ENTROPY OF NONLINEAR BLACK HOLES IN QUADRATIC GRAVITY

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Employing the Noether charge technique and Visser's Euclidean approach the entropy of the nonlinear black hole described by the perturbative solution of the system of coupled equations of the quadratic gravity and nonlinear electrodynamics is constructed. The solution is parametrized by the exact location of the event horizon and charge. Special emphasis is put on the extremal configuration. Consequences of the second choice of the boundary conditions, in which the solution is parametrized by the charge and the total mass as seen by a distant observer is briefly examined.

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1. Introduction

Recently, a great deal of efforts have been devoted to the important issue of regular black holes. One of the most intriguing solutions of this type have been constructed by Ayón-Beato and García [1] and by Bronnikov [2]. In both cases, the line element is a solution of the coupled system of equations of nonlinear electrodynamics and gravity. (We shall refer to the solutions of this type as ABGB geometries.) The former solution describes a regular, static and spherically symmetric configuration with the electric charge, Q_e, whereas the latter one describes a similar geometry characterized by the mass and the magnetic charge Q. For certain values of the parameters both solutions describe black holes. On the other hand, the no-go theorem proved in Ref. [3] (see also [2, 4]) forbids, for the class of electromagnetic Lagrangians with a Maxwell asymptotic in a weak field limit, existence of the electrically charged, static and spherically-symmetric solutions with the regular center. It should be noted, however, that the electric solution is not in conflict with the non existence theorem, as the formulation of the nonlinear electrodynamics [5] employed by Ayón-Beato and García (P framework in the nomenclature of Ref. [2]) differs from the one to which one refers in the assumptions of the no-go theorem. Indeed, the solution of Ayón-Beato and García has been constructed in a formulation of the nonlinear electrodynamics obtained from the original one (F framework) by means of a Legendre transformation (see Ref. [2] for details). Moreover, the no-go theorem does not forbid existence of the solutions with magnetic charge as well as some hybrid configurations in which the electric field does not extend to the central region.

The status of the nonlinear electrodynamics in the model considered here is to provide a static matter source, perhaps the exotic one, to the field equations. That means that the casual structure of the spacetime is still governed by the null geodesics or "ordinary" photons rather than the photons of the nonlinear theory. Actually, the latter move along the geodesics of the effective space [6, 7]. Outside the event horizon the solution of the ABGB-type closely resembles the Reissner–Nordström (RN) geometry both in its global and local structure. Important differences appear near the extremality limit. Consequently, the Penrose diagrams of the ABGB solution are similar to those constructed for the Reissner–Nordström solution, with the one notable distinction: instead of the singularity at $r = 0$ now we have the regular interior.

An attractive feature of the ABGB solutions is possibility to express the location of the horizons in terms of the Lambert special functions [8, 9]. Similarly, the Lambert functions [10, 11] may be used in the discussion of the extremal configurations [12].

According to our present understanding a proper description of the gravitational phenomena should be given by the quantum gravity, being perhaps a part of a more fundamental theory. And although at the present stage we have no clear idea how this theory looks like, we expect that the action functional describing its low-energy approximation should consist of the higher order terms constructed from the curvature tensor, its contractions and covariant derivatives to some required order. Among various generalizations of the Einstein–Hilbert action a special role is played by the quadratic gravity (see for example Refs. [13–21]). Motivations for introducing such terms into the action functional are numerous. When invented, for example, the equations of quadratic gravity have been treated as an exact formulation of the theory of gravitation. On the other hand, it may be considered, quite naturally, as truncation of series expansion of the action of the more general theory. Such terms appear generically in the one-loop calculations of the quantum field theory in curved background [22]. Moreover, from the point of view of the semi-classical gravity, the quadratic terms in the field equations might be treated as some sort of the simplified stress–energy tensor. Such a toy model of the renormalized stress–energy tensor allows to mimic the fairly more complex sources in a relatively simple way. This approach is especially useful when the general pattern that lies behind the calculations of both types is essentially the same. Thus, some general features of the full semi-classical solutions can be analyzed and understood without referring to otherwise intractable equations.

It should be noted that any higher curvature theory contain solutions which are unavailable to the theory based on the classical Einstein–Hilbert Lagrangian. This can most easily be seen by counting the degrees of freedom: the quadratic gravity is known to posses 8 degrees of freedom whereas the General Relativity has only 2. Moreover, there are solutions that are not analytic in the coupling constants, *i.e.*, they do not reduce to solutions of the classical Einstein field equations. (For a comprehensive discussion see for example [23] and the references cited therein.) Unfortunately, because of complexity of the equations of the quadratic gravity it is practically impossible to construct their exact solutions and one is forced to refer either to approximations or to numerical methods. The natural method to obtain reasonable results consists of treating the higher curvature contributions perturbatively. This approach also guarantees that the black hole exists as the perturbative solution of the higher-order solution provided it exists classically [24]. Finally, observe that in the perturbative approach the casual structure is determined by the classical metric, however, the equations of motion of test particles and various characteristics of the solution acquire the first order correction.

Analyses of the spherically-symmetric and static solutions to the higher derivative theory has been carried out in [14,25–30]. Specifically, in Ref. [30] the perturbative solutions of the ABGB-type to the equations of the effective quadratic gravity have been constructed and discussed. In this paper we shall calculate the entropy of such black holes using Wald's approach [31–33] and confirm the final results employing computationally independent but closely related Euclidean techniques propounded by Visser [34–36].

2. Basic equations

The coupled system of the nonlinear electrodynamics and the quadratic gravity considered in this paper is described by the (Lorentzian) action

$$S=\frac{1}{16\pi}\int\Big(R+\alpha R^{2}+\beta R_{ab}R^{ab}+\gamma R_{abcd}R^{abcd}-\mathcal{L}\left(F\right)\Big)\sqrt{-g}\,d^{4}x\,, \quad (1)$$

where $\mathcal{L}\left(F\right)$ is some functional of $F=F_{ab}F^{ab}$ (its exact form will be given later) and all symbols have their usual meaning. The cosmological constant is assumed to be zero. To simplify our discussion from the very beginning we shall relegate the term involving the Kretschmann scalar, $R_{abcd}R^{abcd}$, from the total action employing the Gauss–Bonnet invariant. The coupling constants α and β have the dimension of length squared and throughout the paper we shall assume

$$\frac{\alpha}{L^2} \sim \frac{\beta}{L^2} \ll 1\,, \tag{2}$$

where L is the local curvature scale. Assumption that the mass scales associated with the linearized equations are real may place additional constrains [20, 37, 38] on α and β. Here, however, we shall treat them as small and of comparable order but arbitrary.

The entropy of the black hole may be calculated using various methods. It seems, however, that Wald's technique is especially well suited for calculations in the higher curvature theories. Here we shall follow this very approach. Other competing techniques are the method based on the field redefinition [33, 39] and Visser's Euclidean approach.

For the Lagrangian involving the Riemann tensor and its symmetric derivatives up some finite order, say n, Wald's Noether charge entropy may be compactly written in the form [31–33]

$$\mathcal{S} = -2\pi \int d^2x\, (h)^{1/2} \sum_{m=0}^{n} (-1)^m\, \nabla_{(e_1 \ldots} \nabla_{e_m)} Z^{e_1 \ldots e_m;abcd} \epsilon_{ab}\epsilon_{cd}\,, \tag{3}$$

where

$$Z^{e_1 \ldots e_m;abcd} = \frac{\partial \mathcal{L}}{\partial \nabla_{(e_1 \ldots} \nabla_{e_m)} R_{abcd}}\,, \tag{4}$$

h is the determinant of the induced metric, ϵ_{ab} is the binormal to the bifurcation sphere, and the integration is carried out across the bifurcation surface. Actually $\mathcal{S}$ can be evaluated not only on the bifurcation surface but on an arbitrary cross-section of the Killing horizon. Since $\epsilon_{ab}\epsilon_{cd} = \hat{g}_{ad}\hat{g}_{bc} - \hat{g}_{ac}\hat{g}_{bd}$, where $\hat{g}_{ac}$ is the metric in the subspace normal to cross section on which the entropy is calculated, one can rewrite Eq. (3) in the form

$$\mathcal{S} = 4\pi \int d^2x\, h^{1/2} \sum_{m=0}^{n} (-1)^m\, \nabla_{(e_1 \ldots} \nabla_{e_m)} Z^{e_1 \ldots e_m;abcd} \hat{g}_{ac}\hat{g}_{bd}\,. \tag{5}$$

The tensor $\hat{g}_{ab}$ is related to $V^a = K^a/||K||$ (K^a is the time-like Killing vector) and the unit normal n^a by the formula $\hat{g}_{ab} = V_a V_b + n_a n_b$.

The general expression describing entropy (5) has been applied in numerous cases, mostly for the Lagrangians that are independent of covariant derivatives of the Riemann tensor and its contractions. In Ref. [40], however, Eq. (5) has been employed in calculations of the entropy of the quantum-corrected black hole when the source term is described by the stress–energy tensor of the quantized fields in a large mass limit. Such a tensor is purely geometrical and besides ordinary higher curvature terms it involves also $R\nabla_a\nabla^a R$ and $R_{ab}\nabla_c\nabla^c R^{ab}$.

On the other hand, one can follow an approach propounded by Visser [34–36]. The general formula for the entropy of the stationary black hole with the Hawking temperature $T_{\rm H}$ is given by

$$\mathcal{S} = \frac{A}{4} + \frac{1}{T_{\rm H}} \int\limits_{\Sigma} \left(\rho_{\rm L} - L_{\rm E}\right) K^a d\Sigma_a + \int\limits_{\Sigma} \boldsymbol{s} V^a d\Sigma_a \,, \tag{6}$$

where A is the area of the event horizon, $\boldsymbol{s}$ is the entropy density associated with the fluctuations (ignored in this paper) and finally $\rho_{\rm L}$ and $L_{\rm E}$ are, respectively, the Lorentzian energy density and the Euclideanized Lagrangian of the matter fields surrounding the black hole. (All higher curvature terms have been inserted into the Lagrangian describing matter fields.) For the specific case of the Einstein–Hilbert action augmented with the higher curvature terms (but not covariant derivatives of curvature) Visser's result is equivalent to Wald's formula.

The coupled system of differential equations describing nonlinear electrodynamics in quadratic gravity can be obtained from the variational principle.

Simple calculations indicate that the tensor F^{ab} and its dual ${}^{*}F^{ab}$, satisfy the equations

$$\nabla_a \left(\frac{d\mathcal{L}\left(F\right)}{dF} F^{ab} \right) = 0 \,, \tag{7}$$

$$\nabla_a {}^{*}F^{ab} = 0 \,, \tag{8}$$

respectively. Differentiating functionally the total action S with respect to the metric tensor one obtains equations of the quadratic gravity in the form

$$L^{ab} \equiv G^{ab} - \alpha I^{ab} - \beta J^{ab} = 8\pi T^{ab}, \tag{9}$$

where

$$I^{ab} = 2\nabla^b\nabla^a R - 2RR^{ab} + \frac{1}{2} g^{ab} \left(R^2 - 4\nabla_c\nabla^c R\right), \tag{10}$$

$$J^{ab} = \nabla^b\nabla^a R - \nabla_c\nabla^c R^{ab} - 2R_{cd}R^{cbda} + \frac{1}{2} g^{ab} \left(R_{cd}R^{cd} - \nabla_c\nabla^c R\right) \tag{11}$$

and

$$T_a^b = \frac{1}{4\pi} \left(\frac{d\mathcal{L}\left(F\right)}{dF} F_{ca}F^{cb} - \frac{1}{4}\delta_a^b \mathcal{L}\left(F\right) \right). \tag{12}$$

In this paper we shall concentrate on the static and spherically-symmetric configurations described by the line element of the form

$$ds^2 = -e^{2\psi(r)}f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2, \tag{13}$$

where

$$f(r) = 1 - \frac{2M(r)}{r}. \tag{14}$$

The spherical symmetry places restrictions on the components of F_{ab} tensor and, consequently, its only nonvanishing components compatible with the assumed symmetry are F_{01} and F_{23}. Simple calculations yield

$$F_{23} = Q\sin\theta \tag{15}$$

and

$$r^2 e^{-2\psi}\frac{d\mathcal{L}(F)}{dF}F_{10} = Q_e\,, \tag{16}$$

where Q and Q_e are the integration constants interpreted as the magnetic and electric charge, respectively.

Since the no-go theorem forbids existence of the regular solutions with $Q_e \neq 0$ in the latter we shall assume that the electric charge vanishes. Now, since $F = 2F_{23}F^{23}$, one has

$$F = \frac{2Q^2}{r^4}. \tag{17}$$

The stress–energy tensor (12) calculated for this configuration is

$$T_t^t = T_r^r = -\frac{1}{16\pi}\mathcal{L}(F) \tag{18}$$

and

$$T_\theta^\theta = T_\phi^\phi = \frac{1}{4\pi}\frac{d\mathcal{L}(F)}{dF}\frac{Q^2}{r^4} - \frac{1}{16\pi}\mathcal{L}(F). \tag{19}$$

Further considerations require specification of the Lagrangian $\mathcal{L}(F)$. Following Ayón-Beato, García and Bronnikov let us chose it in the form

$$\mathcal{L}(F) = F\left[1 - \tanh^2\left(s\sqrt[4]{\frac{Q^2F}{2}}\right)\right], \tag{20}$$

where

$$s = \frac{|Q|}{2b}, \tag{21}$$

and b is a free parameter. Inserting Eq. (17) into (20) and making use of Eq. (21) one obtains

$$\mathcal{L}\left(F\right) = \frac{2Q^2}{r^4}\left(1 - \tanh^2 \frac{Q^2}{2br}\right). \tag{22}$$

The system of coupled differential equations of the quadratic gravity with the source term given by (18) and (19) with (22) is rather complicated and cannot be solved exactly. Fortunately, since the coupling constants α and β are expected to be small in a sense of Eq. (2), one can treat the system of the differential equations perturbatively, with the classical solution of the Einstein field equation taken as the zeroth-order approximation. Successive perturbations are therefore solutions of the chain of the differential equations of ascending complexity [41–44]. It should be noted, however, that the higher order equations are probably intractable analytically and the technical difficulties may limit the calculations to the first order.

In the next section, we shall employ perturbative techniques to construct the approximate solution to the equations of the quadratic gravity with the source term being the stress–energy tensor of the Bronnikov type. Such an approach is expected to yield reasonable results and because of complexity of the differential equations, it may be the only way to deal with this problem.

3. Solutions

To keep control of the order of terms in complicated series expansions we shall introduce a dimensionless parameter ε substituting $\alpha \to \varepsilon\alpha$ and $\beta \to \varepsilon\beta$. We shall put $\varepsilon = 1$ at the final stage of calculations. Of functions $M\left(r\right)$ and $\psi\left(r\right)$ we assume that they can be expanded in powers of the auxiliary parameter as

$$M\left(r\right) = M_0\left(r\right) + \varepsilon M_1\left(r\right) + \mathcal{O}\left(\varepsilon^2\right) \tag{23}$$

and

$$\psi\left(r\right) = \varepsilon\psi_1\left(r\right) + \mathcal{O}\left(\varepsilon^2\right). \tag{24}$$

First, consider the left hand side of Eq. (9) calculated for the line element (13) with the functions $M(r)$ and $\psi(r)$ given by (23) and (24), respectively. Making use of the above expansions and subsequently collecting the terms with the like powers of ε, after some rearrangements, one obtains [30]

$$L_t^t = -\frac{2}{r^2}\left(M_0' + \varepsilon M_1' - \varepsilon S_t^t\right), \tag{25}$$

where

$$S_t^t = \beta\left(\frac{2M_0'}{r^2} - \frac{8M_0M_0'}{r^3} + \frac{2M_0'^2}{r^2} - \frac{2M_0''}{r} + \frac{5M_0M_0''}{r^2} - \frac{M_0'M_0''}{r}\right.$$
$$\left. + \frac{M_0''^2}{2} + M_0^{(3)} - \frac{M_0M_0^{(3)}}{r} - M_0'M_0^{(3)} + rM_0^{(4)} - 2M_0\,M_0^{(4)}\right)$$
$$-\alpha\left(\frac{24M_0M_0'}{r^3} - \frac{8M_0'}{r^2} - \frac{4M_0'^2}{r^2} + \frac{8M_0''}{r} - \frac{18M_0M_0''}{r^2} - M_0''^2\right.$$
$$\left. + \frac{2M_0'M_0''}{r} - 4M_0^{(3)} + \frac{6M_0M_0^{(3)}}{r} + 2M_0'M_0^{(3)} - 2rM_0^{(4)} + 4M_0M_0^{(4)}\right) \quad (26)$$

and M_0', M_0'' and $M_0^{(i)}$ for $i \geq 3$ denote first, second and i-th derivatives with respect to the radial coordinate. On the other hand, a simple combination of the components of L_a^b tensor

$$L_r^r - L_t^t = 0 \quad (27)$$

can be easily integrated to yield [30]

$$\psi_1(r) \;=\; (2\alpha+\beta)M_0^{(3)} - \frac{4}{r^2}(3\alpha+\beta)M_0' + C_1\,, \quad (28)$$

where C_1 is the integration constant. It should be noted that contrary to the case of coupled system of the Maxwell equations and quadratic gravity considered in Refs. [25–27, 29], now we have explicit dependence on the parameter α. A comment is in order here regarding the independence of the final result calculated for the Maxwell source on the parameter α. First, observe that the stress–energy tensor of the electromagnetic field for the spherically-symmetric an static configuration with a total charge e assumes simple form

$$T_a^b = -\frac{e^2}{8\pi r^4}\mathrm{diag}[1,1,-1,-1]\,. \quad (29)$$

Therefore, the zeroth-order solution to the (0,0)-component of the equation (9) can be written in the form

$$M_0(r) = -\frac{e^2}{2r} + C\,, \quad (30)$$

where C is the integration constant. Now, substituting (30) into (26) and (28) it can easily be demonstrated that the expression in the second bracket in its right hand side of Eq. (26) as well as the term $M_0^3 - 6M_0'/r^2$ in (28) vanish.

One expects that all characteristics of the black hole, such as the location of the horizons and temperature could also be calculated perturbatively. In the latter, for simplicity, we shall refer to the perturbative solutions of the quadratic gravity using the names of their classical counterparts (the zeroth-order solutions) whenever it will not lead to confusion.

To develop the model further one has to determine the integration constants and the free parameter b. There are, in general, two interesting and physically motivated choices. One can relate the integration constant with the exact location of the event horizon, r_+, and this can easily be done with the aid of the equation

$$M\left(r_+\right) = \frac{r_+}{2}\,. \tag{31}$$

On the other hand it is possible to express solutions of the system of differential equations consisting of $(0,0)$ component of Eqs. (9) and Eq. (27) in terms of the total mass $\mathcal{M}$ as seen by a distant observer

$$\lim_{r\to\infty} M\left(r\right) = \mathcal{M}\,. \tag{32}$$

For the function $\psi\left(r\right)$ we shall always adopt the natural condition

$$\lim_{r\to\infty} \psi\left(r\right) = 0\,. \tag{33}$$

Inspection of Eqs. (25) and (27) reveals their different status. Indeed, Eq. (27) can easily be integrated for a general function $M_0(r)$ and the final solutions is to be obtained by differentiation of the zeroth-order solution and making use of the boundary conditions. On the other hand, the first integral of the differential equation for $M_1\left(r\right)$ cannot be constructed and one has to know the zeroth-order solution to determine M_1.

The assumed expansions of the functions $M\left(r\right)$ and $\psi\left(r\right)$ as given by Eqs. (23) and (24), respectively, suggest that one can rewrite the boundary conditions of the first type in the following form:

$$M_0\left(r_+\right) = \frac{r_+}{2}\,, \qquad M_1\left(r_+\right) = 0\,, \qquad \psi_1\left(\infty\right) = 0\,, \tag{34}$$

whereas for the boundary conditions of the second type one has

$$M_0\left(\infty\right) = \mathcal{M}\,, \qquad M_1\left(\infty\right) = 0\,, \qquad \psi_1\left(\infty\right) = 0\,. \tag{35}$$

Now, let us concentrate on the zeroth-order equations supplemented with the conditions of the first type. Putting $\varepsilon = 0$ in Eq. (25), form (22) and (18) one obtains

$$\frac{dM_0}{dr} = \frac{Q^2}{2r^2}\left(1 - \tanh^2\frac{Q^2}{2br}\right), \tag{36}$$

which can be easily integrated to yield

$$M_0(r) = -b\tanh\frac{Q^2}{2br} + C_2\,. \tag{37}$$

Finally, making use of the conditions (34) one arrives at the desired result

$$M_0\left(r\right) = \frac{r_+}{2} + b\tanh\frac{Q^2}{2br_+} - b\tanh\frac{Q^2}{2br}\,. \tag{38}$$

The thus obtained solution reduces to the Schwarzschild solution for $Q = 0$ and it can be easily demonstrated that, by (26) and the boundary conditions (34) it remains so in the higher-order calculations.

To specify the solution further we shall make use of the well-known relation [45]

$$\mathcal{M} = \frac{\kappa A_{\mathrm{H}}}{4\pi} - \int\limits_{\Sigma}\left(2T_a^b - T\delta_a^b\right)K^a d\Sigma_b\,, \tag{39}$$

where Σ is a constant time hypersurface and K^a is a time-like Killing vector and apply it to the zeroth-order solution. Making use of the explicit form of the stress–energy tensor of the nonlinear electrodynamics one obtains

$$M_{\mathrm{H}} = \frac{r_+}{2} + b\tanh\frac{Q^2}{2br_+}\,, \tag{40}$$

where M_{H} is the mass connected with the zeroth-order solution. We shall refer to M_{H} as to the horizon defined mass of the black hole.

To develop the model further one has to determine the free parameter b. Our choice, which guarantees regularity of the zeroth-order line element at the center, is $b = M_{\mathrm{H}}$, and hence Eq. (38) becomes

$$M_0\left(r\right) = M_{\mathrm{H}}\left(1 - \tanh\frac{Q^2}{2M_{\mathrm{H}}r}\right). \tag{41}$$

Unfortunately, the regularity of the zeroth-order solution does not guarantee regularity of the higher-order perturbative solutions [30].

It should be noted that the $M_{\mathrm{H}} = M_{\mathrm{H}}\left(Q, r_+\right)$ and for fixed Q and r_+ one has to determine M_{H} numerically. On the other hand it is possible to employ the equation $M\left(r_+\right) = r_+/2$ in the zeroth-order calculations and express all the results in (Q, M_{H}) parametrization instead of (Q, r_+). One can, therefore, construct solutions of this equation in terms of the Lambert function. Simple manipulations yield

$$r_+ = -\frac{4M_{\mathrm{H}}Q^2}{4W_+\left(-\rho e^{\rho}\right)M_{\mathrm{H}}^2 - Q^2}\,, \tag{42}$$

where W_+ is a principal branch of the Lambert function and $\rho = Q^2/4M_{\rm H}^2$. Analogous solution for the inner horizon can be written in the form

$$r_- = -\frac{4M_{\rm H}Q^2}{4W_-\left(-\rho e^{\rho}\right)M_{\rm H}^2 - Q^2}\,, \tag{43}$$

where W_- is the second real branch of the Lambert function. (In fact, W_+ and W_- are the only real branches.) Making use of the elementary properties of the Lambert functions one can demonstrate that the principal branch has the expansion

$$W_+(x) = x - x^2 + \frac{3}{2}x^3 - \frac{8}{3}x^4 + \mathcal{O}\left(x^5\right)\,. \tag{44}$$

On the other hand, $W_-(x) \to -\infty$ as $x \to 0$, and, consequently, the location of the event horizon tends to the Schwarzschild value whereas $r_- \to 0$.

A typical run of $M_{\rm H}$ as a function of ξ for a few exemplary values of Q is shown in Fig. 1. For a given Q a line of $M_{\rm H} =$ const. intersects $Q =$ const. curve at one or two points or it has no intersection points at all. The smaller one gives location of the inner horizon whereas the greater is to be identified with the event horizon. The minimum of $M_{\rm H} = M_{\rm H}\left(Q = \text{const.}, \xi\right)$ function represents extremal configurations when the two horizons merge.

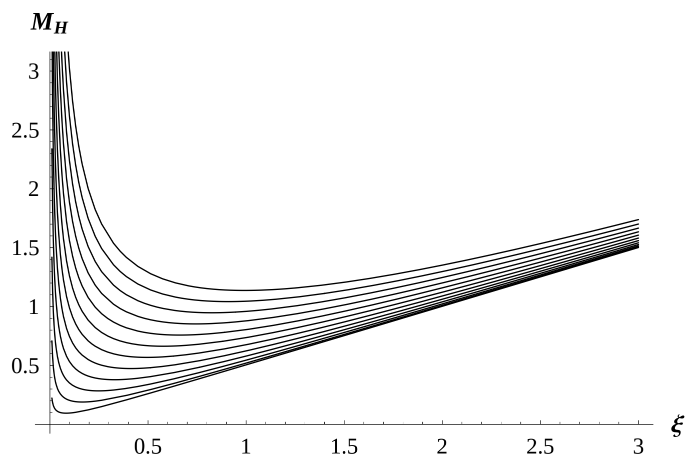

Fig. 1. This graph shows solutions of the equation $M_{\rm H} = \frac{\xi}{2} + M_{\rm H}\tanh\frac{Q^2}{2M_{\rm H}\xi}$ for a few exemplary values of the charge Q. From bottom to top the curves correspond to $Q = 0.1i$, for $i = 1, \ldots, 12$. For $M_{\rm H} =$ const., the greater solution represents location of the event horizon, r_+ whereas the smaller one represents the inner horizon, r_-. The minimum of each curve corresponds to the extremal configuration with $r_+ = r_-$.

It should be noted that the mass $M_{\rm H}$ is not the mass that would be measured for the perturbed black hole by an observer at infinity. Indeed, even for the zeroth-order solutions the meaning of $M_{\rm H}$ and $\mathcal{M}$ is different and the substantial differences are transparent in the first order calculations. This can be easily seen by studying the limit

$$\mathcal{M} = \lim_{r\to\infty} M\left(r\right) = M_{\rm H} + \varepsilon M_{1}\left(\infty\right) \tag{45}$$

and Eq. (32). Identical result can be obtained from Eq. (39). Indeed, in order to apply (39) for the perturbed black hole one has to move the higher curvature terms of Eq. (9) into its right hand side and treat them as a contribution to the total stress–energy tensor. It can be demonstrated explicitly, that making use of Eq. (39) in the first-order calculations one obtains precisely (45).

The function $M_1(r)$ can be expressed in terms of the polylogarithms. Unfortunately, it is rather complicated and, to avoid unnecessary proliferation of long formulas, it will not be displayed here. The first-order solution can be constructed employing the algorithm presented in Appendix of Ref. [30]. It should be noted that the function $M_1(r)$ presented in [30] is calculated for the boundary conditions of the second type.

4. The entropy

Now, let us return to our main theme and calculate the entropy of the ABGB black hole. In doing so we shall put special emphasis on comparison of the results constructed for the nonlinear black hole with the analogous results obtained for the Reissner–Nordström solution. Such a comparison is especially interesting as the geometries of their classical counterparts are practically indistinguishable in two important regimes. To demonstrate this it suffices to expand the metric potentials in powers of $|Q|/r_+$ and r_+/r, respectively. Since the expansion takes the form

$$f(r) \;=\; 1 - \frac{2M_{\rm H}}{r} + \frac{Q^2}{r^2} - \frac{Q^6}{12M_{\rm H}^2 r^4} + \dots\,, \tag{46}$$

the differences in the metric structure between ABGB and RN geometries in the exterior region for $|Q|/r_+ \ll 1$ are small indeed. One has a similar behavior for any (allowable) value of the charge for $r \gg r_+$.

The higher curvature terms in the action functional lead to the appearance of additional terms in the final expression describing entropy, which spoil area/entropy relation. Simple calculations carried out within the Noether charge framework indicate that the contribution of the quadratic part of the action to the entropy is given by

$$\delta\mathcal{S} = 2\pi r_{+}^{2}\left[\alpha R + \frac{1}{2}\beta\left(R_{t}^{t} + R_{r}^{r}\right)\right]_{|r_{+}}. \tag{47}$$

Now, substituting the line element (13) with (14) and (23), into the general expression (47), expanding the right hand side of Eqs. (9) with respect to ε, and, finally, retaining the linear terms only, one gets

$$\mathcal{S} = \pi r_{+}^{2} + 2\pi\varepsilon r_{+}^{2}\left[\frac{4\alpha}{r_{+}^{2}}M_{0}'\left(r_{+}\right) + \frac{2\alpha+\beta}{r_{+}}M_{0}''\left(r_{+}\right)\right] + \mathcal{O}\left(\varepsilon^{2}\right). \tag{48}$$

For the non extreme black hole with the boundary conditions of the first type (34) one has

$$\begin{aligned}\mathcal{S} =& \pi r_{+}^{2} + \frac{2\pi Q^{2}}{M_{\rm H} r_{+}^{3}}\varepsilon\cosh^{-2}\left(\frac{Q^{2}}{2M_{\rm H}r_{+}}\right)\left\{\alpha Q^{2}\tanh\left(\frac{Q^{2}}{2M_{\rm H}r_{+}}\right)\right.\\ &\left.-\beta\left[M_{\rm H}r_{+} - \frac{Q^{2}}{2}\tanh\left(\frac{Q^{2}}{2M_{\rm H}r_{+}}\right)\right]\right\},\end{aligned} \tag{49}$$

where $M_{\rm H} = M_{\rm H}\left(Q, r_{+}\right)$. In this approach the zeroth-order solution (41) determines the first order correction to the entropy of the non extreme black hole completely. Having established $M_{\rm H}$ for given Q and r_{+} one can rewrite Eq. (49) putting $\tilde{\mathcal{S}} = \mathcal{S}/M_{\rm H}^{2}$, $q = |Q|/M_{\rm H}$, $x_{+} = r_{+}/M_{\rm H}$, $\tilde{\alpha} = \alpha/M_{\rm H}^{2}$ and $\tilde{\beta} = \beta/M_{\rm H}^{2}$. Simple manipulations yield

$$\tilde{\mathcal{S}} = \pi x_{+}^{2} + \varepsilon\frac{2\pi q^{2}}{x_{+}^{3}}\cosh^{-2}\frac{q^{2}}{2x_{+}}\left[\tilde{\alpha}Q^{2}\tanh\frac{q^{2}}{2x_{+}} - \tilde{\beta}\left(x_{+} - \frac{q^{2}}{2}\tanh\frac{q^{2}}{2x_{+}}\right)\right]. \tag{50}$$

This results can be contrasted with the analogous result constructed for the Reissner–Nordström black hole

$$\mathcal{S} = \pi r_{+}^{2} - 2\beta\frac{\pi Q^{2}}{r_{+}^{2}} \tag{51}$$

or

$$\tilde{\mathcal{S}} = \pi(1+\sqrt{1-q^{2}})^{2} - 2\tilde{\beta}\frac{\pi q^{2}}{(1+\sqrt{1-q^{2}})^{2}}. \tag{52}$$

To investigate the entropy $\mathcal{S}$ as given by Eq. (49) let us observe that for $|Q|/r_{+} \ll 1$ one has $r_{+} \approx 2M_{\rm H}$. Now, expanding hyperbolic functions in powers of $|Q|/r_{+}$ one obtains

$$\mathcal{S} = \pi r_{+}^{2} - 2\pi\beta\frac{Q^{2}}{r_{+}^{2}} + \mathcal{O}\left(\left(\frac{Q}{r_{+}}\right)^{4}\right). \tag{53}$$

A comparison of Eqs. (53) and (51) shows that for $|Q|/r_+ \ll 1$ the entropies of the ABGB and RN black holes are almost indistinguishable, as expected. It should be noted that contrary to the Reissner–Norström geometry, the entropy of the ABGB black hole depends on α and for $|Q|/r_+ \ll 1$ the leading behavior of this terms is $\sim (Q/r_+)^4$.

The analysis of the extremal configuration is more involved. First, let us return to the zeroth-order solution. It should be emphasized that although we do not ascribe any particular meaning to the zeroth-order solution, some of its features do possess clear and unambiguous meaning. For the boundary conditions (34) such a solution is described by the exact r_+ and Q. The extremality condition places additional relation between the elements of the pair (Q, r_+) or $(Q, \mathcal{M})$. Here we shall confine ourselves to the first pair. Simple considerations yield

$$|Q| = 2w^{1/2} M_{\rm H} \tag{54}$$

and

$$r_+ = \frac{4w}{1+w} M_{\rm H}\,, \tag{55}$$

where $w = W_+(1/e)$, and consequently

$$\frac{|Q|}{r_+} = \frac{1+w}{2w^{1/2}}\,. \tag{56}$$

Returning to the first-order solution we recall the relation valid for the extremal configuration in the Reissner–Nordström geometry

$$r_+ = |Q|\,. \tag{57}$$

In Ref. [12] we have ascribed this simple relation to tracelessness of the stress–energy tensor of the matter fields. As the stress–energy tensor of the nonlinear electrodynamics considered in this paper has a nonzero trace, one expects that the analogous relation between Q and r_+ in the ABGB geometry is more complicated. Indeed, after some algebra, one has

$$r_+ = \frac{2w^{1/2}|Q|}{(1+w)}\left[1 + \varepsilon \frac{\beta + 2\alpha}{16 Q^2 w}(w+3)\left(w^2-1\right)\right]. \tag{58}$$

Now, making use of (58) in (49) gives

$$\mathcal{S}_{\rm extr} = \frac{4\pi w Q^2}{(1+w)^2} - \frac{\pi\varepsilon}{2(1+w)}\left[(2\alpha+\beta)\,w^2 + 2\,(2\alpha-\beta)\,w + 5\alpha + 2\beta\right], \tag{59}$$

and the first term of the right hand side coincides with the Bekenstein–Hawking entropy [46]. Numerically, one has

$$\mathcal{S}_{\rm extr} = \pi Q^2 \times 0.6815 - 2\pi\varepsilon \times (0.0324\alpha + 0.1754\beta)\,, \tag{60}$$

where a common factor 2π has been singled out for convenience. Analogous relation for the extremal Reissner–Nordström black hole reads

$$\mathcal{S}_{\rm extr} = \pi Q^2 - 2\pi\varepsilon\beta\,. \tag{61}$$

Now, let us calculate the entropy of the ABGB black hole employing the Euclidean techniques propounded by Visser. First, observe that if the Lagrangian is arbitrary function of the Riemann tensor (and its contractions) but is independent of its covariant derivatives, both methods, *i.e.* Wald's approach and Visser's method are equivalent. One may wonder, therefore, why we intend to carry out such a calculation. The answer is simple: although both methods should yield identical results, the calculational steps necessary to obtain the final result are quite different and consequently one can consider the calculations carried out within the framework of one method as the useful check of the other. It is especially important in situations when the computational complexity of the considered problem may lead to numerous errors.

The calculations proceed in a few steps. First, incorporate the Euclidean action functional of the quadratic gravity into the matter part of the action. Similarly, the (Lorentzian) energy density is given by

$$\rho = -T_t^t = \frac{1}{16\pi}\mathcal{L}\left(F\right) - \varepsilon\left(\frac{\alpha}{8\pi}I_t^t + \frac{\beta}{8\pi}J_t^t\right). \tag{62}$$

It could easily be demonstrated that $\rho_{\rm L} - L_{\rm E} \sim O\left(\varepsilon\right)$ and consequently it suffices to know the Hawking temperature to the zeroth-order. Moreover, due to subtle cancellations in the integrand of Eq. (6) the final result of the quadratures does not contain polylogarithm functions. Now, substituting

$$T_{\rm H} = \frac{1}{4\pi r_+}\left(1 - \frac{Q^2}{M r_+} + \frac{Q^2}{4M_{\rm H}^2}\right) \tag{63}$$

and (62) into (6), after some algebra, one has

$$\delta\mathcal{S} = \frac{\varepsilon}{r_+^4\left(\eta+1\right)^5}\left[\alpha s_\alpha + \beta s_\beta\right], \tag{64}$$

where $\eta = \exp\left(Q^2/2M_{\rm H}r_+\right)$,

$$s_\alpha = \left(\frac{4Q^6}{r_+ M_{\rm H}^2} - \frac{20Q^4}{M_{\rm H}}\right)\eta^4 - \left(\frac{20Q^4}{M_{\rm H}} - \frac{72Q^4}{r_+} + \frac{12Q^6}{r_+ M_{\rm H}^2} + \frac{8Q^6}{r_+^2 M_{\rm H}}\right)\eta^3 + \left(\frac{20Q^4}{M_{\rm H}} - \frac{12Q^6}{r_+ M_{\rm H}^2} + \frac{56Q^6}{r_+^2 M_{\rm H}}\right)\eta^2 + \left(\frac{4Q^6}{r_+ M_{\rm H}^2} - \frac{72Q^4}{r_+} + \frac{20Q^4}{M_{\rm H}} - \frac{16Q^6}{r_+^2 M_{\rm H}}\right)\eta \tag{65}$$

and

$$s_\beta = \left(8Q^2 r_+ + \frac{2Q^6}{r_+ M_{\rm H}^2} - \frac{12Q^4}{M_{\rm H}}\right)\eta^4 - \left(24\, M_{\rm H}\, Q^2 + \frac{4Q^6}{r_+^2 M_{\rm H}} + \frac{12Q^4}{M_{\rm H}} - 24\, Q^2 r_+ + \frac{6Q^6}{r_+ M_{\rm H}^2} - \frac{36Q^4}{r_+}\right)\eta^3 + \left(\frac{12Q^4}{M_{\rm H}} - \frac{6Q^6}{r_+ M_{\rm H}^2} - \frac{8Q^4}{r_+} + Q^2 r_+ + \frac{28Q^6}{r_+^2 M_{\rm H}} - 48\, M_{\rm H}\, Q^2\right)\eta^2 - \left(24 M_{\rm H}\, Q^2 - 8\, Q^2 r_+ - \frac{12Q^4}{M_{\rm H}} - \frac{2Q^6}{r_+ M_{\rm H}^2} + \frac{44Q^4}{r_+} + \frac{8Q^6}{r_+^2 M_{\rm H}}\right)\eta\,. \tag{66}$$

At first glance this result does not resemble Eq. (49). However, making use of the identity

$$\eta = \frac{4M_{\rm H}}{r_+} - 1\,, \tag{67}$$

one can easily demonstrate that Eqs. (64)–(66) reduce precisely to Eq. (49).

5. Final remarks

In this paper we have constructed the entropy of the nonlinear ABGB-type black holes using the boundary conditions (34). The zeroth-order solution coincides, as expected, with the ABGB line element whereas the first-order correction can be elegantly expressed in terms of the polylogarithm functions. Now, let us briefly discuss the consequences of the second choice, in which the results are expressed in terms of the total mass of the system as measured by a distant observer. To calculate the location of the event horizon to the required order in $(Q, \mathcal{M})$ parametrization one has to solve the first-order equations for $M_1(r)$ and $\psi_1(r)$, and, subsequently, perturbatively solve the equation $g_{tt}(r_+) = 0$ assuming that the event horizon can be expanded as

$$r_+ = r_0 + \varepsilon r_1 + O(\varepsilon^2)\,. \tag{68}$$

Unfortunately, the function $M_1(r)$ is rather complicated (it can be expressed in terms of the polylogarithms) and, once again, to avoid unnecessary proliferation of long formulas it will not be presented here. Interested reader is referred to [30].

Generally, for the nonexterme black hole one has

$$\mathcal{S} = \pi r_0^2 + 2\pi r_1 \varepsilon + 32\pi\varepsilon r_0^2 \left[\frac{4\alpha}{r_0^2} M_0'(r_0) + \frac{2\alpha+\beta}{r_0} M_0''(r_0) \right] + \mathcal{O}\left(\varepsilon^2\right). \quad (69)$$

On the other hand, making use of (68), the equation (69) can be rewritten in the equivalent form

$$\mathcal{S} = \pi r_0^2 + \varepsilon \frac{4\pi M_1(r_0)}{1 - 2M_0'(r_0)} + 32\pi\varepsilon r_0^2 \left[\frac{4\alpha}{r_0^2} M_0'(r_0) + \frac{2\alpha+\beta}{r_0} M_0''(r_0) \right] + \mathcal{O}\left(\varepsilon^2\right). \quad (70)$$

The extremal case should be analyzed separately. The extremal configuration of the ABGB black hole being the solution of the Einstein gravity is described by

$$|Q_c| = 2w^{1/2}\mathcal{M} \quad (71)$$

and

$$r_c = \frac{4w}{1+w}\mathcal{M}. \quad (72)$$

One expects, that the higher-order curvature terms modify these relations, shifting (in a space of the parameters) extremal solution into a slightly different position. Indeed, treating M_0 as a function of Q^2 and r, after some algebra, one concludes that the extremal configuration is still possible and is described by the relations

$$Q^2 = Q_c^2 + \varepsilon\Delta, \qquad r_+ = r_c + \varepsilon\delta, \quad (73)$$

where

$$\Delta = -\left(\frac{\partial}{\partial Q^2} M_0\right)^{-1} M_1. \quad (74)$$

and

$$\delta = -\left(\frac{\partial^2}{\partial r^2} M_0\right)^{-1} \left(\frac{\partial}{\partial r} M_1\right) + \left(\frac{\partial}{\partial Q^2} M_0 \frac{\partial^2}{\partial r^2} M_0\right)^{-1} \left(M_1 \frac{\partial^2}{\partial r \partial Q^2} M_0\right). \quad (75)$$

Both δ and Δ are to be calculated for the parameters describing extremal zeroth-order solution. Numerically, one has

$$\Delta = \frac{1.05314}{\mathcal{M}}\alpha + \frac{0.43288}{\mathcal{M}}\beta \quad (76)$$

and

$$\delta = -\frac{0.05121}{\mathcal{M}}\alpha - \frac{0.57553}{\mathcal{M}}\beta \,. \tag{77}$$

Since the calculations of the entropy follow the general scheme sketched in previous section they will not be presented here.

The purpose of the present paper (besides importance of the quadratic gravity in its own and the natural curiosity) is twofold. First, one can treat the calculations presented in this paper as the first step in understanding of the influence of the higher curvature terms on the entropy of black holes in a more complex setting than Maxwell electrodynamics. The next step would involve, for example, inclusion of the all curvature invariants of the order 4 and 6 and degree 2 and 3 [47–49]. Moreover, it would be interesting to extend this analysis to general D-dimensional manifolds. The natural candidate for a higher-curvature theory is the Lovelock gravity [50]. Moreover, one may consider the more general curvature terms, with arbitrary coefficients rather than those inspired by particular theory. (See, for example [47, 51] and references cited therein.) On the other hand, and this is even more interesting, one can regard this sort of calculations as the preliminary results allowing to analyze and understand the typical subtleties one is likely to encounter when studying the semi-classical equations with the source term given by the renormalized stress–energy tensor of the quantized massive fields. Of course, the semi-classical equations are extremely complex [8, 52], but the general pattern that lies behind the calculations should remain the same. This group of problems are currently actively investigated and the results will be published elsewhere.

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