

ON THE ERNST ELECTRO-VACUUM EQUATIONS AND ERGOSURFACES

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The question of smoothness at the ergosurface of the space-time metric constructed out of solutions $(\mathcal{E}, \varphi)$ of the Ernst electro-vacuum equations is considered. We prove smoothness of those ergosurfaces at which $\Re \mathcal{E}$ provides the dominant contribution to $f = -(\Re \mathcal{E} + |\varphi|^2)$ at the zero-level-set of f . Some partial results are obtained in the remaining cases: in particular we give examples of leading-order solutions with singular isolated “ergocircles”.

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1. Introduction

In recent work [1] we have shown that a vacuum space-time metric is smooth near a “Ernst ergosurface” $E_{\mathcal{E}} = \{\Re \mathcal{E} = 0, \rho \neq 0\}$ if and only if the Ernst potential $\mathcal{E}$ is smooth near $E_{\mathcal{E}}$ and does not have zeros of infinite order there. It is of interest to enquire whether a similar property holds for electro-vacuum metrics. While we have not been able to obtain a complete answer to this question, in this note we present a series of partial results, amongst which:

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THEOREM 1.1 *Consider a smooth solution $(\mathcal{E}, \varphi)$ of the electro-vacuum Ernst equations (2.2)–(2.3) below, and let the Ernst ergosurface $E_{\mathcal{E}, \varphi}$ be defined as the set*

$$E_{\mathcal{E}, \varphi} := \{\mathcal{E} + \bar{\mathcal{E}} + 2\bar{\varphi}\varphi = 0, \quad \rho \neq 0\}. \quad (1.1)$$

Suppose that $\mathcal{E} + \bar{\mathcal{E}}$ has a zero of finite order at $E_{\mathcal{E}, \varphi}$. If the φ terms contribute subleading terms to $\mathcal{E} + \bar{\mathcal{E}} + 2\bar{\varphi}\varphi$ at $E_{\mathcal{E}, \varphi}$, then there exists a neighborhood of $E_{\mathcal{E}, \varphi}$ on which the tensor field (2.1) obtained by solving (2.5)–(2.6) is smooth and has Lorentzian signature.

Theorem 1.1 is proved in Section 3.

To make things clear, consider a point p at which

$$f := -\frac{1}{2}(\mathcal{E} + \bar{\mathcal{E}} + 2\bar{\varphi}\varphi)$$

vanishes. Expanding $\mathcal{E}$ and φ in a Taylor series at p , let m be the order of the leading Taylor polynomial of $\Re \mathcal{E} - \Re \mathcal{E}(p)$, and let k be the corresponding order for $\varphi - \varphi(p)$. Then we say that the φ terms contribute subleading terms to f if $2k > m$.

Under the remaining conditions of Theorem 1.1, the condition of a zero of finite order is *necessary and sufficient*, as smoothness of the metric near $E_{\mathcal{E}, \varphi}$ implies analyticity of $\mathcal{E}$ and φ .

It follows from the analysis in [1] that, in vacuum, a generic point on $E_{\mathcal{E}, \varphi}$ will be a zero of $\mathcal{E}$ of order one. One expects this result to remain true in electro-vacuum, so that Theorem 1.1 should cover generic situations.

A significant application of Theorem 1.1, to solutions obtained by applying a Harrison transformation to a vacuum solution, is given in Section 4 below.

Some partial results, presented in Section 5, are obtained in the cases not covered by Theorem 1.1: We describe completely the leading-order behavior of φ at those ergosurfaces at which φ provides the dominant contribution to f . We show that there exist Taylor polynomials solving the Ernst equation at leading order which result in singularities of the space-time metric on $E_{\mathcal{E}, \varphi}$. This result does not, however, prove that there exist smooth solutions of the electro-vacuum Ernst equations which lead to metrics which are singular at the ergosurface because it is not clear that the “leading-order solutions” that we construct correspond to solutions of the full, non-truncated equations.

2. Preliminaries

We use the same parameterisation of the metric as in [1]:

$$ds^2 = f^{-1} [h (d\rho^2 + d\zeta^2) + \rho^2 d\phi^2] - f (dt + a d\phi)^2, \quad (2.1)$$

with all functions depending only upon ρ and ζ . In electro-vacuum the Ernst equations form a system of two coupled partial differential equations for two complex valued functions $\mathcal{E}$ and φ [5], which we assume to be smooth:

$$(\mathcal{E} + \bar{\mathcal{E}} + 2\bar{\varphi}\varphi) L\mathcal{E} = \left(\frac{\partial \mathcal{E}}{\partial \bar{z}} + 2\bar{\varphi} \frac{\partial \varphi}{\partial \bar{z}} \right) \frac{\partial \mathcal{E}}{\partial z} + \left(\frac{\partial \mathcal{E}}{\partial z} + 2\bar{\varphi} \frac{\partial \varphi}{\partial z} \right) \frac{\partial \mathcal{E}}{\partial \bar{z}}, \quad (2.2)$$

$$(\mathcal{E} + \bar{\mathcal{E}} + 2\bar{\varphi}\varphi) L\varphi = \left(\frac{\partial \mathcal{E}}{\partial \bar{z}} + 2\bar{\varphi} \frac{\partial \varphi}{\partial \bar{z}} \right) \frac{\partial \varphi}{\partial z} + \left(\frac{\partial \mathcal{E}}{\partial z} + 2\bar{\varphi} \frac{\partial \varphi}{\partial z} \right) \frac{\partial \varphi}{\partial \bar{z}}, \quad (2.3)$$

where

$$L = \frac{\partial^2}{\partial z \partial \bar{z}} + \frac{1}{2(z + \bar{z})} \left(\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}} \right),$$

with $z = \rho + i\zeta$. The metric functions are determined from¹

$$f = -\frac{1}{2}(\mathcal{E} + \bar{\mathcal{E}} + 2\bar{\varphi}\varphi), \quad (2.4)$$

$$\frac{\partial h}{\partial z} = (z + \bar{z})h \left(\frac{1}{2} \left(\frac{\partial \mathcal{E}}{\partial z} + 2\bar{\varphi} \frac{\partial \varphi}{\partial z} \right) \left(\frac{\partial \bar{\mathcal{E}}}{\partial z} + 2\varphi \frac{\partial \bar{\varphi}}{\partial z} \right) f^{-2} + 2 \frac{\partial \bar{\varphi}}{\partial z} \frac{\partial \varphi}{\partial z} f^{-1} \right), \quad (2.5)$$

$$\frac{\partial a}{\partial z} = \frac{1}{4}(z + \bar{z}) \left(\frac{\partial \mathcal{E}}{\partial z} + 2\bar{\varphi} \frac{\partial \varphi}{\partial z} - \frac{\partial \bar{\mathcal{E}}}{\partial z} - 2\varphi \frac{\partial \bar{\varphi}}{\partial z} \right) f^{-2}. \quad (2.6)$$

The equations are singular at the *Ernst ergosurface* $E_{\mathcal{E},\varphi}$ defined by (1.1).

Let $\lambda \in \mathbb{C}$, $\mu \in \mathbb{R}$, then the following transformation maps solutions of (2.2)–(2.3) into solutions, *without changing* the right-hand sides of (2.4)–(2.6)

$$\mathcal{E} \rightarrow \mathcal{E} + 2\bar{\lambda}\varphi - |\lambda|^2 + i\mu, \quad \varphi \rightarrow \varphi - \lambda. \quad (2.7)$$

This is easiest seen by noting, first, that both f and $d\mathcal{E} + 2\bar{\varphi}d\varphi$ are left unchanged by (2.7).

3. $\mathcal{E}$ -dominated ergosurfaces

Suppose that $E_{\mathcal{E},\varphi} \neq \emptyset$ and that $\mathcal{E}$ and φ are smooth in a neighborhood of $E_{\mathcal{E},\varphi}$. Let $z_0 = \rho_0 + i\zeta_0 \in E_{\mathcal{E},\varphi}$, we can choose μ and λ so that the potentials transformed as in (2.7) satisfy

$$\mathcal{E}(z_0) = 0, \quad \varphi(z_0) = 0. \quad (3.1)$$

Assume first,

$$Df(z_0) \neq 0.$$

¹ Note that $\mathcal{E}$ here is minus $\mathcal{E}$ in [1].

Performing a Taylor expansion of $\mathcal{E}$ and φ at z_0 and inserting into (2.2)–(2.3), a SINGULAR [2] calculation (and, as a cross-check, a MAPLE one) shows² that either

$$\partial_z \varphi(z_0) = \partial_z \mathcal{E}(z_0) = 0, \quad (3.2)$$

$$0 \neq \partial_{\bar{z}} \mathcal{E}(z_0) = 4\rho_0 \partial_z \partial_{\bar{z}} \mathcal{E}(z_0) = 4\rho_0 \overline{\partial_z^2 \mathcal{E}}(z_0), \quad (3.3)$$

$$\partial_z^2 \mathcal{E}(z_0) \partial_z \partial_{\bar{z}} \varphi(z_0) = \partial_z^2 \varphi(z_0) \partial_z \partial_{\bar{z}} \mathcal{E}(z_0), \quad (3.4)$$

$$\partial_z^2 \mathcal{E}(z_0) \overline{\partial_z^2 \varphi}(z_0) = \overline{\partial_z \partial_{\bar{z}} \varphi}(z_0) \partial_z \partial_{\bar{z}} \mathcal{E}(z_0), \quad (3.5)$$

or that (3.2)–(3.5) is satisfied by the complex conjugates of $(\mathcal{E}, \varphi)$. In the latter case the linear part of the Taylor expansion of $(\mathcal{E}, \varphi)$ is a holomorphic function of z , while it is anti-holomorphic in the former. In the calculations proving smoothness across $E_{\mathcal{E}, \varphi} \cap \{df \neq 0\}$ the equations (3.4)–(3.5) are not used.

Using (3.3) in (2.6) one finds

$$\lim_{z \rightarrow z_0} f^2 \partial_z \left(a + \frac{\rho}{f} \right) = \lim_{z \rightarrow z_0} \partial_z \left[f^2 \partial_z \left(a + \frac{\rho}{f} \right) \right] = \lim_{z \rightarrow z_0} \partial_{\bar{z}} \left[f^2 \partial_z \left(a + \frac{\rho}{f} \right) \right] = 0. \quad (3.6)$$

It follows as in the proof of Theorem 4.1 of [1] that the function $a + \rho/f$ is smooth across $E_{\mathcal{E}, \varphi} \cap \{df \neq 0\}$.

The same argument with $a - \rho/f$ instead of $a + \rho/f$ applies if the complex conjugate solution is used.

A similar calculation with (2.5) shows that

$$\lim_{z \rightarrow z_0} f^2 \partial_z \ln(|h/f|) = \lim_{z \rightarrow z_0} \partial_z (f^2 \partial_z \ln(|h/f|)) = \lim_{z \rightarrow z_0} \partial_{\bar{z}} (f^2 \partial_z \ln(|h/f|)) = 0. \quad (3.7)$$

The remaining arguments of the proof of Theorem 4.1 of [1] apply and we conclude that the metric (2.1) extends smoothly across $E_{\mathcal{E}, \varphi} \cap \{df \neq 0\}$, and has Lorentzian signature in a neighborhood of this set.

Suppose, next, that f has a zero of higher order at $z_0 \in E_{\mathcal{E}, \varphi}$. Since φ enters quadratically in f and in the right-hand sides of (2.5)–(2.6), and through cubic terms in the right-hand sides of (2.2)–(2.3), one would hope that φ will only contribute to subleading terms in Taylor expansions of those equations. But then the analysis of the leading-order behavior of f near $E_{\mathcal{E}, \varphi}$ is reduced to the analysis already done in [1], which would prove smoothness of the space-time metric at the Ernst ergosurface without any provisos.

² See the SINGULAR file `em1.in` and the MAPLE file `em1.mw` at <http://th.if.uj.edu.pl/~szybka/CS/>

It turns out that this is not the case: we shall see in the next section that there exist leading-order Taylor polynomials satisfying the leading-order equations for which the φ terms are *not* dominated by $\mathcal{E}$. Nevertheless, the argument just given establishes that *if* the φ terms are dominated by $\mathcal{E}$, then the analysis of [1] proves smoothness of the metric across $E_{\mathcal{E},\varphi}$, and Theorem 1.1 is proved.

REMARK 3.1 *Consider a $\mathcal{E}$ -dominated zero z_0 of f , after shifting $\Im\mathcal{E}$ by a real constant we can assume that $\mathcal{E}(z_0) = 0$. It then follows from [1, Proposition 5.1] that the order of the zero of $\mathcal{E}$ at z_0 coincides with the order of the zero of $\Re\mathcal{E}$.*

4. Harrison–Neugebauer–Kramer transformations

It is of interest to enquire what happens with Ernst ergosurfaces under Neugebauer–Kramer transformations [5, Equation (34.8e)] (see also [4]) of $(\mathcal{E}, \varphi)$:

$$\begin{aligned}\mathcal{E}' &= \mathcal{E}(1 - 2\bar{\gamma}\varphi - \gamma\bar{\gamma}\mathcal{E})^{-1}, \\ \varphi' &= (\varphi + \gamma\mathcal{E})(1 - 2\bar{\gamma}\varphi - \gamma\bar{\gamma}\mathcal{E})^{-1}.\end{aligned}\tag{4.1}$$

Under (4.1) f is transformed to

$$f' = \frac{f}{|1 - 2\bar{\gamma}\varphi - \gamma\bar{\gamma}\mathcal{E}|^2},\tag{4.2}$$

so that $E_{\mathcal{E},\varphi}$ is mapped into itself. The same remains of course true under Harrison [3] transformations [5, Equation (34.12)], which are a special case of (4.1) when the initial φ vanishes:

$$\mathcal{E}' = \mathcal{E}(1 - \gamma\bar{\gamma}\mathcal{E})^{-1}, \quad \varphi' = \gamma\mathcal{E}(1 - \gamma\bar{\gamma}\mathcal{E})^{-1}.\tag{4.3}$$

As a significant corollary of Theorem 1.1, we obtain

COROLLARY 4.1 *Let $(\mathcal{E}', \varphi')$ be obtained by a Harrison transformation from a smooth solution $(\mathcal{M}, g)$ of the vacuum equations with a non-empty ergo-surface, then the conclusion of Theorem 1.1 holds.*

PROOF: As discussed in [1], the Ernst potential $\mathcal{E}$ is analytic near $E_{\mathcal{E},\varphi}$, hence has a zero of finite order. Clearly, the order of zero of $|\varphi'|^2$ as defined by (4.3) is higher than the order of zero of $\mathcal{E}'$; the latter is the same as the order of zero of $\Re\mathcal{E}'$ by the results in [1]. $\square$

Somewhat more generally, consider $p \in E_{\mathcal{E},\varphi}$, as explained above we can always introduce a gauge so that $\varphi(p) = 0$. In this gauge, let $(\mathcal{E}', \varphi')$ be obtained by a Neugebauer–Kramer transformation from a solution satisfying the hypotheses of Theorem 1.1 near p , then the conclusion of Theorem 1.1 holds near p for the metric constructed by using $(\mathcal{E}', \varphi')$. This follows immediately from (4.1).

5. Some remaining possibilities

It remains to consider the case where the φ terms dominate in f , and the case where all terms are of the same order. The latter case will be referred to as *balanced*.

5.1. Balanced leading-order solutions with singular ergocircles

The simplest such possibility is $Df(z_0) = 0$, $DDf(z_0) \neq 0$ and $\mathcal{E}(z_0) = \varphi(z_0) = 0$. It is easy to completely analyze the first few leading-order equations with the ansatz

$$\partial_z \mathcal{E}(z_0) = \partial_{\bar{z}} \mathcal{E}(z_0) = \partial_z^2 \mathcal{E}(z_0) = \partial_{\bar{z}}^2 \mathcal{E}(z_0) = 0. \quad (5.1)$$

A MAPLE-assisted calculation³ then shows that the leading-order equations do not introduce any constraints on $\partial_z \varphi(z_0)$, and that if we set

$$\alpha := \partial_z \varphi(z_0) \neq 0,$$

then one has

$$\begin{aligned} |\partial_{\bar{z}} \varphi(z_0)|^2 &= |\alpha|^2, \\ \partial_z \partial_{\bar{z}} \mathcal{E}(z_0) &= -4|\alpha|^2. \end{aligned} \quad (5.2)$$

Recall that (2.5)–(2.6) leads to the following equations for the metric function a

$$\frac{f^2}{\rho} \partial_z \left(a + \frac{\rho}{f} \right) = \underbrace{\left(\frac{\partial \mathcal{E}}{\partial z} + 2\bar{\varphi} \frac{\partial \varphi}{\partial z} + \frac{f}{z + \bar{z}} \right)}_{=:\hat{\sigma}_1}, \quad (5.3)$$

$$\frac{f^2}{\rho} \partial_z \left(a - \frac{\rho}{f} \right) = - \underbrace{\left(\frac{\partial \bar{\mathcal{E}}}{\partial z} + 2\varphi \frac{\partial \bar{\varphi}}{\partial z} + \frac{f}{z + \bar{z}} \right)}_{=:\hat{\sigma}_2}. \quad (5.4)$$

In the vacuum case it was shown that one out of $\hat{\sigma}_1/f^2$ and $\hat{\sigma}_2/f^2$ is smooth near $\{f = 0, \rho \neq 0\}$, which then implies smoothness of the ergosurface. (An identical analysis applies to $\mathcal{E}$ -dominated ergosurfaces.) So one can attempt to repeat the argument here. Letting

$$r_0 := \sqrt{(\rho - \rho_0)^2 + (\zeta - \zeta_0)^2},$$

³ See the MAPLE file `em2.mw` at <http://th.if.uj.edu.pl/~szybka/CS/>

the leading terms of $f, \dot{\sigma}_1, \dot{\sigma}_2$ read

$$\begin{aligned}\mathcal{E} &= -4|\alpha z|^2 + O(r_0^3), \\ \varphi &= \alpha z + \bar{\gamma}\bar{z} + O(r_0^2), \\ f &= -\alpha\gamma z^2 + 2|\alpha|^2 z\bar{z} - \bar{\gamma}\bar{\alpha}\bar{z}^2 + O(r_0^3), \\ \dot{\sigma}_1 &= 2\alpha(\gamma z - \bar{\alpha}\bar{z}) + O(r_0^2), \\ \dot{\sigma}_2 &= -2\alpha(\gamma z - \bar{\alpha}\bar{z}) + O(r_0^2),\end{aligned}\tag{5.5}$$

where $\gamma = \overline{\partial_{\bar{z}}\varphi}(z_0)$. Here, for the typesetting convenience, we used the symbol z for $z - z_0$. Those examples clearly lead to a singularity both in $\dot{\sigma}_1/f^2$ and in $\dot{\sigma}_2/f^2$, therefore a different strategy is needed.

Now,

$$f = |\alpha z - \bar{\gamma}\bar{z}|^2 + (|\alpha|^2 - |\gamma|^2)|z|^2 + O(r_0^3),$$

so that if $|\alpha| > |\gamma|$ we obtain an isolated zero of f , an “ergocircle”. More precisely, the intersection of the set where f vanishes with a neighborhood of z_0 will be $\{z_0\}$. This, at any given value of t , corresponds to an isolated null orbit of the isometry group of the metric generated by ∂_ϕ provided that the metric is non-singular there.

Still assuming $|\alpha| > |\gamma|$, we claim that the metric will be singular at z_0 . Indeed, adding (5.3) and (5.4) one finds that ∂a is uniformly bounded near z_0 , hence a can be extended by continuity to a Lipschitz continuous function defined on a neighborhood of z_0 . But then $g(\partial_\phi, \partial_\phi)$ blows up as ρ_0^2/f at z_0 .

5.2. Balanced solutions with radial $\mathcal{E}_{2k}$

The solutions of Section 5.1 are a special case of a family of solutions in which the leading terms in $\mathcal{E}$ take the form

$$\mathcal{E}_{2k} = \mu_1 e^{i\mu_0} (z - z_0)^k (\bar{z} - \bar{z}_0)^k, \quad \mu_0 \in \mathbb{R}, \quad \mu_1 \in \mathbb{R}^*. \tag{5.6}$$

Let us write

$$\varphi_k = \sum_{m=0}^k \alpha_m (z - z_0)^m (\bar{z} - \bar{z}_0)^{k-m}, \tag{5.7}$$

where all the α_m ’s do not vanish simultaneously. Inserting (5.6)–(5.7) into (2.2)–(2.3) one obtains

$$(\mathcal{E}_{2k} + \bar{\mathcal{E}}_{2k}) \frac{\partial^2 \mathcal{E}_{2k}}{\partial \bar{z} \partial z} - 2 \frac{\partial \mathcal{E}_{2k}}{\partial \bar{z}} \frac{\partial \mathcal{E}_{2k}}{\partial z} = 2 \bar{\varphi}_k \left(\frac{\partial \varphi_k}{\partial \bar{z}} \frac{\partial \mathcal{E}_{2k}}{\partial z} + \frac{\partial \varphi_k}{\partial z} \frac{\partial \mathcal{E}_{2k}}{\partial \bar{z}} \right) - 2 \bar{\varphi}_k \varphi_k \frac{\partial^2 \mathcal{E}_{2k}}{\partial \bar{z} \partial z}, \tag{5.8}$$

$$(\mathcal{E}_{2k} + \bar{\mathcal{E}}_{2k}) \frac{\partial^2 \varphi_k}{\partial \bar{z} \partial z} - \left(\frac{\partial \varphi_k}{\partial \bar{z}} \frac{\partial \mathcal{E}_{2k}}{\partial z} + \frac{\partial \varphi_k}{\partial z} \frac{\partial \mathcal{E}_{2k}}{\partial \bar{z}} \right) = 4 \bar{\varphi}_k \frac{\partial \varphi_k}{\partial \bar{z}} \frac{\partial \varphi_k}{\partial z} - 2 \bar{\varphi}_k \varphi_k \frac{\partial^2 \varphi_k}{\partial \bar{z} \partial z}. \tag{5.9}$$

The right-hand side of (5.8) vanishes, and the vanishing of the left-hand side implies $\sin \mu_0 = 0 \implies \mu_0 = j\pi$, where $j \in \mathbb{N}$. Changing μ_1 to $-\mu_1$ if necessary we can without loss of generality assume $\mu_0 = 0$. Setting $\alpha_i = 0$ for $i < 0$ or $i > k$, and working out the coefficients of the terms $(z - z_0)^{k-1+l}(\bar{z} - \bar{z}_0)^{2k-1-l}$ in (5.9) we obtain for $-k+1 \leq l \leq 2k-1$

$$\mu_1 \alpha_l ((k-l)^2 + l^2) = - \sum_{\substack{-m+n+i=l \\ 0 \leq m, n, i \leq k}} 2\bar{\alpha}_m \alpha_n \alpha_i (k-i)(2n-i). \quad (5.10)$$

We expect that a complete description of such solutions should be possible (for example, it immediately follows for $2k-1 > k$ (*i.e.*, $k > 1$) that $\bar{\alpha}_0 \alpha_k \alpha_{k-1} = 0$), but we have not attempted to do that. Instead we list here all such leading-order solutions for $k=2$ and $k=3$, as calculated⁴ using MAPLE:

$$\begin{aligned} k=2, \mathcal{E}_4 &= -|\alpha|^2 |z|^4 : & \varphi_2 &= \alpha |z|^2, & \alpha &\in \mathbb{C}^*, \\ \mathcal{E}_4 &= -4|\alpha|^2 |z|^4 : & \varphi &= \alpha z^2 + \bar{\gamma} \bar{z}^2, & \alpha, \gamma &\in \mathbb{C}^*, |\alpha| = |\gamma|, \\ k=3, \mathcal{E}_6 &= -\frac{4}{5}|\alpha|^2 |z|^6 : & \varphi_3 &= \alpha z |z|^2 \text{ or } \varphi_3 = \alpha \bar{z} |z|^2, & \alpha &\in \mathbb{C}^*, \\ \mathcal{E}_6 &= -4|\alpha|^2 |z|^6 : & \varphi_3 &= \alpha z^3 + \bar{\gamma} \bar{z}^3, & \alpha, \gamma &\in \mathbb{C}^*, |\alpha| = |\gamma|. \end{aligned}$$

As before, for typesetting convenience, we used the symbol z for $z - z_0$. (We have not included the solutions with $\varphi_k = 0$, as they are not balanced.)

The above suggests the following solutions, for all $k \geq 1$,

$$\mathcal{E}_{2k} = -4|\alpha|^2 |z|^{2k} : \quad \varphi_k = \alpha z^k + \bar{\gamma} \bar{z}^k, \quad \alpha, \gamma \in \mathbb{C}^*, |\alpha| = |\gamma|, \quad (5.11)$$

$$\mathcal{E}_{4k} = -|\alpha|^2 |z|^{4k} : \quad \varphi_{2k} = \alpha |z|^{2k}, \quad \alpha \in \mathbb{C}^*, \quad (5.12)$$

$$\begin{aligned} \mathcal{E}_{4k+2} &= -\frac{2k(k+1)|\alpha|^2}{2k^2+2k+1} |z|^{4k+2} : \\ \varphi_{2k+1} &= \alpha z |z|^{2k} \text{ or } \varphi_{2k+1} = \alpha \bar{z} |z|^{2k}, \quad \alpha \in \mathbb{C}^*. \end{aligned} \quad (5.13)$$

Those can be verified by a direct calculation.

Regularity of the metric can be established by showing that $g_{\phi t} = -af$, $\ln g_{\zeta \zeta} = \ln g_{\rho \rho} = \ln(hf^{-1})$, $g_{\phi \phi} = (\rho^2 - (af)^2)/f$ are smooth across $\{f=0, \rho>0\}$ and that af does not vanish whenever f does. All solutions with leading-order behavior (5.12), if any, have a zero of f which is of order higher than $4k$. Thus f vanishes to higher order there, and any analysis of the metric near $\{f=0\}$ requires knowledge of the higher-order Taylor coefficients of $\mathcal{E}$ and φ there.

⁴ See the MAPLE file `em3.mw` at <http://th.if.uj.edu.pl/~szybka/CS/>

On the other hand, the solution $\mathcal{E}_6 = -4/5|\alpha|^2|z|^6$, $\varphi_3 = \alpha z|z|^2$ leads to a singularity in the metric. (The same is true for its conjugate pair, namely $\bar{\mathcal{E}}$, $\bar{\varphi}$.) For this solution we have, using (2.4)–(2.6),

$$f = -\frac{1}{5}|\alpha|^2 z^3 \bar{z}^3 + \dots, \quad (5.14)$$

$$\frac{1}{h} \frac{\partial h}{\partial z} = -56 \frac{\rho_0}{z^2} + \dots, \quad (5.15)$$

$$\frac{\partial a}{\partial z} = 25 \frac{\rho_0}{|\alpha|^2 z^4 \bar{z}^3} + \dots. \quad (5.16)$$

(Eq. (5.14) shows that f vanishes at an isolated point in the (ρ, ζ) plane, leading again to an ergocircle.) Integrating we obtain

$$\ln(-h) = 112\rho_0 \frac{\rho - \rho_0}{(\rho - \rho_0)^2 + (\zeta - \zeta_0)^2} + \dots, \quad (5.17)$$

$$a = \frac{-25}{3|\alpha|^2} \frac{\rho_0}{((\rho - \rho_0)^2 + (\zeta - \zeta_0)^2)^3} + \dots, \quad (5.18)$$

hence

$$af = \frac{5}{3}\rho_0 + \dots, \quad (5.19)$$

$$\begin{aligned} \ln(hf^{-1}) &= 112\rho_0 \frac{\rho - \rho_0}{(\rho - \rho_0)^2 + (\zeta - \zeta_0)^2} \\ &\quad - \ln \left(\frac{1}{5}|\alpha|^2 ((\rho - \rho_0)^2 + (\zeta - \zeta_0)^2)^3 \right) + \dots, \end{aligned} \quad (5.20)$$

$$g_{\phi\phi} = \frac{80}{9|\alpha|^2} \frac{\rho_0^2}{((\rho - \rho_0)^2 + (\zeta - \zeta_0)^2)^3} + \dots. \quad (5.21)$$

Even though af is regular at leading order, the metric is singular at the point (ρ_0, ζ_0) . This is not merely a coordinate singularity, since (5.21) shows that the norm $g_{\phi\phi} = g(\partial_\phi, \partial_\phi)$ of the Killing vector ∂_ϕ is unbounded.

5.3. φ -dominated ergocircles

We consider now those solutions where φ dominates in f . It follows immediately from Theorem 5.2 below that such solutions correspond to isolated points of $\{f = 0\}$, hence to ergocircles within the level sets of the coordinate t .

The simplest solutions in this class would have $\mathcal{E}$ vanishing altogether, or vanishing to very high order. In this context, symbolic algebra calculations⁵ show that there are no non-trivial solutions such that

⁵ See the SINGULAR files `em4a.in`, `em4b.in` at <http://th.if.uj.edu.pl/~szybka/CS/>

- $\varphi = O(|z - z_0|)$ with non-zero gradient at z_0 , and $\mathcal{E} = O(|z - z_0|^4)$,
- $\varphi = O(|z - z_0|^2)$ with non-zero Hessian at z_0 , and $\mathcal{E} = O(|z - z_0|^9)$.

In other words the assumption that $\varphi = O(|z - z_0|)$ and $\mathcal{E} = O(|z - z_0|^4)$ implies $\varphi = O(|z - z_0|^2)$; similarly $\varphi = O(|z - z_0|^2)$ and $\mathcal{E} = O(|z - z_0|^9)$ implies $\varphi = O(|z - z_0|^3)$. Those results require the analysis of the Taylor series of φ to higher order.

More systematically, let us assume that the leading-order Taylor polynomial φ_k of φ is of order k , with the corresponding Taylor polynomial for $\mathcal{E}$ is of order ℓ , while $\Re \mathcal{E} = O(|z - z_0|^m)$. The following shows that both, for balanced and for φ -dominated solutions the order of $\mathcal{E}$ cannot be smaller than that of $|\varphi|^2$ (compare Remark 3.1):

PROPOSITION 5.1 *Suppose that $\mathcal{E} = O(|z - z_0|^\ell)$, $\varphi = O(|z - z_0|^k)$, and $\Re \mathcal{E} = O(|z - z_0|^m)$ with $m \geq 2k$, then*

$$\ell \geq 2k. \quad (5.22)$$

PROOF: Assume that $\ell < 2k$, then inspection of (2.2) gives

$$\partial_z \mathcal{E}_\ell \partial_{\bar{z}} \mathcal{E}_\ell = 0.$$

Since $\mathcal{E}_\ell$ is purely imaginary this reads $|d\mathcal{E}_\ell|^2 = 0$, and the result follows. $\square$

Clearly $m \geq \ell$ under the hypotheses of Proposition 5.1, so (5.22) implies $m \geq \ell \geq 2k$. We conclude that at a zero which is balanced we must have $m = \ell$; equivalently the order of $\mathcal{E}$ equals that of $\Re \mathcal{E}$. The same is true for $\mathcal{E}$ -dominated solutions by Remark 3.1. It follows that the hypothesis that φ dominates in f is equivalent to

$$2k < \ell. \quad (5.23)$$

Supposing that f vanishes at $(\rho_0, \zeta_0) = z_0$, (2.3) becomes

$$\bar{\varphi}_k \varphi_k L \varphi_k = 2\bar{\varphi}_k \frac{\partial \varphi_k}{\partial \bar{z}} \frac{\partial \varphi_k}{\partial z} + O(r_0^{k+\ell-2}) + O(r_0^{3k-3}). \quad (5.24)$$

By (5.23) the second term can be absorbed into the first one. Since the first derivatives part of L contributes terms which vanish faster than the second derivative ones, inspection of the leading-order terms leads to the equation

$$\varphi_k \Delta_2 \varphi_k = 2|d\varphi_k|^2 \iff \Delta_2 \varphi_k^{-1} = 0, \quad (5.25)$$

on the set $\{\varphi_k \neq 0\}$, where Δ_2 is the Laplace operator of the metric $d\rho^2 + d\zeta^2$. (Similarly, $(\mathcal{E} \equiv 0, \varphi)$ is a solution of (2.2)-(2.3) if and only if $\Delta_3 \varphi^{-1} = 0$, where Δ_3 is the Laplace operator of the metric $d\rho^2 + d\zeta^2 + \rho^2 d\phi^2$.)

We have the following:

THEOREM 5.2 *Homogeneous polynomial solutions of (5.25) are either holomorphic or anti-holomorphic.*

PROOF: Let φ_k be a homogeneous polynomial of order k solving (5.25), conveniently parameterized as

$$\varphi_k = \sum_{m=0}^k \alpha_m (z - z_0)^m (\bar{z} - \bar{z}_0)^{k-m}. \quad (5.26)$$

In complex notation the truncated Ernst–Maxwell equation (5.25) reads

$$\varphi_k \frac{\partial^2 \varphi_k}{\partial z \partial \bar{z}} = 2 \frac{\partial \varphi_k}{\partial z} \frac{\partial \varphi_k}{\partial \bar{z}}. \quad (5.27)$$

Inserting (5.26) into (5.27) we obtain

$$\sum_{1 \leq m+j \leq 2k-1} (k-m)(m-2j) \alpha_m \alpha_j (z-z_0)^{m+j-1} (\bar{z}-\bar{z}_0)^{2k-m-j-1} = 0. \quad (5.28)$$

Hence, for $1 \leq \ell \leq 2k-1$:

$$\sum_{m+j=\ell, m \leq k} (k-m)(m-2j) \alpha_m \alpha_j = 0. \quad (5.29)$$

For $\ell \leq k$ this equation can be written in the form

$$\sum_{m=0}^{\ell} (k-m)(3m-2\ell) \alpha_m \alpha_{\ell-m} = 0. \quad (5.30)$$

We consider $\ell \leq k$. For $\ell = 1$ we have

$$(k+1) \alpha_0 \alpha_1 = 0.$$

Assume, first, that $\alpha_0 \neq 0$. Then $\alpha_1 = 0$, and for $\ell = 2$ we obtain

$$2(k+2) \alpha_0 \alpha_2 = 0,$$

thus $\alpha_2 = 0$. More generally, if we assume for some ℓ_0 that $\alpha_m = 0$ for $0 < m < \ell_0$ we have from (5.30)

$$\ell_0(k + \ell_0) \alpha_0 \alpha_{\ell_0} = 0 \implies \alpha_{\ell_0} = 0.$$

We can repeat this argument for $\ell = \ell_0 + 1$ and continue up to $\ell = k$. Therefore, assumption $\alpha_0 \neq 0$ leads to $\alpha_m = 0$ for $0 < m \leq k$ and φ_k is holomorphic. Similarly, replacing above φ_k with its complex conjugate reveals that $\alpha_k \neq 0$ implies anti-holomorphicity of φ_k . Note that for $k = 1$ we are done.

Next, we assume $k \geq 2$ and we turn to the case $\alpha_0 = 0, \alpha_k = 0$. Again, we consider $\ell \leq k$. The equation with $\ell = 1$ has already been shown to be satisfied, but for $\ell = 2$ we have

$$(k-1)\alpha_1^2 = 0,$$

thus $\alpha_1 = 0$ since $k \neq 1$. The value of $\ell = 3$ gives no new conditions but for $\ell = 4$

$$(k-2)\alpha_2^2 = 0,$$

thus $\alpha_2 = 0$.

More generally, let us assume that $\alpha_m = 0$ for $0 \leq m < m_0 \leq k/2$, then (5.30) for $\ell = 2m_0$ implies

$$(k-m_0)\alpha_{m_0}^2 = 0,$$

hence we have a contradiction. We conclude that $\alpha_0 = 0$ implies $\alpha_m = 0$ for $0 \leq m \leq k/2$.

The above result applied to the complex conjugate of φ_k shows that $\alpha_k = 0$ implies $\alpha_m = 0$ for $k/2 \leq m < k$, as desired. $\square$

5.3.1. φ -dominated leading-order solutions with singular ergocircles

We continue our analysis of φ of order $k \geq 1$, with the leading term of $\mathcal{E}$ of order $2k+1$ or higher, so that f is $O(r_0^{2k})$. (Note that some possibilities for $k = 1$ and $k = 2$ have already been eliminated at the beginning of Section 5.3.) Since the Ernst–Maxwell equations are invariant under transformation $\varphi \rightarrow c\varphi$, $\mathcal{E} \rightarrow \bar{c}c\mathcal{E}$, where c is a complex constant, we can without loss of generality assume that the Taylor development $\tilde{\varphi}$ of φ , as truncated at order $k+1$, takes the form

$$\tilde{\varphi} = (z - z_0)^k + \sum_{m=0}^{k+1} \alpha_m (z - z_0)^m (\bar{z} - \bar{z}_0)^{k+1-m}. \quad (5.31)$$

Similarly, we have

$$\mathcal{E}_{2k+1} = \sum_{m=0}^{2k+1} \iota_m (z - z_0)^m (\bar{z} - \bar{z}_0)^{2k+1-m}. \quad (5.32)$$

The function f takes the form

$$f = -(z - z_0)^k (\bar{z} - \bar{z}_0)^k + O\left(r_0^{2k+1}\right). \quad (5.33)$$

The leading terms in the Ernst–Maxwell equations appear in order $4k - 1$ and $3k - 1$, respectively

$$\tilde{\varphi} \frac{\partial^2 \mathcal{E}_{2k+1}}{\partial z \partial \bar{z}} = \frac{\partial \tilde{\varphi}}{\partial z} \frac{\partial \mathcal{E}_{2k+1}}{\partial \bar{z}}, \quad (5.34)$$

$$2\bar{\varphi} \left\{ \tilde{\varphi} \left(\frac{\partial^2 \tilde{\varphi}}{\partial z \partial \bar{z}} + \frac{1}{2(z + \bar{z})} \frac{\partial \tilde{\varphi}}{\partial z} \right) - 2 \frac{\partial \tilde{\varphi}}{\partial z} \frac{\partial \tilde{\varphi}}{\partial \bar{z}} \right\} = \frac{\partial \mathcal{E}_{2k+1}}{\partial \bar{z}} \frac{\partial \tilde{\varphi}}{\partial z}. \quad (5.35)$$

It follows from (5.34) that

$$\frac{\partial \mathcal{E}_{2k+1}}{\partial \bar{z}} = \hat{C}(\bar{z}) \tilde{\varphi}, \quad (5.36)$$

where $\hat{C}(\bar{z})$ is arbitrary function of $\bar{z}$. However, we have assumed that $\mathcal{E}$ has leading term of order $2k + 1$. The comparison of (5.36) with (5.32) gives

$$\frac{\partial \mathcal{E}_{2k+1}}{\partial \bar{z}} = (k + 1) \iota_k (z - z_0)^k (\bar{z} - \bar{z}_0)^k, \quad (5.37)$$

thus, $\iota_m = 0$ for $m \neq k$ and $m \neq 2k + 1$.

(Somewhat more generally, an identical argument proves that if $\mathcal{E} = O(|z - z_0|^\ell)$ and $\varphi = O(|z - z_0|^k)$, with $2k < \ell$, φ holomorphic to leading order, then there exists $c \in \mathbb{C}$ such that $\mathcal{E}_\ell$ takes the form $\mathcal{E}_\ell = c(z - z_0)^k (\bar{z} - \bar{z}_0)^{\ell-k}$.)

The field equations imply

$$\frac{f^2}{\rho} \partial_z \ln \left(\left| \frac{h}{f} \right| \right) = \hat{\kappa}, \quad (5.38)$$

where

$$\begin{aligned} \hat{\kappa} := & \frac{1}{2} \left(\left(\frac{\partial \bar{\mathcal{E}}}{\partial z} + 2\varphi \frac{\partial \bar{\varphi}}{\partial z} + \frac{2f}{z + \bar{z}} \right) \left(\frac{\partial \mathcal{E}}{\partial z} + 2\bar{\varphi} \frac{\partial \varphi}{\partial z} \right) \right. \\ & + \left(\frac{\partial \mathcal{E}}{\partial z} + 2\bar{\varphi} \frac{\partial \varphi}{\partial z} + \frac{2f}{z + \bar{z}} \right) \left(\frac{\partial \bar{\mathcal{E}}}{\partial z} + 2\varphi \frac{\partial \bar{\varphi}}{\partial z} \right) \\ & \left. - 4 \frac{\partial \bar{\varphi}}{\partial z} \frac{\partial \varphi}{\partial z} (\mathcal{E} + \bar{\mathcal{E}} + 2\bar{\varphi}\varphi) \right), \end{aligned} \quad (5.39)$$

and recall that the functions $\mathring{\sigma}_1$ and $\mathring{\sigma}_2$ have been defined in (5.3)–(5.4). We are going to show that if the conditions mentioned at the beginning of this section hold, then (5.35), (5.34) imply that

$$\mathring{\sigma}_2 = d\mathring{\sigma}_2 = \dots = d^{2k}\mathring{\sigma}_2 = 0$$

and

$$\hat{\kappa} = d\hat{\kappa} = \dots = d^{4k-2}\hat{\kappa} = 0$$

on $E_{\mathcal{E},\varphi}$ but $d^{4k-1}\hat{\kappa} = 0$ only for special solutions.

Inserting (5.31) and (5.37) into (5.35) gives

$$\begin{aligned} & \sum_{m=0}^{k-1} (k+1-m)(m-2k)\alpha_m(z-z_0)^{k+m-1}(\bar{z}-\bar{z}_0)^{k-m} \\ & - k \left(\alpha_k + \frac{k+1}{2}\iota_k - \frac{1}{4\rho_0} \right) (z-z_0)^{2k-1} = 0. \end{aligned} \quad (5.40)$$

The comparison of the coefficients in front of powers of $(z-z_0)$ and $(\bar{z}-\bar{z}_0)$ allows us to read off that $\alpha_m = 0$ for $m = 0, \dots, k-1$. Moreover,

$$\alpha_k + \iota_k(k+1)/2 = \frac{1}{4\rho_0}$$

and there are no restrictions in the leading order on α_{k+1} , ι_{2k+1} . Hence

$$\begin{aligned} \tilde{\varphi} &= (z-z_0)^k + \alpha_k(z-z_0)^k(\bar{z}-\bar{z}_0) + \alpha_{k+1}(z-z_0)^{k+1}, \\ \mathcal{E}_{2k+1} &= \iota_k(z-z_0)^k(\bar{z}-\bar{z}_0)^{k+1}. \end{aligned}$$

Keeping this result in mind, we write down the leading terms of $\mathring{\sigma}_2$:

$$\begin{aligned} \mathring{\sigma}_2 &= -\frac{\partial \bar{\mathcal{E}}_{2k+1}}{\partial z} - 2\tilde{\varphi} \left(\frac{\partial \bar{\varphi}}{\partial z} - \frac{1}{2} \frac{\bar{\varphi}}{z+\bar{z}} \right) + O(r_0^{2k+1}) \\ &= -2 \left(\sum_{m=0}^k (k+1-m)\bar{\alpha}_m(\bar{z}-\bar{z}_0)^m(z-z_0)^{2k-m} \right. \\ &\quad \left. + \left(\frac{k+1}{2}\bar{\iota}_k - \frac{1}{4\rho_0} \right) (\bar{z}-\bar{z}_0)^k(z-z_0)^k \right) + O(r_0^{2k+1}) \\ &= O(r_0^{2k+1}). \end{aligned} \quad (5.41)$$

Therefore, $\mathring{\sigma}_2$ is at least $O(r_0^{2k+1})$. Moreover, it follows from the identity

$$-2\frac{\partial f}{\partial z} = \mathring{\sigma}_1 - \mathring{\sigma}_2 - \frac{2f}{z+\bar{z}}, \quad (5.42)$$

that $\dot{\sigma}_1$ is $O(r_0^{2k-1})$ but *not better*, because it has to compensate for the lowest terms of $\partial_z f$, see (5.33).

Now, we turn to $\hat{\kappa}$. Firstly, we rewrite (5.39) in terms of $\dot{\sigma}_1, \dot{\sigma}_2$

$$\hat{\kappa} = -\dot{\sigma}_1 \dot{\sigma}_2 - \frac{f^2}{(z + \bar{z})^2} + 4 \frac{\partial \bar{\varphi}}{\partial z} \frac{\partial \varphi}{\partial z} f. \quad (5.43)$$

It follows from our previous results and (5.33) that

$$\hat{\kappa} = - \left(\frac{1}{\rho_0} - 2(k+1)\bar{l}_k \right) k(z - z_0)^{2k-1}(\bar{z} - \bar{z}_0)^{2k} + O(r_0^{4k}). \quad (5.44)$$

Therefore, $\hat{\kappa}$ is only $O(r_0^{4k-1})$ for any

$$\iota_k \neq (2(k+1)\rho_0)^{-1}$$

and any solution with the above leading-order behavior, if it exists, will lead to a singular space-time metric (note, however, that this could be a coordinate singularity).

On the other hand if $\iota_k = (2(k+1)\rho_0)^{-1}$ then $\alpha_k = 0$ and φ is holomorphic also in the order $k+1$. For such solutions $\hat{\kappa}$ is at least $O(r_0^{4k})$, which is *not* incompatible in an *obvious* way with smoothness of the space-time metric at the ergosurface.

6. Concluding remarks

Our results are far from satisfactory, with the following questions open:

1. Which “solutions at leading order”, as constructed above using Taylor series expansions (whether balanced, φ - or $\mathcal{E}$ -dominated), *do arise* from real solutions of the Ernst–Maxwell equations which are smooth across the zero-level set of f ? Here we mean that the associated harmonic map is smooth, without (in a first step) requesting that the associated space-time metric be smooth as well. The non-existence results mentioned at the beginning of Section 5.3 are instructive: there *do* exist Taylor polynomials solving the leading-order equations with $\varphi = O(|z - z_0|)$ with non-zero gradient at z_0 and with, say, $\mathcal{E} = 0$, and one has to go a few orders more in the Taylor series to show that the coefficients of the leading-order Taylor polynomial are all zero. The same mechanism applies to leading-order solutions with $\varphi = O(|z - z_0|^2)$ with non-zero Hessian at z_0 .
2. Can one exhaustively describe the balanced leading-order solutions? The question seems hard. There does not seem, however, to be any good reason to invest a lot of energy therein as long as the previous question remains open.

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