_x H^{\dagger^*}_y - E^{\dagger}_y H^{\dagger^*}_x)(z)\,. \tag{21a}$$

Taking into account (18a,b), a simple calculation gives

$$\begin{aligned} S^{\dagger}_z(z) \;&=\; n_0/2\varepsilon_0 \mathrm{Im}[\{A_y B^*_y \sin^2(\omega n_0 z) - A^*_y B_y \cos^2(\omega n_0 z)\} \\ &\quad +\{A_x B^*_x \sin^2(\omega n_0 z) - A^*_x B_x \cos^2(\omega n_0 z)\}]\,. \end{aligned} \tag{21b}$$

Now for $z = d$ we have according to (20b)

$$\mathrm{Im}(A_y B^*_y) = -\mathrm{Im}(A^*_y B_y)\,, \qquad \mathrm{Im}(A_x B^*_x) = -\mathrm{Im}(A^*_x B_x) \tag{22}$$

so that

$$S^{\dagger}_z(z) = n_0/2\varepsilon_0 \mathrm{Im}(A_y B^*_y + A_x B^*_x)\,. \tag{23a}$$

But, still using (20b):

$$\begin{aligned} \mathrm{Im}(A_y B^*_y) \;&=\; \varepsilon_0/n_0[E_x(d)H^*_y(d) + E^*_x(d)H_y(d)]\,, \\ \mathrm{Im}(A_x B^*_x) \;&=\; -\varepsilon_0/n_0[E_y(d)H^*_x(d) + E^*_y(d)H_x(d)]\,, \end{aligned} \tag{23b}$$

implying

$$\mathrm{Im}(A_y B^*_y + A_x B^*_x) = 2\varepsilon_0/n_0 S_z(d)\,, \tag{24}$$

and substituting (24) into (23a) gives finally $S^{\dagger}_z(d) = S_z(d)$.

The electromagnetic flow in free space surrounding the Tellegen 1D-slab with zero permittivity is constant in the z direction with the amplitude of the inner energy flow reaching the surface $z = d$. This excited structure is an antenna with an ultra-narrow radiation pattern.

3. Tellegen circular metacylinder medium with zero permittivity

We now consider a circular metacylinder Tellegen medium with zero permittivity, centered along the z-axis materialized by an electric current filament (ρ, ϕ, z are the cylindrical coordinates)

$$j_z = I_0 \delta(\rho)/2\pi\rho\,, \qquad j_\rho = j_\varphi = 0\,. \tag{25}$$

The components of the electromagnetic field inside such a medium are obtained in Appendix B and we get $H_\rho = E_\rho = 0$ and

$$H_z(\rho) = (\omega\xi I_0/4)[v_Y(\rho)J_0(\omega\xi\rho) - v_J(\rho)Y_0(\omega\xi\rho)]\,, \tag{26a}$$

$$H_\varphi(\rho) = (\omega\xi I_0/4)[v_Y(\rho)J_1(\omega\xi\rho) - v_J(\rho)Y_1(\omega\xi\rho)]\,, \tag{26b}$$

in which $J_{0,1}$, $Y_{0,1}$ are the Bessel functions of first and second kind, of order zero, one and

$$v_Y(\rho) = \partial^{-1}[Y_0(\omega\xi\rho)\delta(\rho)]\,, \qquad v_J(\rho) = \partial^{-1}[J_0(\omega\xi\rho)\delta(\rho)]\,. \tag{26c}$$

∂^{-1} is the primitive operator defined in Appendix A. The components of the electric field are

$$\begin{aligned} E_z &= i\mu/\xi H_z(\rho) - 2i\omega^2\xi\mu[h_Y(\rho)J_0(\omega\xi\rho) - h_J(\rho)Y_0(\omega\xi\rho)]\,, \\ E_\phi &= (1/\omega\xi)\partial_\rho E_z^1 = 2i\omega^2\xi\mu[h_Y(\rho)J_1(\omega\xi\rho)h_J(\rho)Y_1(\omega\xi\rho)\,, \end{aligned} \tag{27a}$$

with

$$h_Y(\rho) = \partial^{-1}[H_z(\rho)\, Y_0(\omega\xi\rho)]\,, \qquad h_J(\rho) = \partial^{-1}[H_z(\rho)\, J_0(\omega\xi\rho)]\,. \tag{27b}$$

The only nonnull component of the Poynting vector is

$$S_\rho = (1/2)\mathrm{Re}[E_\phi H_z^* - E_z H_\phi^*]\,. \tag{28}$$

The energy flow is radial with a rather intricate analytical expression.

This Tellegen medium is now supposed to be a tube of radius a, surrounded by free space with the time harmonic current $\boldsymbol{j}$ moving along the z-axis of this tube and generating the electromagnetic field with the components (26a), (27a), the only field present inside the cylinder since the symmetry of the structure makes null the reflected field at boundaries.

In the surrounding free space, the Maxwell equations in cylindrical geometry for fields that depend only on ρ reduce to $E_\rho^\dagger = H_\rho^\dagger = 0$ and to

$$-\partial_\rho E_z^\dagger + i\omega\mu_0 H_\phi^\dagger = 0\,, \qquad \rho^{-1}\partial_\rho(\rho E_\phi^\dagger) + i\omega\mu_0 H_z^\dagger = 0\,, \tag{29a}$$

$$\partial_\rho H_z^\dagger + i\omega\varepsilon_0 E_\phi^\dagger = 0\,, \qquad \rho^{-1}\partial_\rho(\rho H_\phi^\dagger) - i\omega\varepsilon_0 E_z^\dagger = 0\,, \tag{29b}$$

and we look for the solutions of these equations in the form

$$H^{\dagger}_{\phi,z} = h_{\phi,z}(\rho)\,, \quad i\omega\varepsilon_0 E^{\dagger}_{\phi} = -\partial_\rho h_z(\rho), \quad i\omega\varepsilon_0 E^{\dagger}_{z} = \rho^{-1}\partial_\rho[\rho h_\phi(\rho)] \quad \text{(30a)}$$

supplying the wave equations

$$\begin{aligned} &\partial^2_\rho h_z + \rho^{-1}\partial_\rho h_z + \omega^2 n_0^2 h_z = 0\,, \qquad n_0^2 = \varepsilon_0\mu_0\,, \\ &\partial^2_\rho h_\phi + \rho^{-1}\partial_\rho h_\phi - \rho^{-2} h_\phi + \omega^2 n_0^2 h_\phi = 0\,. \end{aligned} \quad \text{(30b)}$$

The Bessel functions J_0, Y_0, J_1, Y_1 are the respective solutions of (30b), so according to (30a)

$$\begin{aligned} H^{\dagger}_z &= A_z J_0(n_0\omega\rho) + B_z Y_0(n_0\omega\rho)\,, \\ H^{\dagger}_\phi &= A_\phi J_1(n_0\omega\rho) + B_\phi Y_1(n_0\omega\rho)\,, \\ i\varepsilon_0 E^{\dagger}_\phi &= n_0[A_z J_1(n_0\omega\rho) + B_z Y_1(n_0\omega\rho)]\,, \\ i\varepsilon_0 E^{\dagger}_z &= n_0[A_\phi J_0(n_0\omega\rho) + B_\phi Y_0(n_0\omega\rho)]\,, \end{aligned} \quad \text{(31a)}$$

since [8]

$$\begin{aligned} \partial_t\{J_0(t),\, Y_0(t)\} &= -\{J_1(t),\, Y_1(t)\}\,, \\ t^{-1}\partial_t[t\{J_0(t),\, Y_0(t)\}] &= \{J_0(t),\, Y_0(t)\}\,. \end{aligned} \quad \text{(31b)}$$

The amplitudes A_z, B_z, A_ϕ, B_ϕ, are determined by the boundary conditions on the surface $\rho = a$ of the Tellegen cylinder

$$H^{\dagger}_{\phi,z}(a) = H_{\phi,z}(a)\,, \qquad E^{\dagger}_{\phi,z}(a) = E_{\phi,z}(a) \quad \text{(32)}$$

and, taking into account (31a), we get the four relations

$$\begin{aligned} A_z J_0(n_0\omega a) + B_z Y_0(n_0\omega a) &= H_z(a)\,, \\ A_\phi J_1(n_0\omega a) + B_\phi Y_1(n_0\omega a) &= H_\phi(a)\,, \\ (-in_0/\varepsilon_0)[A_z J_1(n_0\omega a) + B_z Y_1(n_0\omega a)] &= E_\phi(a) \\ (-in_0/\varepsilon_0)[A_\phi J_0(n_0\omega a) + B_\phi Y_0(n_0\omega a)] &= E_z(a)\,. \end{aligned} \quad \text{(33)}$$

Using the Wronskian [8]: $J_1(n_0\omega a)Y_0(n_0\omega a) - J_0(n_0\omega a)Y_1(n_0\omega a) = 2/(\pi n_0\omega a)$ and deleting the arguments of the functions since no confusion is possible, the solution of (33) is

$$\begin{aligned} A_z &= -(\pi n_0\omega a/2)[Y_1 H_z - (i\varepsilon_0/n_0)Y_0 E_\phi]\,, \\ B_z &= (\pi n_0\omega a/2)[J_1 H_z - (i\varepsilon_0/n_0)J_0 E_\phi)]\,, \\ A_\phi &= (\pi n_0\omega a/2)[Y_0 H_\phi - (i\varepsilon_0/n_0)Y_1 E_z]\,, \\ B_\phi &= -(\pi n_0\omega a/2)[J_0 H_\phi - (i\varepsilon_0/n_0)J_1 E_z]\,. \end{aligned} \quad \text{(34)}$$

The only nonnull component of the Poynting vector in free space is

$$S_\rho^\dagger = (1/2)\mathrm{Re}(E_\phi^\dagger H_z^{\dagger\,*} - E_z^\dagger H_\phi^{\dagger\,*})\,, \tag{35a}$$

and using (33) a simple calculation gives

$$\begin{aligned} S_\rho^\dagger \;=\; & n_0/\varepsilon_0\,\mathrm{Im}[B_\phi A_\phi^* J_1 Y_0 + B_\phi^* A_\phi J_0 Y_1 \\ & -B_z A_z^* J_0 Y_1 - B_z^* A_z J_1 Y_0]\,. \end{aligned} \tag{35b}$$

But, according to (34)

$$\mathrm{Im}(B_\phi A_\phi *) = -\mathrm{Im}(B_\phi^* A_\phi)\,, \quad \mathrm{Im}(B_z A_z^*) = -\mathrm{Im}(B_z^* A_z)\,, \tag{36a}$$

and substituting (36a) into (35b) we get since $J_1 Y_0 - J_0 Y_1 = 2/(\pi\omega a n_0)$

$$S_\rho^\dagger = [2/(\pi\omega a\varepsilon_0)][\mathrm{Im}(B_\phi A_\phi^*) + \mathrm{Im}(B_z A_z^*)]\,. \tag{36b}$$

Then, still using (34) and the Wronskian, we have

$$\begin{aligned} \mathrm{Im}(B_\phi A_\phi^*) \;&=\; -(\pi^2\omega^2 a^2 n_0 \varepsilon_0/4)\mathrm{Re}[H_\phi E_z^* J_0 Y_1 - H_\phi^* E_z J_1 Y_0] \\ &=\; (\pi\omega a\varepsilon_0/2)\mathrm{Re}(H_\phi^* E_z)\,, \end{aligned} \tag{37a}$$

and similarly

$$\mathrm{Im}(B_z A_z^*) = -(\pi\omega a\varepsilon_0/2)\mathrm{Re}(H_z^* E_\phi)\,. \tag{37b}$$

These two relations imply

$$\mathrm{Im}(B_\phi A_\phi^*) + \mathrm{Im}(B_z A_z^*) = (\pi\omega a\varepsilon_0/2) S_\rho(a)\,. \tag{38}$$

Substituting (38) into (36b) gives finally $S_\rho^\dagger(a) = S_\rho(a)$. The electromagnetic energy flow in the outward free space is radial, constant with the value of the inner energy flow on the surface $\rho = a$ of the cylinder. This excited structure is, as the Tellegen slab, an antenna with an ultra-narrow radiation pattern.

4. Tellegen metaparaboloid with zero permittivity

We consider twin paraboloids of revolution around the z-axis, with apex at $x = y = z = 0$. Then, using the polar coordinates ρ, ϕ, z, the equation of this structure is $z = \pm\rho^2/2R$ in which $R/2$ is the distance apex focus along $0z$. This paraboloid is supposed made of a Tellegen meta material with zero permittivity in a specified frequency band and characterized by the constitutive relations (1), the excitation current (25) being directed along $0z$.

4.1. Field inside the upper paraboloid

We get from Eqs. (B.4b), (B.5b) of Appendix B the wave equations satisfied by the components of the magnetic field

$$\begin{aligned}
&\partial_z^2 H_\rho + (\partial_\rho^2 + \rho^{-1}\partial_\rho - \rho^{-2} + \omega^2\xi^2)H_\rho = 0\,,\\
&\partial_z^2 H_\phi + (\partial_\rho^2 + \rho^{-1}\partial_\rho - \rho^{-2} + \omega^2\xi^2)H_\phi = 0\,,\\
&\partial_z^2 H_z + (\partial_\rho^2 + \rho^{-1}\partial_\rho + \omega^2\xi^2)H_z = -\omega\xi j_z\,.
\end{aligned} \tag{39}$$

As an important difference with the situations met in the previous two sections, we have to take into account the reflected field on the paraboloid surface and consequently, we look for the solutions of (39) in the form

$$H_\rho = H_\rho^0(\rho, z)\,, \qquad H_{\phi,z} = H_{\phi,z}^0(\rho, z) + H_{\phi,z}^1(\rho)\,, \tag{40}$$

in which the components of $\boldsymbol{H}^0$ are solutions of (39) with $j_z = 0$ while the (φ, z)-components of $\boldsymbol{H}^1$ are the solutions (26a). Now, the solutions $\boldsymbol{H}^0$ bounded for $\rho = 0$ may be written

$$\begin{aligned}
H_{\rho,\phi}^0,(\rho, z) &= \int_0^\infty d\lambda \exp(-\lambda z) f_{\rho,\phi}^0(\lambda) J_1(\gamma\rho)\,, \quad \gamma^2 = \omega^2\xi^2 + \lambda^2\,,\\
H_z^0(\rho, z) &= \int_0^\infty d\lambda \exp(-\lambda z) f_z^0(\lambda) J_0(\gamma\rho)\,.
\end{aligned} \tag{41a}$$

Substituting (41a) into (B.5b) and into the first equation $\partial_z H_\varphi + \omega\xi H_\rho = 0$ of the set (B.4b), still using the relations (31b) gives $f_{z,\phi}^0$ in terms of f_ρ^0

$$\lambda f_\phi^0 = \omega\xi f_\rho^0\,, \qquad \lambda f_z^0 = \gamma f_\rho^0\,. \tag{41b}$$

To sum up, taking into account (26a), we have with v_J, v_Y given by (26c)

$$\begin{aligned}
H_\rho^0(\rho, z) &= \int_0^\infty d\lambda \exp(-\lambda z) f_\phi^0(\lambda) J_1(\gamma\rho)\,,\\
H_z(\rho, z) &= H_z^0(\rho, z) + (\omega\xi I_0/4)[v_Y(\rho)J_0(\omega\xi\rho) - v_J(\rho)Y_0(\omega\xi\rho)]\,,\\
H_\phi(\rho, z) &= H_\phi^0(\rho, z) + (\omega\xi I_0/4)[v_Y(\rho)J_1(\omega\xi\rho) - v_J(\rho)Y_1(\omega\xi\rho)]\,,
\end{aligned} \tag{42a}$$

in which according to (41a) and (41b)

$$\begin{aligned}
H_z^0(\rho, z) &= \int_0^\infty \gamma\lambda^{-1} d\lambda \exp(-\lambda z) f_\rho^0(\lambda) J_0(\gamma\rho)\,,\\
H_\phi^0(\rho, z) &= \int_0^\infty \omega\xi\lambda^{-1} d\lambda \exp(-\lambda z) f_\rho^0(\lambda) J_1(\gamma\rho)\,.
\end{aligned} \tag{42b}$$

Similarly, we look for the electric field in the form

$$\boldsymbol{E} = \boldsymbol{E}^0(\rho, z) + \boldsymbol{E}^1(\rho) \tag{43a}$$

in which $H^1_\rho(\rho) = 0$ while the $E^1_{\phi,z}(\rho)$'s are the solutions (27a) and

$$\begin{aligned} E^0_{\rho,\phi}(\rho, z) &= \int_0^\infty d\lambda \exp(-\lambda z) g^0_{\rho,\phi}(\lambda) J_1(\gamma\rho)\,, \\ E^0_z(\rho, z) &= \int_0^\infty d\lambda \exp(-\lambda z) g^0_z(\lambda) J_0(\gamma\rho)\,. \end{aligned} \tag{43b}$$

Substituting (43a) into the Maxwell equations (B.4a) of Appendix B, taking into account (43b) (27, 41) and using the relations (31b) gives a set of equations supplying the amplitudes g^0 in terms of f^0:

$$\begin{aligned} -\lambda g^0_\phi - \omega\xi g^0_\rho + i\omega\mu f^0_\rho &= 0\,, \\ -\lambda g^0_\rho + \gamma g^0_z - \omega\xi g^0_\phi + i\omega\mu f^0_\phi &= 0\,, \\ -\lambda g^0_\phi - \omega\xi g^0_z + i\omega\mu f^0_z &= 0\,, \end{aligned} \tag{44}$$

a system easy to solve. According to (43a) and (27a), the components of the electric field are

$$\begin{aligned} E^0_\rho(\rho, z) &= \int_0^\infty d\lambda \exp(-\lambda z) g^0_\rho(\lambda) J_1(\gamma\rho)\,, \\ E_z(\rho, z) &= E^0_\rho(\rho, z) + (i\mu/\xi) H^1_z - 2i\omega^2\xi\mu[h_Y(\rho) J_0(\omega\xi\rho) - h_J(\rho) Y_0(\omega\xi\rho)]\,, \\ E_\phi(\rho, z) &= E^0_\phi(\rho, z) + 2i\omega^2\xi\mu[h_Y(\rho) J_1(\omega\xi\rho) - h_J(\rho) Y_1(\omega\xi\rho)\,, \end{aligned} \tag{45}$$

with h_J, h_Y given by (27b).

Taking into account (41b) and (44), we see that the electromagnetic field inside the paraboloid structure is not fully determined, as in slabs and cylinders, but depends on a arbitrary constant f^0_ρ to be determined by the boundary conditions at the surface of the paraboloid. And, the boundary conditions to be satisfied come from the continuity imposed on the tangential components of the $\boldsymbol{E}$, $\boldsymbol{H}$ fields and on the normal component of the $\boldsymbol{B}$ field. Now, at the altitude z on the surface of the paraboloid $z = \rho^2/2R$, we have $\rho_s = (2Rz)^{1/2}$ and the tangential components are

$$\begin{aligned} &\{E_\varphi, H_\varphi\}(\rho_s, z)\,, \\ &\{E_\mathrm{T}, H_\mathrm{T}\}(\rho_s, z) = [\rho_s\{H_z, E_z\} + R\{(H_\rho, E_\rho\}](\rho_s, z)\,, \end{aligned} \tag{46a}$$

while the normal component $B_{\mathrm{N}} = -RB_z + \rho_s B_\rho$ of $\boldsymbol{B}$ becomes taking into account (1)

$$B_{\mathrm{N}}(\rho_s, z) = -R(\mu H_z + i\xi E_z)(\rho_s, z) + \rho_s(\mu H_\rho + i\xi E_\rho)(\rho_s, z)\,. \qquad (46\text{b})$$

4.2. Outside field, boundary conditions

To get the electromagnetic field in free space surrounding the twin paraboloids, we use the equations (B.1), (B.2) of Appendix B which become

$$\begin{aligned}
\partial_z E_\phi^\dagger - i\omega\mu_0 H_\rho^\dagger &= 0\,, \qquad \partial_z H_\phi^\dagger + i\omega\varepsilon_0 E_\rho^\dagger = 0\,,\\
\partial_z E_\rho^\dagger - \partial_\rho E_z^\dagger + i\omega\mu_0 H_\phi^\dagger &= 0\,, \qquad \partial_z H_\rho^\dagger - \partial_\rho H_z^\dagger - i\omega\varepsilon_0 E_\phi^\dagger = 0\,,\\
\rho^{-1}\partial_\rho(\rho E_\varphi^\dagger) + i\omega\mu_0 H_z^\dagger &= 0\,, \qquad \rho^{-1}\partial_\rho(\rho H_\varphi^\dagger) - i\omega\varepsilon_0 E_z^\dagger = 0\,, \qquad (47\text{a})
\end{aligned}$$

and

$$\rho^{-1}\partial_\rho(\rho H_\rho^\dagger) + \partial_z H_z^\dagger = 0\,, \qquad \rho^{-1}\partial_\rho(\rho E_\rho^\dagger) + \partial_z E_z^\dagger = 0\,. \qquad (47\text{b})$$

It is easy to get the wave equations satisfied by the components of the magnetic field

$$(\partial_z^2 + \partial_\rho^2 + \rho^{-1}\partial_\rho - \rho^{-2} + \omega^2 n_0^2)H_{\rho,\phi}^\dagger = 0\,, \qquad n_0^2 = \varepsilon_0\mu_0\,, \qquad (48\text{a})$$

$$(\partial_z^2 + \partial_\rho^2 + \rho^{-1}\partial_\rho + \omega^2 n_0^2)H_z^\dagger = 0\,, \qquad (48\text{b})$$

from which the components of the electric field are obtained by the relations

$$i\omega\varepsilon_0 E_\phi^\dagger = -\partial_z H_\phi^\dagger\,, \quad i\omega\varepsilon_0 E_z^\dagger = \rho^{-1}\partial_\rho(\rho H_\phi^\dagger)\,, \quad i\omega\varepsilon_0 E_\phi^\dagger = \partial_z H_\rho^\dagger - \partial_\rho H_z^\dagger\,. \quad (49)$$

We may write the solutions of (48a) in terms of the Bessel functions J_1, Y_1 for $z > 0$

$$\begin{aligned}
&H_{\rho,\phi}^\dagger(\rho, z) = \int_0^\infty d\lambda \exp(-\lambda z)[f_{\rho,\phi}^\dagger(\lambda)J_1(\nu\rho) + g_{\rho,\phi}^\dagger(\lambda)Y_1(\nu\rho)]\,,\\
&\nu^2 \cong \omega^2 n_0^2 + \lambda^2\,, \qquad (50\text{a})
\end{aligned}$$

while the solutions of (48b) depend on J_0, Y_0

$$H_z^\dagger(\rho, z) = \int_0^\infty d\lambda \exp(-\lambda z)[f_z^\dagger(\lambda)J_1(\nu\rho) + g_z^\dagger(\lambda)Y_0(\nu\rho)]\,. \qquad (50\text{b})$$

Then, substituting (50a), (50b) into (49) and using the relations (31a), (31b) give for the electric field

$$i\omega\varepsilon_0 E_\rho^\dagger = \int_0^\infty \lambda d\lambda \exp(-\lambda z)[f_\phi^\dagger(\lambda)J_1(\nu\rho) + g_\phi^\dagger(\lambda)Y_1(\nu\rho)]\,, \tag{51a}$$

$$i\omega\varepsilon_0 E_z^\dagger = \int_0^\infty \nu d\lambda \exp(-\lambda z)[f_\phi^\dagger(\lambda)J_0(\nu\rho) + g_\phi^\dagger(\lambda)Y_0(\nu\rho)]\,, \tag{51b}$$

$$\begin{aligned} i\omega\varepsilon_0 E_\phi^\dagger = & \int_0^\infty d\lambda(\exp(-\lambda z)[\{\nu f_z^\dagger(\lambda) - \lambda f_\rho^\dagger(\lambda)\}J_0(\nu\rho) \\ & +\{\nu g_z^\dagger(\lambda) - \lambda g_\rho^\dagger(\lambda)\}Y_0(\nu\rho)]\,. \end{aligned} \tag{51c}$$

But all the functions $f^\dagger$, $g^\dagger$, in (50), (51) are not independent and substituting (50a), (51c) into the equation $\partial_z E_\phi^\dagger - i\omega\mu_0 \boldsymbol{H}_\rho^\dagger = 0$ of the set (47a) gives the relations

$$\nu f_\rho^\dagger(\lambda) = \lambda f_z^\dagger(\lambda)\,, \qquad \nu g_\rho^\dagger(\lambda) = \lambda g_z^\dagger(\lambda)\,, \tag{52}$$

and the solution (51c) becomes

$$i\omega\varepsilon_0 E_\phi^\dagger = \int_0^\infty \omega^2 n_0^2 \nu^{-1} d\lambda \exp(-\lambda z)[f_z^\dagger(\lambda)J_1(\nu\rho) + g_z^\dagger(\lambda)Y_1(\nu\rho)]\,. \tag{53}$$

It is easily checked that for $\lambda = 0$ these expressions give the electromagnetic field outside the Tellegen cylindrical antenna of Section 3.

So, we are left with four unknown functions $f_{z,\phi}^\dagger$, $g_{z,\phi}^\dagger$ obtained from the boundary conditions on the surface of the parabolic structure which supplies in addition, as previously noticed, the amplitude f_ρ^0 characteristic of the inner field. The tangential components of the $\boldsymbol{E}^\dagger$, $\boldsymbol{H}^\dagger$ fields at the altitude z on the surface of the paraboloid $z = \rho^2/2R$, are with $\rho_s = (2Rz)^{1/2}$

$$\begin{aligned} &\{E_\varphi^\dagger,\, H_\varphi^\dagger\}(\rho_s, z)\,, \\ &\{E_{\mathrm{T}}^\dagger,\, H_{\mathrm{T}}^\dagger\}(\rho_s, z) = [\rho_s\{H_z^\dagger,\, E_z^\dagger\} + R\{H_\rho^\dagger\,, E_\rho^\dagger\}](\rho_s, z)\,, \end{aligned} \tag{54a}$$

and the normal component of the $\boldsymbol{B}$ field is

$$B_{\mathrm{N}}^\dagger(\rho_s, z) = -\mu_0 R H_z^\dagger(\rho_s, z) + \mu_0 \rho_s H_\rho^\dagger(\rho_s, z)\,. \tag{54b}$$

Then, taking into account (46a,b) the boundary conditions are

$$\{E_{\phi,T}^\dagger,\, H_{\phi,T}^\dagger\}(\rho_s, z) = \{E_{\phi,T},\, H_{\phi,T}\}(\rho_s, z)\,, \quad B_{\mathrm{N}}^\dagger(z, \rho_s) = B_{\mathrm{N}}(\rho_s, z)\,. \tag{55}$$

We illustrate these boundary conditions on the -component of the magnetic field: taking into account (50a), we get

$$\int_0^\infty d\lambda \exp(-\lambda z)[f_\phi^\dagger(\lambda) J_1(\nu\rho_s) + g_\phi^\dagger(\lambda) Y_1(\nu\rho_s)] = H_\phi(\rho_s, z)\,, \qquad (56)$$

in which $H_\phi(\rho_s, z)$ is the expression (42a) on the paraboloid surface.

This integral equation is not easy to solve and, with the only objective to make clear the type of difficulties to be met, we suppose null the $g_\phi^\dagger(\lambda)$ function so that (56) reduces to

$$\int_0^\infty d\lambda \exp(-\lambda z) f_\phi^\dagger(\lambda) J_1(\nu\rho_s) = H_\phi(\rho_s, z)\,. \qquad (57)$$

Now on the paraboloid surface $z = \rho_s^2/2R$, then multiplying (57) by ρ_s^2 and performing the ρ_s integration gives

$$\int_0^\infty d\lambda f_\phi^\dagger(\lambda) \int_0^\infty \rho_s^2 d\rho_s \exp(-\lambda\rho_s^2/2R) J_1(\nu\rho_s) = \alpha\,,$$

$$\alpha = \int_0^\infty \rho_s^2 d\rho_s H_\phi(\rho_s, \rho_s^2/2R)\,. \qquad (58)$$

But [9]

$$\int_0^\infty \rho_s^2 d\rho_s \exp(-\lambda\rho_s^2/2R) J_m(\nu\rho_s) = \nu^m (R/\lambda)^{m+1} \exp(-\nu^2 R/2\lambda)\,, \qquad (59)$$

so that the equation (58) becomes

$$\int_0^\infty d\lambda f_\phi^\dagger(\lambda) \nu (R/\lambda)^2 \exp(-\nu^2 R/2\lambda) = \alpha\,, \qquad (60a)$$

with the solution since $\nu^2 \cong \omega^2 n_0^2 + \lambda^2$

$$f_\phi^\dagger(\lambda) = \alpha\lambda^2/2n_0 R \exp(\omega^2 n_0^2 R/2\lambda)\,. \qquad (60b)$$

Unfortunately, relations similar to (59) do not exist for the Bessel functions Y_m and clearly the boundary conditions (54a), (54b) put a challenge.

4.3. Poynting vector

The Poynting vector $\boldsymbol{S}(\rho, z)$ inside the Tellegen paraboloids has the components

$$\begin{aligned} S_\phi(\rho,z) &= (1/2)\mathrm{Re}\{E_z H_\rho^* - E_\rho H_z^*\}(\rho,z)\,, \\ S_\rho(\rho,z) &= (1/2)\mathrm{Re}\{E_\phi H_z^* - E_z H_\phi^*\}(\rho,z)\,, \\ S_z(\rho,z) &= (1/2)\mathrm{Re}\{E_\rho H_\phi^* - E_\phi H_\rho^*\}(\rho,z)\,. \end{aligned} \tag{61}$$

The part of the energy flow able to radiate outside the Tellegen structure is supplied by the normal component $S_\mathrm{N} = -RS_z + \rho S_\rho$ which takes the value on the surface of paraboloids

$$S_\mathrm{N}(\rho_s, z) = -RS_z(\rho_s, z) + \rho_s S_\rho(\rho_s, z)\,, \qquad z = \rho_s^2/2R\,. \tag{62a}$$

Substituting (61) into (62a) and taking into account (46a) gives

$$S_\mathrm{N}(\rho_s, z) = (1/2)\mathrm{Re}\{E_\phi H_\mathrm{T}^* - E_\mathrm{T} H_\phi^*\}(\rho_s, z)\,. \tag{62b}$$

The Poynting vector $\boldsymbol{S}^\dagger(\rho, z)$ in the free space outside the Tellegen structure has components formally similar to (61) and the radiation in the far field comes from the normal component of $\boldsymbol{S}^\dagger$ which takes the values on the surface of the Tellegen structure

$$S_\mathrm{N}^\dagger(\rho_s, z) = (1/2)\mathrm{Re}\{E_\phi^\dagger {H_\mathrm{T}^\dagger}^* - E_\mathrm{T}^\dagger {H_\phi^\dagger}^*\}(\rho_s, z)\,, \tag{63}$$

and the boundary conditions (55) imply

$$S_\mathrm{N}^\dagger(\rho_s, z) = S_\mathrm{N}(\rho_s, z) \tag{64}$$

a result expected from those obtained in the previous two sections. This meta-Tellegen structure behaves as a parabolic antenna with a narrow radiation pattern.

5. Discussion

So, the theoretical calculations performed in the previous three sections, for longitudinally unbounded slabs, circular cylinders, paraboloids of revolution, made of chiral Tellegen material with zero permittivity, prove that these structures become directive antennas with a narrow radiation pattern when they are excited by an electric filament along their symmetry axis. When the excitation current is constant, inward and outward electromagnetic fields have simple analytical expressions for slabs but, concerning cylinders, coherent approximations will be necessary to make manageable these fields obtained in the form of integrals requiring Bessel functions.

Only the equations satisfied by the field components are given for paraboloids of revolution and numerical codes will have to be developed to get their solution. To conclude on a practical design, we may think of a meta Tellegen long straight (tape) antenna with a length appreciable compared with wavelength as in most of practical antennas. Several other types of constitutive relations exist for isotropic chiral media proposed by Drude, Born, Fedorov, Condon, Post... with debatable merits. It has been shown to be equivalent to each other for time harmonic fields [10], an equivalence not necessary valid for chiral materials of zero permittivity and fields generated with an electric filament. Then if narrow pattern antennas appear to become an important tool in future technology, it could be interesting to devote further works to materials with different constitutive relations. To assume a constant excitation current is a bit restrictive which leads to consider what happens with a time dependent current. With this objective, we discuss in Appendix C a Tellegen chiral metaslab of zero permittivity excited by a current with time history $\boldsymbol{J}(t)$

$$J_x = \underline{J}(t)\delta(z)\,, \qquad J_y = J_z = 0\,. \tag{65}$$

Using the Laplace transform [11, 12] $f(s) = L[F(t)]$ shows that, roughly speaking, we have just to change in the previous calculations: $i\omega$ into s and j into $j(s)$ to obtain the fields $\{\boldsymbol{e}(s),\, \boldsymbol{h}(s)\}$. Of course, the inverse Laplace transform is necessary to get the time dependent fields $\{\boldsymbol{E}(t),\, \boldsymbol{H}(t)\}$ but there now exist powerful techniques to do this job efficiently [13]. Concerning slabs and, assuming the chirality parameter ξ real positive, the inverse Laplace transform of fields inside the slab have simple analytical expressions, for instance

$$H_x = -(i/2)J(t - \xi z/c)\,, \quad H_y = -(1/2)J(t - \xi z/c)\,, \quad 0 \le \xi z/c < t \tag{66}$$

(see (C.8) for electric field components). Outward fields are sums of similar functions such as $\underline{J}[t - n_0 z/c \pm (n_0 \pm \xi)d/c]$ with different delays $(n_0 \pm \xi)d/c$, $2d$ being the slab thickness.

Thus, for metachiral structures with zero permittivity, conveniently excited with currents respecting the symmetry source-structure there is a potential application as highly directive antennas and this result carries on theoretical and numerical works [3] on the design of directive antennas.

The antennas discussed here, infinite along $0z$, should be truncated to represent realistic structures. From a mathematical point of view, it suffices to multiply the field expressions by the function $U(z-z_0)-U(z-z_1)$ in which U is the Heaviside function, z_0, z_1, the lower and upper coordinates, with as consequence, to make calculations a bit more intricate. But, one should have to introduce boundary conditions at $z = z_0$ and $z = z_1$. An interesting situation happens for z-periodic antennas since it has been proved [14] that these structures support infinite wavelength.

Appendix A

Primitives

Using the relations where $\delta(z)$, $U(z)$ are the Dirac distribution and the Heaviside unit function

$$\delta(z)dz = d[U(z)]\,, \qquad U(z)dz = d[zU(z)]\,, \tag{A.1}$$

and integrating by parts, we get for the primitive $\partial^{-1}[f(z)\delta(z)] = \int f(z)\delta(z)dz$

$$\int f(z)\delta(z)dz = g_d(z)U(z)\,, \qquad g_d(z) = \sum_{n=0}^{\infty}(-1)^n z^n/n!\partial_z^n f(z)\,, \tag{A.2}$$

and similarly for $\partial^{-1}[f(z)U(z)] = \int f(z)U(z)dz$

$$\int f(z)U(z)dz = g_u(z)U(z), \quad g_u(z) = \sum_{n=0}^{\infty}(-1)^n z^{n+1}/(n+1)!\partial z^n f(z)\,. \tag{A.3}$$

There exist similar relations with $U(-z)$. In particular for $\exp(az)$ we get from (A.2) and (A.3)

$$\begin{aligned} \partial^{-1}[\exp(az)\delta(z)] &= U(z)\,, \\ \partial^{-1}[\exp(az)U(z)] &= a^{-1}[\exp(az) - 1]U(z)\,, \end{aligned} \tag{A.4}$$

these relations imply

$$\partial^{-1}[\cos(\omega\xi z)\delta(z)] = U(z)\,, \qquad \partial^{-1}[\sin(\omega\xi z)\delta(z)] = 0\,, \tag{A.5}$$

$$\begin{aligned} \partial^{-1}[\sin(2\omega\xi z)U(z)] &= (1/\omega\xi)\sin^2(\omega\xi z)U(z)\,, \\ \partial^{-1}[\cos(2\omega\xi z)U(z)] &= (1/\omega\xi)\ sin(2\omega\xi z)U(z)\,, \end{aligned} \tag{A.6}$$

and

$$\partial^{-1}[\sin^2(\omega\xi z)U(z)] = [z/2 - (1/4\omega\xi)\sin(2\omega\xi z)]U(z)\,. \tag{A.7}$$

These simple results are not the general rule, they do not hold for the Bessel functions $Y_0(\omega\xi\rho)$, $J_0(\omega\xi\rho)$ solutions of the cylindrical wave equation, we get in this case

$$\partial^{-1}[Y_0(\omega\xi\rho)\delta(\rho)] = v_Y(\rho)U(\rho)\,, \quad \partial^{-1}[J_0(\omega\xi\rho)\delta(\rho)] = v_J(\rho)U(\rho)\,, \tag{A.8}$$

$$v_Y(\rho) = \sum_{n=0}^{\infty} (-1)^n \rho^n / n! \partial_\rho^n Y_0(\omega\xi\rho)\,,$$
$$v_J(\rho) = \sum_{n=0}^{\infty} (-1)^n \rho^n / n! \partial_\rho^n J_0(\omega\xi\rho)\,. \tag{A.9}$$

It is difficult to get consistent approximations of these sums, even with small ρ, specially for $v_Y(\rho)$ because of its logarithmic behavior in this domain.

Appendix B

Electromagnetic field in a zero permittivity cylinder

For fields that do not depend on azimuth, Maxwell's equations in cylindrical coordinates ρ, ϕ, z with a current $\boldsymbol{j}$ and a charge e are

$$\begin{aligned} -\partial_z E_\phi + i\omega B_\rho &= 0\,, \\ \partial_z E_\rho - \partial_\rho E_z + i\omega B_\phi &= 0\,, \\ \rho^{-1}\partial_\rho(\rho E_\varphi) + i\omega B_z &= 0\,, \end{aligned} \tag{B.1a}$$
$$\begin{aligned} -\partial_z H_\phi - i\omega D_\rho &= j_\rho\,, \\ \partial_z H_\rho - \partial_\rho H_z - i\omega D_\phi &= j_\phi\,, \\ \rho^{-1}\partial_\rho(\rho E_\varphi) - i\omega D_z &= j_z\,, \end{aligned} \tag{B.1b}$$

with the divergence equations

$$\rho^{-1}\partial_\rho(\rho B_\rho) + \partial_z B_z = 0\,, \tag{B.2a}$$
$$\rho^{-1}\partial_\rho(\rho D_\rho) + \partial_z D_z = 0\,. \tag{B.2b}$$

For the zero permittivity Tellegen medium with constitutive relations (1) and with the electric current

$$j_z = I_0 \delta(\rho)/2\pi\rho\,, \qquad j_\rho = j_\varphi = 0 \tag{B.3}$$

these equations become

$$\begin{aligned} -\partial_z E_\phi - \omega\xi E_\rho + i\omega\mu H_\rho = 0\,, \\ \partial_z E_\rho - \partial_\rho E_z - \omega\xi E_\phi + i\omega\mu H_\phi = 0\,, \\ \rho^{-1}\partial_\rho(\rho E_\phi) - \omega\xi E_z + i\omega\mu H_z = 0\,, \end{aligned} \tag{B.4a}$$
$$\begin{aligned} \partial_z H_\phi + \omega\xi H_\rho = 0\,, \\ \partial_z H_\rho - \partial_\rho H_z - \omega\xi H_\phi = 0\,, \\ \rho^{-1}\partial_\rho(\rho H_\varphi) - \omega\xi H_z = j_z\,, \end{aligned} \tag{B.4b}$$

and

$$\rho^{-1}\partial_\rho(\rho E_\rho) + \partial_z E_z = 0\,, \tag{B.5a}$$
$$\rho^{-1}\partial_\rho(\rho H_\rho) + \partial_z H_z = 0\,, \tag{B.5b}$$

the charge e est null and the electromagnetic field depends only on ρ, so that the equations (B.4a), (B.4b) imply $H_\rho = 0$ and reduce to

$$\rho^{-1}\partial_\rho(\rho H_\varphi) - \omega\xi H_z = -\omega\xi I_0 \delta(\rho)/2\pi\rho, \tag{B.6a}$$
$$\partial_\rho H_z + \omega\xi H_\phi = 0\,, \tag{B.6b}$$

while we get from (B.4a) $E_\rho = 0$ and

$$\partial_\rho E_z + \omega\xi E_\phi = i\omega\mu H_\phi, \qquad \rho^{-1}\partial_\rho(\rho E_\varphi) - \omega\xi E_z = -i\omega\mu H_z\,. \tag{B.7}$$

Eliminating H_ϕ from (B.6) and E_ϕ from (B.7) gives the inhomogeneous equations

$$\partial_\rho^2 H_z + \rho^{-1}\partial_\rho H_z + \omega^2\xi^2 H_z \;=\; -\omega\xi I_0 \delta\rho/2\pi\rho\,, \tag{B.8a}$$
$$\partial_\rho^2 E_z + \rho^{-1}\partial_\rho E_z + \omega^2\xi^2 E_z \;=\; i\omega\mu(\partial_\rho H_\varphi + \rho^{-1} H_\varphi + \omega\xi H_z)\,. \tag{B.8b}$$

We first look for the solutions of Eq.(8b): consider the inhomogeneous differential equation

$$y'' + \rho^{-1} y' + \omega^2\xi^2 = f(\rho) \tag{B.9}$$

the homogeneous wave equation $y'' + \rho^{-1}y' + \omega^2\xi^2 = 0$ has the Bessel functions of the first and second kind of order zero $J_0(\omega\xi\rho)$, $Y_0(\omega\xi\rho)$ as solutions so that since $\partial_\rho(J_0, Y_0) = -(J_1, Y_1)$ and since the Wronskian $J_1Y_0 - J_0Y_1 = 2/(\pi\omega\xi\rho)$ [8] the solution of (B.9) is

$$y = C_1(\rho)J_0(\omega\xi\rho) + C_2(\rho)Y_0(\omega\xi\rho)\,, \tag{B.10a}$$

in which the amplitudes $C_{1,2}(\rho)$ are defined by their derivatives

$$C_1'(\rho) = -\pi\rho/2 f(\rho) Y_0(\omega\xi\rho)\,, \qquad C_2'(\rho) = \pi\rho/2 f(\rho) J_0(\omega\xi\rho)\,. \tag{B.10b}$$

For $f(\rho) = -\omega\xi I_0\delta(\rho)/2\pi\rho$ which is the right hand side of (B.8a), we get

$$C_1(\rho) \;=\; (\omega\xi I_0/4)v_Y\,, \qquad C_2(\rho) = -(\omega\xi I_0/4)v_J\,, \tag{B.11a}$$
$$v_Y(\rho) \;=\; \partial^{-1}[Y_0(\omega\xi\rho)\delta(\rho)]\,, \qquad v_J(\rho) = \partial^{-1}[J_0(\omega\xi\rho)\delta(\rho)]\,, \tag{B.11b}$$

obtained in the form of two infinite series in Appendix A. Substituting (B.11a) into (B.10a) gives the solution of (B.8a)

$$H_z(\rho) = (\omega\xi I_0/4)[v_Y(\rho)J_0(\omega\xi\rho) - v_J(\rho)Y_0(\omega\xi\rho)]\,, \tag{B.12a}$$

and taking into account (B.12a), we get from (B.6b)

$$H_\varphi(\rho) = (\omega\xi I_0/4)[v_Y(\rho)J_1(\omega\xi\rho) - v_J(\rho)Y_1(\omega\xi\rho)]\,. \tag{B.12b}$$

We now look for the solution of (B.8b) which becomes, taking into account (B.6a),

$$\partial_\rho^2 E_\mathrm{Z} + \rho^{-1}\partial_\rho E_\mathrm{Z} + \omega^2\xi^2 E_\mathrm{Z} = i\omega\mu[2\omega\xi H_\mathrm{Z} + I_0\delta(\rho)/2\pi\rho] \tag{B.13}$$

and we write the right hand side of (B.13)

$$f(\rho) = f^0(\rho) + f^1(\rho)\,, \quad f^0(\rho) = i\omega\mu I_0\delta(\rho)/2\pi\rho\,, \quad f^1(\rho) = 2i\omega^2\xi\mu H_z\,. \tag{B.14}$$

Then, the solution of (B.13) takes the form $E_z = E_z^0 + E_z^1$ and comparing $f^0(\rho)$ with the right hand side of (B.6a) gives at once

$$E_z^0(\rho) = i\mu/\xi H_z(\rho)\,, \tag{B.15}$$

while for $f^1(\rho)$

$$C_1'(\rho) = -2i\omega\xi\mu H_z(\rho)Y_0(\omega\xi\rho)\,, \qquad C_2'(\rho) = 2i\omega\xi\mu H_z(\rho)J_0(\omega\xi\rho)\,, \tag{B.16}$$

with the primitives

$$C_1(\rho) = -2i\omega^2\xi\mu h_y\,, \quad C_2(\rho) = 2i\omega^2\xi\mu h_j\,, \tag{B.17a}$$

where

$$h_Y(\rho) = \partial^{-1}[H_z(\rho)(Y_0(\omega\xi\rho)]\,, \quad h_J(\rho) = \partial^{-1}[H_z(\rho)(J_0(\omega\xi\rho)]\,, \tag{B.17b}$$

so that according to (B.10a)

$$E_z^1 = -2i\omega^2\xi\mu[h_Y(\rho)J_0(\omega\xi\rho) - h_J(\rho)Y_0(\omega\xi\rho)]\,, \tag{B.18a}$$

and since $E_z = E_z^0 + E_z^1$ we get, taking into account (B.15),

$$E_z = i\mu/\xi H_z(\rho) - 2i\omega^2\xi\mu[h_Y(\rho)J_0(\omega\xi\rho) - h_J(\rho)Y_0(\omega\xi\rho)]\,. \tag{B.18b}$$

Now, according to (B.7a)

$$E_\phi = (1/\omega\xi)\partial_\rho E_z^0 - (i\mu/\xi)H_\phi + (1/\omega\xi)\partial_\rho E_z^1 \tag{B.19a}$$

but, according to (B.6b) and (B.15) $(1/\omega\xi)\partial_\rho E_z^0 = (i\mu/\xi)H_\phi$, so that

$$E_\phi = (1/\omega\xi)\partial_\rho E_z^1 = 2i\omega^2\xi\mu[h_y(\rho)J_1(\omega\xi\rho) - h_j(\rho)Y_1(\omega\xi\rho)]\,. \tag{B.19b}$$

Appendix C

Time dependent excitation current

For fields depending only on z and for a source driven according to

$$J_x = \underline{J}(t)\delta(z)\,, \qquad J_y = J_z = 0\,, \tag{C.1}$$

the Maxwell equations inside a Tellegen metamedium with the constitutive relations (1) reduce to

$$\begin{aligned} &\partial_z H_y - i\xi c^{-1}\partial_t H_x = -\underline{J}(t)\delta(z)\,, \qquad \partial_z H_x + i\xi c^{-1}\partial_t H_y = 0\,,\\ &\partial_z E_y - c^{-1}\partial_t(\mu H_x + i\xi E_x) = 0\,, \qquad \partial_z E_x + c^{-1}\partial_t(\mu H_y + i\xi E_y) = 0\,. \end{aligned} \tag{C.2}$$

Using the Laplace transform [11, 12] $f(s) = L[F(t)]$, Eqs. (C.2) become

$$\partial_z h_y - i\xi c^{-1} s h_x = -j(s)\delta(z)\,, \qquad \partial_z h_x + i\xi c^{-1} s h_y = 0\,, \tag{C.3a}$$

$$\partial_z e_y - c^{-1} s(\mu h_x + i\xi e_x) = 0\,, \qquad \partial_z e_x + c^{-1} s(\mu h_y + i\xi e_y) = 0\,. \tag{C.3b}$$

The comparison of the relations (C.3a,b) and (C.5a,b) shows that we have just to change ω into $-isc^{-1}$ and j_0 into $j(s)$ in the expressions of Section 2 to get the solutions of (C.3a,b) and this substitution applied to (8a), (8b), assuming to simplify ξ real, positive gives

$$h_x = ij(s)\sinh(s\xi z/c)U(z)\,, \qquad h_y = -j(s)\cosh(s\xi z/c)U(z)\,, \tag{C.4}$$

while according to (14) and (15b)

$$\begin{aligned} e_x &= [(-i\mu/\xi)j(s)\sinh(s\xi z/c) - \mu s c^{-1} z j(s)]U(z)\,,\\ e_y &= (i\mu/\xi)j(s)[2\cosh(s\xi z/c) - 1]U(z)\,. \end{aligned} \tag{C.5}$$

Now, we have the inverse Laplace transforms [11]

$$\begin{aligned} L^{-1}[\exp(\pm s\xi z/c)] &= \delta(t \pm \xi z/c)\,,\\ L^{-1}[j(s)\exp(\pm s\xi z/c)] &= \int_0^t d\tau J(t-\tau)\delta(\tau \pm \xi z/c)\,. \end{aligned} \tag{C.6}$$

Assuming the source launched at $t = 0 : \delta(t + \xi z/c) = 0$ in the half space $z > 0$ while with $\delta(t - \xi z/c)$ the convolution integral is nonnull for $0 \le \xi z/c \le t$ so that the inverse Laplace transform of the (C.4) fields is

$$H_x = -(i/2)\underline{J}(t - \xi z/c)U(z)\,, \qquad H_y = (1/2)\underline{J}(t - \xi z/c)U(z)\,, \tag{C.7}$$

and from (C.5) with the current derivative $\underline{J}'(t)$

$$\begin{aligned} E_x &= [(-\mu/\xi)\underline{J}(t-\xi z/c) - \mu c^{-1} z \underline{J}'(t-\xi z/c)]U(z)\,, \\ E_y &= [(i\mu/\xi)\underline{J}(t-\xi z/c) - \underline{J}(t)]U(z)\,. \end{aligned} \tag{C.8}$$

We have a similar result in the half space $z < 0$ with $J(t+\xi z/c)$ and an opposite sign for H_x. In the free space surrounding the Tellegen metaslab, the Maxwell equations have the Laplace transform

$$\begin{aligned} \partial_z h_y^\dagger + \varepsilon_0 c^{-1} s e_x^\dagger &= 0\,, \qquad \partial_z e_y^\dagger - \mu_0 c^{-1} s h_x = 0\,, \\ \partial_z h_x^\dagger - \varepsilon_0 c^{-1} s e_y^\dagger &= 0\,, \qquad \partial_z e_x^\dagger + \mu_0 c^{-1} s h_y^\dagger = 0\,. \end{aligned} \tag{C.9}$$

Here also, the comparison of (C.9) and (17a,b) shows that we have only to change $i\omega$ into s/c in the relations (18a,b) to get the solutions of (C.9)

$$\begin{aligned} h_{x,y}^\dagger &= A_{x,y}\cosh(sn_0 z/c) + B_{x,y}\sinh(sn_0 z/c)\,, \\ i\varepsilon_0 e_x^\dagger &= n_0 A_y \sinh(sn_0 z/c) - n_0 B_y \cosh(sn_0 z/c)\,, \\ i\varepsilon_0 e_y^\dagger &= -n_0 A_x \sinh(sn_0 z/c) + n_0 B_x \cosh(sn_0 z/c)\,. \end{aligned} \tag{C.10}$$

The boundary conditions $h_{x,y}^\dagger(d) = h_{x,y}(d)$, $e_{x,y}^\dagger(d) = e_{x,y}(d)$ supply the amplitudes $A_{x,y}$, $B_{x,y}$ and we get for instance from (20b)

$$A_y = h_y(d)\cosh(sn_0 d/c) - (\varepsilon_0/n_0)e_y(d)\sinh(sn_0 d/c)\,. \tag{C.11}$$

To illustrate the form of the electromagnetic field in the outward free space, we consider the truncated expression $h_y^\dagger = A_y \cosh(sn_0 z/c)$ in (C.10), with as approximation of A_y, the first term of (C.11) which becomes according to (C.4)

$$A_y = h_y(d)\cosh(sn_0 d/c) = -j(s)\cosh(s\xi d/c)\cosh(sn_0 d/c)\,. \tag{C.12}$$

Substituting (C.12) in the first relation of the set (C.10) and also only keeping the first term of the resultant expression give

$$h_y^\dagger = -j(s)\cosh[s(n_0 z/c - n_0 d/c - \xi d/c)] \tag{C.13}$$

with, according to (C.6) the inverse Laplace transform,

$$H_y^\dagger = -\underline{J}[t - (n_0 z/c - n_0 d/c - \xi d/c)]\,. \tag{C.14}$$

So, the electromagnetic field in the outward free space has the same form as inside the Tellegen metaslab and is made of a sum of terms similar to (C.14) with $n_0 z/c$ instead of $\xi z/c$ and different delays $\pm n_0 d/c \pm \xi d/c$).

REFERENCES

[1] S. Enoch, G. Tayeb, D. Maystre, *Opt. Commun.* **161**, 171 (1999).

[2] S. Enoch, G. Tayeb, P. Sabouroux, N. Guerin, P. Vincent, *Phys. Rev. Lett.* **89**, 213902 (2002).

[3] G. Tayeb, S. Enoch, P. Sabouroux, N. Guerin, P. Vincent, 12th Inter. Symp. on Antennas, Nice 2002.

[4] G. Tayeb, S. Enoch, P. Vincent, P. Sabouroux, International Conference on Electromagnetics for Advanced Applications, Torino 2003.

[5] D. Maystre, S. Enoch, G. Tayeb, *Proc. SPIE* **5359**, 64 (2004).

[6] R.W. Ziolkowski, *Phys. Rev.* **E70**, 046608 (2004).

[7] A. Lakhtakia, *Spec. Sci. Technol.* **14**, 2 (1991).

[8] M. Abramowitz, I.A. Stegun, *Handbook of Mathematical Functions*, Dover, New York 1968.

[9] G.N. Watson, *A Treatise on the Theory of Bessel Functions*, Univ. Press, Cambridge 1962.

[10] A. Lakhtakia, V.K. Varadan, V.V. Varadan, *J. Opt. Soc. Am.* **5**, 175 (1988).

[11] G. Doetsch, *Guide to the Applications of the Laplace and Z-transforms*, Van Nostrand, London 1971.

[12] P.M. Makila, *Int. J. Control* **79**, 36 (2006).

[13] G. Dahlquist, *BIT* **33**, 85 (1993).

[14] A. Lai, K.M.K.H. Leong, T. Itoh, *IEEE Transl. Anten. Propag.* **55**, 868 (2007).