ON THE POSITIVITY OF MATTER ENERGY

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The positivity of the matter-energy density in the Universe, defined from the energy-momentum tensor as $\rho \equiv T_0^0$ — one of the initial assumptions in the positive-energy conjecture of Arnowitt *et al.* — is related to the Lorentzian signature of space-time, assumed to be spatially flat, via the Friedmann equation, by applying the Faddeev (Newton–Wigner) propagator K for the cosmological Schrödinger equation in the semi-classical approximation, the corresponding Euclidean propagator, which allows negative ρ, decaying on the Planck time-scale. A corollary of this result is that the masses of all elementary particles, and hence of all astrophysical bodies and black holes, are positive semi-definite.

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1. Introduction

The field equations of the Einstein theory of general relativity[1, 2],

$$R_{ij} - \tfrac{1}{2} g_{ij} R = \kappa^2 T_{ij} \,, \tag{1}$$

admit solutions for which the source is characterised by either positive or negative energy-density ρ (and pressure p), assuming for simplicity the perfect-fluid form of the energy-momentum tensor,

$$T_{ij} = (\rho + p) u_i u_j - p g_{ij} \,. \tag{2}$$

Here g_{ij} is the metric of a time-orientable four-space whose signature is $(+ - - -)$, R_{ij} is the Ricci tensor, $\kappa^2 \equiv 8\pi G_{\mathrm{N}}$ is the gravitational coupling,

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$G_\mathrm{N} \equiv M_\mathrm{P}^{-1/2}$ being the Newton gravitational constant and M_P the Planck mass, and u^i is a time-like unit normal vector which satisfies

$$u_i u^i = 1\,. \tag{3}$$

Writing the line element in the form

$$ds^2 = dt^2 - \gamma_{\alpha\beta} dx^\alpha dx^\beta\,, \tag{4}$$

where t is comoving time and $\gamma_{\alpha\beta}$ is the spatial three-metric, we have

$$u^i = (1, 0, 0, 0)\,, \tag{5}$$

so that $\rho \equiv T_0^0$, and it becomes a fundamental question why this quantity is always observed (and therefore also assumed) to be positive semi-definite. The pressure, however, is not subject to the same constraint, since it can be related to ρ by the adiabatic index γ as

$$p = (\gamma - 1)\rho\,, \tag{6}$$

where $-\rho \leq p \leq \rho$ if causality is to hold, so that $0 \leq \gamma \leq 2$. (We restrict consideration to theories for which the kinetic-energy terms of scalar fields ϕ_n are linear in $(\nabla\phi_n)^2$.)

Similarly, the Dirac equation[3] for a fermionic spinor of mass m and spin $\frac{1}{2}$ in flat space-time,

$$(i\gamma^k \partial_k - m)\psi = 0\,, \tag{7}$$

where γ^k are the Dirac matrices, allows the existence of particles of positive and negative electrical charge $\pm q$, which led to the hole theory[4] and the prediction of the positron, named and confirmed experimentally by Anderson [5] from observations of cosmic-ray tracks, and implying the phenomena of vacuum polarisation and pair creation. In the theory of nuclear matter[6], the rôle of the infinite sea of negative-energy states remains a subject of debate — they are either ignored in the no-sea approximation or taken into account via non-linear and derivative scalar-potential terms.

The mass of an electron, say, enters linearly in Eq. (7), and therefore its sign is chosen to be positive, but the argument leading to the prediction of anti-particles proceeds via pre-multiplication of Eq. (7) by the conjugate operator $(i\gamma^k\partial_k + m)$ to produce the Klein–Gordon equation (see §§ 67, 73 of the fourth part of Refs. [3])

$$(\Box + m^2)\psi = 0\,, \tag{8}$$

which likewise admits solutions of positive- and negative-energy for a particle of momentum $\boldsymbol{p}$,

$$E = \pm\sqrt{\boldsymbol{p}^2 + m^2}\,. \tag{9}$$

The negative-energy solutions for an electron can be re-interpreted as positive-energy solutions for a positron, whose existence can be understood from the hole theory[4], in which all the negative-energy states are assumed to be occupied, the vacation of such a state bringing the positron into being. But the equation of motion (8) and its solution (9) are invariant under the operator $\mathcal{M}$, defined by

$$m \to -\tilde{m}\,, \tag{10}$$

so the theory could equally well apply to particles of negative-energy, if there were a sea of occupied positive-energy states. The question then recurs of why all the negative-energy states are excluded, while positive and negative electrical charges are permitted equally.

Now solutions to the Einstein equations (1) involving charged objects invariably require the square of the charge, the simplest example being the Reissner–Nordström space-time[7] for a black hole of mass M and charge Q, expressed in co-ordinates (t', r, θ, ϕ),

$$\begin{aligned} ds^2 \;=\; & \left(1 - 2Mr^{-1} + Q^2 r^{-2}\right) dt'^2 - \left(1 - 2Mr^{-1} + Q^2 r^{-2}\right)^{-1} dr^2 \\ & - r^2 (d\theta^2 + \sin^2\theta d\phi^2)\,, \end{aligned} \tag{11}$$

and are thus invariant under the charge-conjugation operator $\mathcal{C}$. Therefore, it is easy to guess that the absence of negative energy in the Universe has something to do with the geometry, since the metric (11) is not invariant under the operator (10). The local geometry, in turn, is determined by the entire mass distribution in the Universe, and thus the non-linearity of the Einstein theory leads us to seek an explanation for the positivity of mass-energy by applying Machian precepts, some feature of the global geometry now being postulated first — the interrelationship between geometry (that is, g_{ij}) and matter (that is, T_{ij}) on the global scale is discussed in pp. 241–243 of the second part of Refs. [2], where Einstein introduced what he called "Mach's principle", thus generalising the requirement of Mach that inertia be due to an interaction between bodies.

2. The Friedmann Universe

Let us, therefore, consider the Friedmann solution[8] describing the expansion of the spatially isotropic Universe (4) with the perfect-fluid source (2). The line element is now

$$ds^2 = dt^2 - a^2(t)\, d\boldsymbol{x}^2\,, \tag{12}$$

where $a(t) \equiv a_0 \exp[\alpha(t) - \alpha(t_0)]$ is the radius function of the three-space of curvature k, expressible in polar co-ordinates (r, θ, ϕ) as

$$d\boldsymbol{x}^2 = \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2\theta d\phi^2)\,, \tag{13}$$

while the field equations (1) reduce to

$$\xi^2 = \tfrac{1}{3}\kappa^2\rho - \frac{k}{a^2} \tag{14}$$

and

$$\dot{\xi} = -\tfrac{1}{2}\kappa^2\gamma\rho\,, \tag{15}$$

where $\xi \equiv \dot{\alpha}$ and $\dot{} \equiv d/dt$. Solutions to this system of equations are discussed by Heckmann[9], assuming that $\rho \geq 0$ and allowing all three curvatures $k = 0, \pm 1$, completing the analyses of the cases $k = \pm 1$ carried out by Friedmann[8].

For simplicity, suppose that $k = 0$ so that $d\boldsymbol{x}^2$ is flat — as previously argued from quantum cosmology[10] — and ask what happens if ρ becomes negative. As long as κ^2 remains positive, Eq. (14) then becomes inconsistent, the only possible resolution being to Wick-rotate the time into the Euclidean sector via the operator $\mathcal{T}$, defined by

$$t \to \pm i\,\tilde{t}\,. \tag{16}$$

More generally, we find that Eqs. (14), (15) are invariant under the combined operators $\mathcal{M}$, $\mathcal{T}$ and $\mathcal{K}$ (which reverses the spatial curvature k) applied both to ρ and p, that yield the transformations

$$\left.\begin{aligned} \rho &\to -\tilde{\rho}\,, \\ p &\to -\tilde{p}\,, \\ u^i &\to \mp i\tilde{u^i}\,, \\ k &\to -\tilde{k}\,. \end{aligned}\right\} \tag{17}$$

They can then be written as

$$\tilde{\xi}^2 = \tfrac{1}{3}\kappa^2\tilde{\rho} - \tilde{k}/a^2 \tag{18}$$

and

$$\tilde{\xi}' = -\tfrac{1}{2}\kappa^2\gamma\tilde{\rho}\,, \tag{19}$$

where $\tilde{\xi} = \alpha'$ and $' \equiv d/d\tilde{t}$, leading to exactly the same solution as before, but referred to the Euclidean line element corresponding to (12) and (13),

$$ds^2 = -d\tilde{t}^{\,2} - a^2\left(\tilde{t}\right)\left[\frac{dr^2}{1 + \tilde{k}r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]\,, \tag{20}$$

in which $\tilde{u}^i = (1, 0, 0, 0), \tilde{u}_i = (-1, 0, 0, 0)$.

Thus, it is now clear that negative energies in cosmology involve Euclidean spaces and vice versa. "Time" is on the same footing as space and the usual physical interpretation in terms of "evolution" and "propagation" becomes untenable, since all four co-ordinates have the same basic geometrical character. From the classical viewpoint, however, the Euclidean solution (18)–(20) is just as good as the Lorentzian solution (12)–(15), raising the question why one is selected rather than the other. For either allows a possible mathematical description of gravity, both globally and locally. As mentioned by Witten[11], the Schwarzschild solution is equally valid for mass parameters M or $-\tilde{M}$, the Euclidean, zero-charge, negative-mass version of Eq. (11) being

$$ds^2 = -\left(1 + 2\tilde{M}r^{-1}\right)d\tilde{t}^2 - \left(1 + 2\tilde{M}r^{-1}\right)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)\,. \quad (21)$$

In discussions[11, 12] of the positive-energy conjecture of Arnowitt *et al.*[13] for asymptotically flat spaces, however, there is the same initial assumption that $T_0^0 \geq 0$ locally.

3. The signature of space-time

Classically, there is no way of transforming the Euclidean spaces (20) or (21) into the Lorentzian space-times (12) or (11), respectively, and the quantum theory is therefore necessary to justify the assumption that $\rho \geq 0$, which is the purpose of this paper. In fact, the problem of why the Lorentzian signature is favoured over the Euclidean was analysed previously in Ref. [14], hereafter called paper I, by deriving[15] the Wheeler–DeWitt equation[16] for the wave function of the Friedmann Universe (12) (the mini-superspace approximation) in the heterotic superstring theory of Gross *et al.*[17], from which the Faddeev[18] (Newton–Wigner[19]) propagators $K_\pm$ can be obtained, referring to positive- and negative-energy solutions of the Schrödinger wave equation. In the Euclidean space (20), K_+ increases exponentially on the Planck time-scale $\sim t_\mathrm{P} \equiv M_\mathrm{P}^{-1} \approx 5 \times 10^{-44}$s, which is physically unacceptable, while K_- decreases exponentially on the same scale, effectively ruling out the Euclidean signature and thus proving that space-time has to be Lorentzian.

4. The quantum cosmological propagator

For completeness, we give an expanded version of the argument of paper I, with further clarification, although the principal results are not affected. The starting point is the dimensionally-reduced, effective four-action for the bosonic sector of the heterotic string, which can be written as the power series in the Regge slope parameter α' [15],

$$S = \int d^4x\sqrt{-g}\left[-\frac{R}{2\kappa^2} + B\left(R^2 - R_{ij}R^{ij}\right) + \ldots\right], \tag{22}$$

where the gravitational coupling, up to renormalisation[20, 21] (see Eq. (54) below), is

$$\kappa^2 \approx \kappa_0^2 = \frac{\alpha'}{4}, \tag{23}$$

and the coefficient of the higher-derivative terms $\mathcal{R}^2$ is given by

$$B = A_{\rm r} B_{\rm r}^{-2} \bar{B}. \tag{24}$$

The dilaton $A_{\rm r}$ and modulus $B_{\rm r}$ are the real parts of the two complex chiral superfields $\mathcal{A}$, $\mathcal{B}$ which arise from the reduction by Witten[22] of the ten-action $\hat{S}$ to four dimensions, the string line-element (I16) being

$$d\hat{s}^2 = A_{\rm r}^{-1} g_{ij} dx^i dx^j + B_{\rm r} \overline{g}_{\mu\nu} dy^\mu dy^\nu. \tag{25}$$

At the tree level $A_{\rm r} \approx g_{\rm s}^{-2}$, where $g_{\rm s}$ is the gauge-coupling parameter, and $\sqrt{B_{\rm r}\alpha'}$ is the radius of the internal space $\overline{g}_{\mu\nu}$, which is assumed to be isotropic. Previously, we have argued[23] that the modulus cannot be less than the Hagedorn value[17, 24] $B_{\rm r}^{(\rm H)} = (1 + 1/\sqrt{2})^2 \approx 2.914$, and requiring the four-theory to be supersymmetric results in the estimate[25]

$$B_{\rm r} \approx 3.5, \tag{26}$$

using the non-linear formulation of supersymmetry due to Volkov and Akulov[26], and assuming the three-generation Calabi–Yau manifold[27]. The constant $\bar{B}$ is given by the integral over the internal space

$$\bar{B} = \frac{\zeta(3)\alpha'^2}{128} \frac{\int d^6y\sqrt{\overline{g}}\,\overline{R}_{\mu\nu\xi o}\,\overline{R}^{\mu\nu\xi o}}{\int d^6y\sqrt{\overline{g}}}, \tag{27}$$

where $\zeta(3) \approx 1.202$ is the Riemann zeta function, leading to the estimate

$$B \approx 1. \tag{28}$$

Consider first the Lorentzian signature. The Wheeler–DeWitt equation can be written as[1]

$$i a^{-1/2} \frac{\partial \Psi}{\partial a} \approx \left[-\frac{A}{\zeta}\frac{\partial^2}{\partial \zeta^2} + Z\zeta\right]\Psi, \tag{29}$$

[1] There are minor errors in Eqs. (I12), (I15), (I18), (I26) and the subsequent expressions, which are corrected here in Eqs. (29), (31), (33), (40) and (41), respectively.

where the line element (12) has been expressed, rescaling the time t, as (I13),

$$ds^2 = a^3(\eta)d\eta^2 - a^2(\eta)\, d\boldsymbol{x}^2 \,, \tag{30}$$

and $\zeta = a^{-1}da/d\eta$. The positive constant A is defined by

$$A^{-1} = 96B = \frac{3}{4}\zeta(3)A_{\rm r}B_{\rm r}^{-2}\alpha'^2\, \frac{\int d^6y\sqrt{\overline{g}}\,\overline{R}_{\mu\nu\xi o}\overline{R}^{\mu\nu\xi o}}{\int d^6y\sqrt{\overline{g}}} \,, \tag{31}$$

and in the semi-classical approximation, where $B\mathcal{R}^2 \ll R/\kappa^2$, the geometrical and matter contributions to the potential in Eq. (29) (that is, the pseudo-Hamiltonian) can be equated, so that (see (I17))

$$Z \approx \frac{6}{\kappa^2} \,. \tag{32}$$

Introducing new co-ordinates (τ, χ), defined by

$$d\tau = \zeta a^{1/2} da \,, \qquad d\chi = \zeta d\zeta \,, \tag{33}$$

we rewrite Eq. (29), ignoring operator-ordering factors, as the Schrödinger equation (I19),

$$i\frac{\partial \Psi}{\partial \tau} \approx \left[-A\frac{\partial^2}{\partial \chi^2} + Z\right]\Psi \equiv \mathcal{H}_{\rm ps}\,\Psi \,, \tag{34}$$

where $\mathcal{H}_{\rm ps}$ is the pseudo-Hamiltonian.

The derivation of the Faddeev propagator for Eq. (34), $K(\tau, \chi; \tau_0, \chi_0)$ is given in paper I and leads to the answer (I23) in the Lorentzian régime,

$$K(\tau,\chi;\tau_0,\chi_0) = \lim_{\epsilon\to 0^+}\left(\frac{1}{4\pi i A(\triangle\tau - i\epsilon)}\right)^{1/2} \exp\left[\frac{Z(\triangle\tau - i\epsilon)^2 - (\triangle\chi)^2/4A}{i(\triangle\tau - i\epsilon)}\right], \tag{35}$$

which is oscillatory, and hence physically allowed. Now let us perform the Wick rotation (16) into the Euclidean space (20), which implies that

$$\eta = \pm\, i\tilde{\eta} \,, \qquad \zeta = \mp\, i\tilde{\zeta} \,, \qquad \tau = \mp\, i\tilde{\tau} \,, \qquad \chi = -\tilde{\chi} \,. \tag{36}$$

(Note that the "spatial" co-ordinate χ undergoes a parity transformation when the "time" τ is Wick-rotated.) For self-consistency, we have to apply the transformations (17), taking the source into the negative-energy region, and work from the semi-classical equations (18), (19). Expression (35) is then transformed into the Euclidean propagator (I24), now setting $\epsilon = 0$,

$$\tilde{K}_{\pm}\,(\tilde{\tau}, \tilde{\chi}; \tilde{\tau}_0, \tilde{\chi}_0) = \left(\frac{\mp 1}{4\pi A \triangle\tau}\right)^{1/2} \exp\left[\pm\left(\frac{Z(\triangle\tilde{\tau})^2 + (\triangle\tilde{\chi})^2/4A}{\triangle\tilde{\tau}}\right)\right] \,. \tag{37}$$

Choosing $\triangle\tilde{\tau} > 0$, the only physically acceptable solution is $\tilde{K}_-$, which decreases exponentially at least as fast (setting $\triangle\tilde{\chi} = 0$) as (I25),

$$\tilde{K}_- \sim \exp\left(\frac{-3\triangle\tilde{\tau}}{4\pi t_{\mathrm{P}}^2}\right) , \tag{38}$$

showing that the characteristic decay time-scale is $\sim t_{\mathrm{P}}$.

More precisely, in the Euclideanized Friedmann space (20) with $k = 0$, the scale factor is $a(\tilde{t}) = a_0(\tilde{t}/\tilde{t}_0)^{2/3\gamma}$ if the negative energy-density $-\tilde{\rho}$ and negative pressure $-\tilde{p}$ are related via the adiabatic index $\gamma = 1 + \tilde{p}/\tilde{\rho}$. By continuity with the Lorentzian space-time, we then set $a_0 = (4\pi/3)^{1/3} l(\tilde{t}_0)$, where $l(\tilde{t})$ is the Euclideanized "particle horizon", defined by

$$l(\tilde{t}\,) = \frac{3\gamma\tilde{t}}{3\gamma - 2} , \tag{39}$$

yielding

$$\frac{\triangle\tilde{\tau}}{t_{\mathrm{P}}^2} = a\left(\frac{da}{d\tilde{t}}\right)^2 \frac{\triangle\tilde{t}}{t_{\mathrm{P}}^2} = \frac{16\pi\gamma}{(3\gamma - 2)^3}\left(\frac{\tilde{t}_0}{\tilde{t}}\right)^{(3\gamma-2)/\gamma}\left(\frac{\tilde{t}}{t_{\mathrm{P}}}\right)\left(\frac{\triangle\tilde{t}}{t_{\mathrm{P}}}\right) . \tag{40}$$

For $\gamma = 1$ (implying $\zeta =$ constant) and $\tilde{t} = \tilde{t}_0$, Eq.(38) reduces to

$$\tilde{K}_-(\tilde{t}_0) \sim \exp\left(\frac{-12\tilde{t}_0\triangle\tilde{t}}{t_{\mathrm{P}}^2}\right) . \tag{41}$$

And since $\triangle\tilde{\tau}/t_{\mathrm{P}}^2 \sim \tilde{t}^{-2(\gamma-1)/\gamma}$, the suppression factor is even greater at earlier times for all $\gamma > 1$. More generally, at the Planck time, including both signs, we have

$$\tilde{K}_\pm(t_{\mathrm{P}}) \sim \exp\left[\pm\frac{12\gamma}{(3\gamma - 2)^3}\left(\frac{\tilde{t}_0}{t_{\mathrm{P}}}\right)^{(3\gamma-2)/\gamma}\frac{\triangle\tilde{t}}{t_{\mathrm{P}}}\right] . \tag{42}$$

The interpretation of expression (42) thus depends upon the value of γ. When $\gamma > 2/3$, the e-folding time for $\tilde{K}_-(t_{\mathrm{P}})$ is

$$\left(\triangle\tilde{t}\,\right)_e = \frac{(3\gamma - 2)^3}{12\gamma}\left(\frac{t_{\mathrm{P}}}{t_0}\right)^{(3\gamma-2)/\gamma} t_{\mathrm{P}} \ll t_{\mathrm{P}} , \tag{43}$$

while $\tilde{K}_+(t_{\mathrm{P}})$ increases exponentially on the same scale. As $\gamma - 2/3 \to 0_+$, $(\triangle\tilde{t}\,)_e \to 0$. When $0 < \gamma < 2/3$, the rôles of K_+ and K_- are interchanged, with the same limit $\left(\triangle\tilde{t}\,\right)_e \to 0$ as $\gamma - 2/3 \to 0_-$. The limiting case $\gamma = 0$, corresponding to Euclidean anti-de Sitter space, requires a separate treatment, since $a = a_0 \exp(\tilde{\xi}\tilde{t})$ and expression (40) is replaced by

$$\frac{\triangle\tilde{\tau}}{t_{\mathrm{P}}^2} = a^3\tilde{\xi}^2\frac{\triangle\tilde{t}}{t_{\mathrm{P}}^2}\,, \tag{44}$$

so that

$$\tilde{K}_{\pm}(\tilde{t}) \sim \exp\left[\frac{\pm 3a^3(\tilde{t})\tilde{\xi}^2\triangle\tilde{t}}{4\pi t_{\mathrm{P}}^2}\right]. \tag{45}$$

Since $a(\tilde{t}) > t_{\mathrm{P}}\,\forall\,\tilde{t}$, the e-folding time for $K_-(\tilde{t})$ is

$$(\triangle\tilde{t}\,)_e = \frac{4\pi t_{\mathrm{P}}^2}{3a^3(\tilde{t}\,)\tilde{\xi}^2}\,. \tag{46}$$

In the situations of interest, however, we shall have $a^3(\tilde{t}\,)\tilde{\xi}^2 \gtrsim t_{\mathrm{P}}$, and again find that $(\triangle\tilde{t}\,)_e \lesssim t_{\mathrm{P}}$, allowing us to rule out both the cases $\pm$ by the same reasoning as above.

Thus, only the Lorentzian signature is physically allowed.

For the metric coefficient g_{00} is not a dynamical variable, as emphasised by Ellis *et al.*[28], being the square of the lapse function N (when $g_{0\alpha} = 0$) which occurs as a Lagrange multiplier in the Hamiltonian formulation of general relativity, although it becomes dynamical when higher-derivative terms $\mathcal{R}^2$ are included — see the footnote on p. 314 of Ref. [29]. Ignoring $\mathcal{R}^2$, we can still evolve the space-time from one signature to another if the total energy-density $\rho_{\mathrm{t}} \equiv \rho - 3k/\kappa^2a^2$ changes sign at some point, and consequently the examples of such spaces constructed in Ref. [28] all require $k = +1$, so that the curvature contribution to ρ_{t} is negative. But the change-over cannot occur unless $\rho_{\mathrm{t}} = 0$, which is usually interpreted to mean that the Friedmann expansion has ceased, to be succeeded by a contracting phase, ρ_{t} remaining positive semi-definite throughout. In this case, the curvature density $\rho_{\mathrm{c}} \equiv -3k/\kappa^2a^2$ becomes significant far from the Planck era (where it is ignorable), but grows faster than $\rho \approx 4/3\gamma^2\kappa^2t^2$, since $a \sim t^{2/3\gamma}$, assuming that $\gamma > 2/3$.

5. Discussion

We have seen that the cosmological Faddeev propagator K can be applied to explain why space-time is Lorentzian rather than Euclidean. The proof, starting from the action (22), assumes the semi-classical approximation $B|\mathcal{R}^2| \ll |R|/2\kappa^2$, so that higher-derivative terms can be ignored in the Friedmann equation, although their presence is essential to the derivation of the Schrödinger equation (34). From the estimate (28) that $B \approx 1$, it follows that this inequality is satisfied except at the Planck era, where decompactification effects become important, and consequently a more exact analysis taking the term $\mathcal{R}^2$ into account in the classical equation of motion would not significantly affect our result. Nor shall we discuss the curved spaces $k = \pm 1$ further, since, in addition to the quantum cosmological argument of Ref. [10], the measurements of the anisotropy of the cosmic microwave background radiation on sub-degree angular scales by the BOOMERANG and MAXIMA-I balloon experiments[30] suggest that $k = 0$, the first peak in the angular power spectrum being at multipole $l \approx 200$.

It is nevertheless of interest that the energy-density of geometrical origin from $\mathcal{R}^2$ is not necessarily positive definite, even though the scalar and tensor masses are both real for the heterotic superstring theory, which is thus free of space-time tachyons. The modified field equations (1) read, setting $\mu = 1$ in Eq. (22) of Ref.[31],

$$\begin{aligned} R_{ij} - \frac{1}{2} g_{ij} R \; = \; \kappa^2 \Big\{ T_{ij} + B \Big[- \left(R^2 - R_{kl} R^{kl} \right) g_{ij} \\ + 4 \left(R R_{ij} - R_{kl} R_{i\ j}^{\ k\ l} \right) + 3 \Box R g_{ij} - 2 \Box R_{ij} - 2 R_{;ij} \Big] \Big\}, \end{aligned} \tag{47}$$

and in the Friedmann space-time (12) the geometrical contribution to the energy-density and pressure are given by expressions (26) and (28) of Ref.[31], respectively,

$$\rho_{\mathcal{R}^2} = 24B \left(\ddot{\alpha}^2 - 2\dot{\alpha}\dddot{\alpha} - 6\dot{\alpha}^2\ddot{\alpha} \right) \tag{48}$$

and

$$p_{\mathcal{R}^2} = 8B \left(9\ddot{\alpha}^2 + 12\dot{\alpha}\dddot{\alpha} + 18\dot{\alpha}^2\ddot{\alpha} + 2\ddddot{\alpha} \right) . \tag{49}$$

The term in B on the right-hand side of Eq. (47), deriving from $\mathcal{R}^2$, vanishes not only in Minkowski space, but also in de Sitter space or anti-de Sitter space, defined by

$$R_{ijkl} = \frac{1}{3} \wedge \left(g_{il} g_{jk} - g_{ik} g_{jl} \right) , \tag{50}$$

the maximally symmetric space with cosmological constant $\wedge$ positive or negative, respectively, that constitutes the natural cosmological vacuum[32] from which the gravitational energy can be defined[33].

As long as $B|\mathcal{R}^2| \ll |R|/2\kappa^2$, however, it follows from the analysis of Section 2 that the cosmological energy-density ρ must be positive definite everywhere for the Friedmann space-time (which can be imposed as a symmetry), so that $\rho = \rho(t) > 0$. (More precisely, this inequality refers to ρ_{t}, but we are assuming that $k = 0$.) Since this matter is made up of elementary particles of various types, it is reasonable to assume that they all have positive semi-definite rest-mass and energy — one might call this the positive-mass hypothesis, since it underlies the positive-energy conjecture. These are the particles which go into the formation of astrophysical bodies and black holes, explaining why only the positive-mass Schwarzschild solution is relevant in the Universe in which we live.

This result is important in quantum cosmology for defining the vacuum wave-function of the Universe Ψ_0. The positive semi-definiteness of the matter energy-density ρ implies, via the Friedmann equation, that the semi-classical potential $V \propto +(\rho a^3)^{1/2}$ in the Schrödinger equation (29) — which is rescaled to the (positive) constant Z in Eq. (34) — is also positive semi-definite. This allows a global vacuum state $V_0 = 0$, which is Minkowski space, associated with non-vanishing probability density $\Psi_0 \Psi_0^*$, and as a corollary, we can argue that the cosmological constant has to be set to zero[34]. The existence of this vacuum state is a necessary prerequisite for the interpretation[35] of the neutron-diffraction experiment of Colella *et al.*[36] as evidence for the correctness of the quantisation procedure leading to the cosmological Wheeler–DeWitt equation in the form of the Schrödinger equation (8) of Ref. [35],

$$i\frac{\partial \Psi}{\partial t} \approx \mathcal{H}_{\mathrm{ps}} \Psi \,. \tag{51}$$

The local matter Schrödinger-equation, Eq. (13) of Ref. [35],

$$i\frac{\partial \Psi_1}{\partial t} = \mathcal{H}_1 \Psi_1 \,, \tag{52}$$

is then obtained by factorisation of the wave function, $\Psi = \Psi_{\mathrm{g}} \Psi_1$, where the suffix g refers to the background gravitational component and $\mathcal{H}_{\mathrm{ps}} = \mathcal{H}_{\mathrm{g}} + \mathcal{H}_1$.

We have assumed throughout that the gravitational coupling κ^2 is positive, in agreement with present-day experiment, but which requires theoretical justification. For the bare value obtained after reduction of the initial ten-dimensional Einstein–Hilbert term $\hat{R}/2\hat{\kappa}^2$, given by the formula

$$\kappa_0^2 = \frac{\hat{\kappa}^2}{\int d^6 y \sqrt{\overline{g}}} \,, \tag{53}$$

is renormalised by contributions originating from the higher-derivative terms $\alpha'^3 \hat{\mathcal{R}}^4$, yielding the formula[21]

$$\kappa^2 = \kappa_0^2 \left[1 - \frac{15\zeta(3)\overline{\chi}}{16\lambda B_{\mathrm{r}}^3} - \frac{g_{\mathrm{s}}^2 \overline{\chi}}{384\pi^2} \right]^{-1} \,. \tag{54}$$

Here $\overline{\chi}$ is the Euler characteristic of the Ricci-flat internal space $\overline{g}_{\mu\nu}$, whose volume $\overline{V}_6$ is parametrised in terms of that of the unit six-sphere by λ,

$$\overline{V}_6 = \int d^6 y \sqrt{\overline{g}} = \frac{16\pi^3 \lambda \alpha'^3}{15}\,, \tag{55}$$

and $g_s^2 \approx 1/A_r \approx 1/2$ at tree level. The second and third terms in the bracket on the right-hand side of Eq. (54) derive from the tree-level[37] and one-loop[38] corrections, respectively.

The only known, three-generation Calabi–Yau manifold[27] is characterised by $\overline{\chi} = \pm 6$ (taking into account the mirror manifold), so that these corrections are proportionately

$$\left.\frac{\delta\kappa^2}{\kappa^2}\right|_{\text{tree-level}} \approx \pm\frac{6.76}{\lambda B_r^3}\,, \qquad \left.\frac{\delta\kappa^2}{\kappa^2}\right|_{\text{1-loop}} \approx \pm 8\times 10^{-4}\,, \tag{56}$$

in the first of which we have to substitute the result (26) for B_r, and which now depends upon the compactification volume. The simplest approximation [22] in reducing the ten-action to the standard four-dimensional supergravity form is the six-torus, for which $\overline{V}_6 = 64\pi^6\alpha'^3$, so that $\lambda = 60\pi^3$, in which case the tree-level and one-loop corrections to κ_0^2 are approximately

$$\left.\frac{\delta\kappa^2}{\kappa^2}\right|_{\text{tree-level}} \approx \pm 1.35\times 10^{-4}\left(\frac{3}{B_r}\right)^3, \qquad \left.\frac{\delta\kappa^2}{\kappa^2}\right|_{\text{1-loop}} \approx \pm 7.92\times 10^{-4}\left(\frac{2}{A_r}\right), \tag{57}$$

both being of the same sign as that of $\overline{\chi}$ and of the same order of magnitude. Thus, the sign of κ_0^2 remains unchanged by the renormalisation, as it should, since the positivity of the gravitational constant is linked to the Lorentzian signature of space-time[39], and hence to the positivity of matter energy in the Universe.

Finally, it is illuminating to relate the phenomenon of metric-signature change to the theory of quantum tunnelling, applied by Gamow[40], for example, to explain the emission of alpha particles from atomic nuclei of energy less than the height of the Coulomb barrier. Thus, consider, for simplicity, the time-independent problem of a particle of mass m, momentum p and energy E in a one-dimensional region of flat space-time where the potential is $V(x)$. Upon quantisation via the operator replacement $p \to -id/dx$, the non-relativistic equation for the conservation of energy

$$\frac{p^2}{2m} + V = E \tag{58}$$

yields the Schrödinger equation for the wave function ψ,

$$\frac{1}{2m}\frac{d^2\psi}{dx^2} + (E - V)\psi = 0\,. \tag{59}$$

In the classically-allowed region $E \geq V$, the solutions to Eq. (59) are oscillatory

$$\psi = \psi_0 \exp\left\{\pm i \left[2m(E - V)\right]^{1/2} x\right\} , \tag{60}$$

where V is constant, while in the classically-forbidden region under the potential barrier, $E < V$, the physical solution is

$$\psi = \psi_0 \exp\left\{- \left[2m(V - E)\right]^{1/2} x\right\} , \tag{61}$$

which describes tunnelling through the barrier if this is of finite width.

Eq. (58) is the Hamiltonian constraint $H = 0$, the cosmological version of which in the Friedmann space-time (12) is Eq. (14). The analogues of the one-dimensional energy and potential are the terms $\kappa^2\rho/3$ and k/a^2, respectively, the classically-allowed and -forbidden regions being $\xi^2 \geq 0$ and $\xi^2 < 0$. There is an alternative interpretation, however, for the classically-forbidden region becomes classically allowed if we Wick-rotate the time coordinate according to Eq. (16), changing the metric signature from $(+ - --)$ to $(- - --)$. If we set $k = 0$ and $V = 0$, the two regions are defined by $E \geq 0$ and $E < 0$, respectively, while the operators d^2/dx^2 in Eq. (58) and $\partial^2/\partial\chi^2$ in Eq. (34) remain invariant under the signature change. Thus, the decaying propagator (38) for the negative-energy solution can be regarded either as classically forbidden in Lorentzian space-time or classically allowed in Euclidean space. From either standpoint, the occurrence of negative energies is restricted by the indeterminacy principle to the Planck era.

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