ASYMPTOTIC CONFORMAL YANO–KILLING TENSORS FOR ASYMPTOTIC ANTI-DE SITTER SPACE-TIMES AND CONSERVED QUANTITIES

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Conformal rescaling of conformal Yano–Killing tensors and relations between Yano and CYK tensors are discussed. Pullback of these objects to a submanifold is used to construct all solutions of a CYK equation in anti-de Sitter and de Sitter space-times. Properties of asymptotic conformal Yano–Killing tensors are examined for asymptotic anti-de Sitter space-times. Explicit asymptotic forms of them are derived. The results are used to construct asymptotic charges in asymptotic AdS space-time. Well known examples like Schwarzschild–AdS, Kerr–AdS and NUT–AdS are examined carefully in the construction of the concept of energy, angular momentum and dual mass in asymptotic AdS space-time.

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1. Introduction

According to [28] one can define, in terms of space-time curvature, two kinds of conserved quantities with the help of conformal Yano–Killing tensors (see [39, 40]). Sometimes they are also called conformal Killing forms or twistor forms (see *e.g.* [6, 30, 36, 37]). The first kind is linear and the second quadratic with respect to the Weyl tensor but a basis for both of them is the Maxwell field. Conserved quantities which are linear with respect to CYK tensor were investigated many times (*cf.* [1, 16, 17, 26–28, 31, 32]). On the other hand, quadratic charges are less known and usually examined in terms of the Bel–Robinson tensor (see *e.g.* [8–10, 12]).

In electrodynamics the linear quantity corresponds to electric or magnetic charge and the quadratic one expresses the energy, linear momentum or angular momentum of the Maxwell field. In gravity both kinds of charges play a role of energy. The linear conserved quantities (as two-surface integrals) correspond to ADM mass and linear or angular momentum but bilinear ones are not obviously related to energy. They rather play a role of energy estimates like in [10] (*cf.* [3]). In this paper we analyze the existence and the properties of the linear charges for asymptotic anti-de Sitter space-times.

Let M be an n-dimensional ($n > 1$) manifold with a Riemannian or pseudo-Riemannian metric $g_{\mu\nu}$. The covariant derivative associated with the Levi–Civita connection will be denoted by ∇ or just by "$;$". By $T_{\dots(\mu\nu)\dots}$ we will denote the symmetric part and by $T_{\dots[\mu\nu]\dots}$ the skew-symmetric part of tensor $T_{\dots\mu\nu\dots}$ with respect to indices μ and ν (analogous symbols may be used for more indices).

Let $Q_{\mu\nu}$ be a skew-symmetric tensor field (two-form) on M and let us denote by $\mathcal{Q}_{\lambda\kappa\sigma}$ a (three-index) tensor which is defined as follows:

$$\mathcal{Q}_{\lambda\kappa\sigma}(Q,g) := Q_{\lambda\kappa;\sigma} + Q_{\sigma\kappa;\lambda} - \frac{2}{n-1}\left(g_{\sigma\lambda}Q^{\nu}{}_{\kappa;\nu} + g_{\kappa(\lambda}Q_{\sigma)}{}^{\mu}{}_{;\mu}\right) . \tag{1.1}$$

The object $\mathcal{Q}$ has the following algebraic properties

$$\mathcal{Q}_{\lambda\kappa\mu}g^{\lambda\mu} = 0 = \mathcal{Q}_{\lambda\kappa\mu}g^{\lambda\kappa}\,, \qquad \mathcal{Q}_{\lambda\kappa\mu} = \mathcal{Q}_{\mu\kappa\lambda}\,, \tag{1.2}$$

i.e. it is traceless and partially symmetric.

Definition 1. *A skew-symmetric tensor $Q_{\mu\nu}$ is a conformal Yano–Killing tensor (or simply CYK tensor) for the metric g if $\mathcal{Q}_{\lambda\kappa\sigma}(Q,g) = 0$.*

In other words, $Q_{\mu\nu}$ is a conformal Yano–Killing tensor if it fulfills the following equation:

$$Q_{\lambda\kappa;\sigma} + Q_{\sigma\kappa;\lambda} = \frac{2}{n-1}\left(g_{\sigma\lambda}Q^{\nu}{}_{\kappa;\nu} + g_{\kappa(\lambda}Q_{\sigma)}{}^{\mu}{}_{;\mu}\right) \tag{1.3}$$

(first proposed by Tachibana and Kashiwada, *cf.* [39]).

A more abstract way with no indices of describing a CYK tensor can be found in [7,30,36] or [37], where it is considered as the element of the kernel of the twistor operator $Q \to \mathcal{T}_{\text{wist}}Q$ defined as follows:

$$\forall X \;\; \mathcal{T}_{\text{wist}}Q(X) := \nabla_X Q - \frac{1}{p+1} X \lrcorner\, dQ + \frac{1}{n-p+1} g(X) \wedge d^*Q\,.$$

However, to simplify the exposition, we prefer abstract index notation which also seems to be more popular.

The paper is organized as follows: In Section 2 we prove the Theorem: if Q is a CYK tensor of the ambient $(n+1)$-dimensional space-time metric $\overset{(n+1)}{g}$ (in a special form), then its pullback to (the correctly chosen) submanifold is a CYK tensor of the induced metric $\overset{(n)}{g}$, which may be easily applied for anti-de Sitter embedded in flat pseudo-Riemannian five-dimensional manifold. In the next section we use the preceding results to construct all CYK tensors in de Sitter and anti-de Sitter space-times. Section 4 is devoted to the anti-de Sitter space-time together with some important examples. In particular, we construct (more explicitly than usual) Fefferman–Graham canonical coordinates for the Kerr–AdS solution. Next section contains analysis of symplectic structure at scri and, finally, in Section 6 we analyze asymptotic charges. To clarify the exposition some of the technical results and proofs have been shifted to the appendix.

2. Pullback of CYK tensor to submanifold of codimension one

Let N be a differential manifold of dimension $n+1$ and $\overset{(n+1)}{g}$ its metric tensor (the signature of the metric plays no role). Moreover, we assume that there exists a coordinate system (x^A), where $A = 0, \ldots, n$, in which $\overset{(n+1)}{g}$ takes the following form:

$$\overset{(n+1)}{g} = f(u)h + s\,du^2\,, \tag{2.1}$$

where s is equal to 1 or -1, $u \equiv x^n$, f is a certain function, and h is a certain tensor, which does not depend on u. The metric (2.1) possesses a conformal Killing vector field[1] $\sqrt{f}\partial_u$. Tensor $f(u)h$ is a metric tensor on a submanifold $M := \{u = \text{const.}\}$. We will denote it by $\overset{(n)}{g}$. We will distinguish all objects associated with the metric $\overset{(n)}{g}$ by writing (n) above their symbols. Similar notation will be used for objects associated with the metric $\overset{(n+1)}{g}$.

It turns out that:

Theorem 1. *If Q is a CYK tensor of the metric $\overset{(n+1)}{g}$ in N, then its pullback to the submanifold M is a CYK tensor of the metric $\overset{(n)}{g}$.*

Proof. In order to show this, we need to derive some helpful formulae. Let us notice that in coordinates x^A we have[2]:

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[1] A conformally rescaled metric $\tilde{g} = \frac{1}{f}\overset{(n+1)}{g} = h + s\frac{du^2}{f(u)} = h + s\,dv^2$ (where $dv := f(u)^{-1/2}\,du$) has the Killing vector $\partial_v = \sqrt{f}\partial_u$, which is a conformal Killing vector field for the original metric (2.1).

[2] In this chapter we will use the convention that indices denoted by capital letters of the Latin alphabet go from 0 to n and Greek indices go from 0 to $n-1$. The index u denotes n-th component of a tensor.

$$\overset{(n+1)}{g}_{uu} = s\,, \qquad \overset{(n+1)}{g}_{u\mu} = 0\,, \qquad \overset{(n+1)}{g}_{\mu\nu} = \overset{(n)}{g}_{\mu\nu}\;. \tag{2.2}$$

It means that the only non-vanishing derivatives of the metric are the following:

$$\overset{(n+1)}{g}_{\mu\nu,u} = \Phi(u)\,\overset{(n)}{g}_{\mu\nu} \quad \text{and} \quad \overset{(n+1)}{g}_{\mu\nu,\rho} = \overset{(n)}{g}_{\mu\nu,\rho}\,, \tag{2.3}$$

where $\Phi(u) = \frac{d}{du}\left(\log f(u)\right)$. Using the formula

$$\overset{(n+1)}{\Gamma}{}^{A}{}_{BC} = \frac{1}{2}\,g^{AD}(g_{DB,C} + g_{DC,B} - g_{BC,D})\,,$$

we compute all non-vanishing Christoffel symbol of the metric $\overset{(n+1)}{g}$:

$$\overset{(n+1)}{\Gamma}{}^{A}{}_{\nu u} = \frac{1}{2}\,\Phi(u)\delta^{A}{}_{\nu}\,, \qquad \overset{(n+1)}{\Gamma}{}^{u}{}_{\mu\nu} = -\frac{s}{2}\Phi(u)g_{\mu\nu}\,, \qquad \overset{(n+1)}{\Gamma}{}^{\mu}{}_{\nu\rho} = \overset{(n)}{\Gamma}{}^{\mu}{}_{\nu\rho}\,. \tag{2.4}$$

Using formulae (2.4) we compute:

$$\begin{aligned} \overset{(n+1)}{\nabla}_{\mu} Q_{\nu\rho} &= Q_{\nu\rho,\mu} - Q_{A\rho}\,\overset{(n+1)}{\Gamma}{}^{A}{}_{\nu\mu} - Q_{\nu A}\,\overset{(n+1)}{\Gamma}{}^{A}{}_{\rho\mu} \\ &= \overset{(n)}{\nabla}_{\mu} Q_{\nu\rho} - Q_{u\rho}\,\overset{(n+1)}{\Gamma}{}^{u}{}_{\nu\mu} - Q_{\nu u}\,\overset{(n+1)}{\Gamma}{}^{u}{}_{\rho\mu} \\ &= \overset{(n)}{\nabla}_{\mu} Q_{\nu\rho} + \frac{s}{2}\,\Phi(u)\,Q_{u\rho}g_{\mu\nu} + \frac{s}{2}\,\Phi(u)\,Q_{\nu u}g_{\rho\mu}\,. \end{aligned} \tag{2.5}$$

Let us denote $\overset{(n)}{\xi}_{\rho} := g^{\mu\nu}\,\overset{(n)}{\nabla}_{\mu} Q_{\nu\rho}$. The formula (2.5) directly implies that:

$$g^{\mu\nu}\,\overset{(n+1)}{\nabla}_{\mu} Q_{\nu\rho} = \overset{(n)}{\xi}_{\rho} + s\frac{n-1}{2}\,\Phi(u)\,Q_{u\rho}\,. \tag{2.6}$$

Tensor Q satisfies the CYK equation, *i.e.*

$$\overset{(n+1)}{\nabla}_{A} Q_{BC} + \overset{(n+1)}{\nabla}_{B} Q_{AC} = \frac{2}{n}\left(g_{AB}\,\overset{(n+1)}{\xi}_{C} - g_{C(B}\,\overset{(n+1)}{\xi}_{A)}\right). \tag{2.7}$$

Substituting $A = B = u$ and $C = \rho$ we get:

$$\overset{(n+1)}{\nabla}_{u} Q_{u\rho} = \frac{s}{n}\,\overset{(n+1)}{\xi}_{\rho}\;. \tag{2.8}$$

Using formulae (2.6) and (2.8) we compute:

$$\begin{aligned}\overset{(n+1)}{\xi}{}_\rho &= g^{AC} \overset{(n+1)}{\nabla}{}_C Q_{A\rho} = g^{uu} \overset{(n+1)}{\nabla}{}_u Q_{u\rho} + g^{\mu\nu} \overset{(n+1)}{\nabla}{}_\mu Q_{\nu\rho} \\ &= \frac{1}{n} \overset{(n+1)}{\xi}{}_\rho + \overset{(n)}{\xi}{}_\rho + s\frac{n-1}{2} \Phi(u)\, Q_{u\rho} \,, \end{aligned} \tag{2.9}$$

which implies:

$$\overset{(n+1)}{\xi}{}_\rho = \frac{n}{n-1} \overset{(n)}{\xi}{}_\rho + s\frac{n}{2} \Phi(u)\, Q_{u\rho} \,. \tag{2.10}$$

Using formulae (2.5) and (2.10) we get:

$$\begin{aligned} &\overset{(n+1)}{\nabla}{}_\sigma Q_{\lambda\kappa} + \overset{(n+1)}{\nabla}{}_\lambda Q_{\sigma\kappa} - \frac{2}{n}\left(g_{\sigma\lambda} \overset{(n+1)}{\xi}{}_\kappa - g_{\kappa(\lambda} \overset{(n+1)}{\xi}{}_{\sigma)} \right) \\ &= \overset{(n)}{\nabla}{}_\sigma Q_{\lambda\kappa} + \overset{(n)}{\nabla}{}_\lambda Q_{\sigma\kappa} - \frac{2}{n-1}\left(g_{\sigma\lambda} \overset{(n)}{\xi}{}_\kappa - g_{\kappa(\lambda} \overset{(n)}{\xi}{}_{\sigma)} \right) . \end{aligned} \tag{2.11}$$

Left-hand side of the equation (2.11) vanishes because Q is a CYK tensor of the metric $\overset{(n+1)}{g}$. This implies that the right-hand side is also equal to zero, hence the pullback of Q to the surface $u = \text{const.}$ is a CYK tensor of the metric $\overset{(n)}{g}$. □

3. CYK tensors in the de Sitter (and anti-de Sitter) space-time

In this section we will discuss the problem of existence and basic properties of CYK tensors for de Sitter and anti-de Sitter metrics. These metrics are solutions to the vacuum Einstein equations with the cosmological constant Λ having a maximal symmetry group. Therefore, they can be treated as a generalization of the flat Minkowski metric to the case of nonzero cosmological constant. De Sitter metric is a solution of Einstein equations with positive cosmological constant. Anti-de Sitter metric corresponds to negative cosmological constant. We will restrict ourselves to the case of four dimensional metrics, although they can be defined for manifolds of any dimension (*cf.* [21, 34]). In dimension four we can express these metrics with the use of coordinate system (t, r, θ, ϕ) as follows

$$\tilde{g} = -\left(1 - s\frac{r^2}{l^2}\right) dt^2 + \frac{1}{1 - s\frac{r^2}{l^2}}\, dr^2 + r^2\, d\Omega_2 \,, \tag{3.1}$$

where $d\Omega_2 := d\theta^2 + \sin^2\theta\, d\phi^2$ is a unit sphere metric and l is a certain constant related to the cosmological constant by the formula $\Lambda = 3s/l^2$.

Moreover, s is equal to 1 for de Sitter metric and -1 for anti-de Sitter metric. When l goes to infinity, the cosmological constant goes to zero and the metric (3.1) approaches a flat metric, as one could expect.

Quite often it is more convenient to use another (rescaled) coordinate system together with the following notation: $\bar{t} = t/l$, $\bar{r} = r/l$. In coordinates $(\bar{t}, \bar{r}, \theta, \phi)$ (anti-)de Sitter metric has the following form:

$$\tilde{g} = l^2 \left[(-1 + s\bar{r}^2)\, d\bar{t}^2 + \frac{1}{1 - s\bar{r}^2}\, d\bar{r}^2 + \bar{r}^2\, d\Omega_2 \right] . \tag{3.2}$$

(Anti-)de Sitter manifold (denoted by $\tilde{M}$) is (by definition) Einstein space-time *i.e.* its Ricci tensor is proportional to the metric. Its Weyl tensor vanishes which means it is a conformally flat metric (*i.e.* there exists a conformal rescaling which brings it to the flat Minkowski metric). As we have shown in [24], four-dimensional Minkowski space-time admits twenty-dimensional space of solutions of the CYK equation. Moreover, the following

Theorem 2. *If $Q_{\mu\nu}$ is a CYK tensor for the metric $g_{\mu\nu}$, then $\Omega^3 Q_{\mu\nu}$ is a CYK tensor for the conformally rescaled metric $\Omega^2 g_{\mu\nu}$.*

(proved in [24]) implies that (anti-)de Sitter also admits precisely twenty independent CYK tensors[3]. We will show in the sequel how to obtain them in an independent way and examine their basic properties.

In order to do that we use the immersion of our four-dimensional (anti-)de Sitter space-time in five-dimensional flat pseudo-Riemmannian manifold with signature $(s, 1, 1, 1, -1)$. In order to make formulae more legible we will use the following convention: Greek indices $\mu, \nu, \ldots$ label space-time coordinates in $\tilde{M}$ and run from 0 to 3; Latin indices $i, j, \ldots$ label space coordinates and run from 1 to 3, and finally indices denoted by capital letters of the Latin alphabet go from 0 to 4 and they label coordinates in N.

Let N be a five-dimensional differential manifold with a global coordinate system (y^A). We define the metric tensor η of the manifold N by the formula:

$$\begin{aligned} \eta &= \eta_{AB}\, dy^A \otimes dy^B \\ &= s\, dy^0 \otimes dy^0 + dy^1 \otimes dy^1 + dy^2 \otimes dy^2 + dy^3 \otimes dy^3 - dy^4 \otimes dy^4 \,. \end{aligned} \tag{3.3}$$

Let $\tilde{M}$ be a submanifold of N defined by:

$$\eta_{AB} y^A y^B = s l^2 \,. \tag{3.4}$$

[3] In general, this might be true only locally, *e.g.* in [23] one can find some global topological difficulties with the conformally covariant solutions of Maxwell and Dirac equations for cosmological models conformal to Minkowski space-time.

The metric η restricted to $\tilde{M}$ is just the (anti-)de Sitter metric (*cf.* [33]). In order to see this, let us introduce a coordinate system $(\bar{t}, \bar{r}, \theta, \phi)$ on $\tilde{M}$. However, we need to consider the cases $s = 1$ and $s = -1$ separately. For $s = 1$ a parametrization of $\tilde{M}$ takes the following form:

$$y^0 = l\sqrt{1-\bar{r}^2}\cosh\bar{t}\,, \tag{3.5}$$
$$y^1 = l\bar{r}\sin\theta\cos\phi\,, \tag{3.6}$$
$$y^2 = l\bar{r}\sin\theta\sin\phi\,, \tag{3.7}$$
$$y^3 = l\bar{r}\cos\theta\,, \tag{3.8}$$
$$y^4 = l\sqrt{1-\bar{r}^2}\sinh\bar{t}\,. \tag{3.9}$$

If $s = -1$, the analogous formulae are the following:

$$y^0 = l\sqrt{1+\bar{r}^2}\cos\bar{t}\,, \tag{3.10}$$
$$y^1 = l\bar{r}\sin\theta\cos\phi\,, \tag{3.11}$$
$$y^2 = l\bar{r}\sin\theta\sin\phi\,, \tag{3.12}$$
$$y^3 = l\bar{r}\cos\theta\,, \tag{3.13}$$
$$y^4 = l\sqrt{1+\bar{r}^2}\sin\bar{t}\,. \tag{3.14}$$

Let us notice that functions l, $\bar{t}$, $\bar{r}$, θ and ϕ can be considered as the local coordinate system on N. Substituting formulae (3.5)–(3.9) or (3.10)–(3.14) into definition (3.3) of the metric η we get:

$$\eta = s\,dl^2 + l^2\left[(-1+s\bar{r}^2)\,d\bar{t}^2 + \frac{1}{1-s\bar{r}^2}\,d\bar{r}^2 + \bar{r}^2\,d\Omega_2\right]. \tag{3.15}$$

In particular, formula (3.15) implies that η restricted to the surface $M := \{l = \text{const.}\} \subset N$ has the same form as the metric $\tilde{g}$ (*cf.* (3.2)).

Identifying the (anti-)de Sitter space-time with the submanifold $\tilde{M}$ enables one to find all Killing vector fields of the metric $\tilde{g}$. The vector fields

$$L_{AB} := y_A\frac{\partial}{\partial y^B} - y_B\frac{\partial}{\partial y^A}$$

(where $y_A := \eta_{AB}y^B$) are the Killing fields of the metric η. However, the formulae defining the fields L_{AB} depend on the sign s. For $s = 1$ we get:

$$L_{40} = -\frac{\partial}{\partial\bar{t}}\,, \tag{3.16}$$

$$L_{i4} = \frac{x^i}{\sqrt{1-\bar{r}^2}}\cosh\bar{t}\frac{\partial}{\partial\bar{t}} + \sqrt{1-\bar{r}^2}\sinh\bar{t}\frac{\partial}{\partial x^i}\,, \tag{3.17}$$

$$L_{i0} = -\frac{x^i}{\sqrt{1-\bar{r}^2}}\sinh\bar{t}\frac{\partial}{\partial\bar{t}} - \sqrt{1-\bar{r}^2}\cosh\bar{t}\frac{\partial}{\partial x^i}\,, \tag{3.18}$$

$$L_{ij} = x^i\frac{\partial}{\partial x^j} - x^j\frac{\partial}{\partial x^i}\,, \tag{3.19}$$

where in the coordinate system on N instead of spherical coordinates $\bar{r}, \theta, \phi$ we use Cartesian $x^k := y^k/l = \bar{r}n^k$, $k = 1, 2, 3$.

If $s = -1$ in coordinate system $(l, \bar{t}, x^k)$ we have:

$$L_{40} = \frac{\partial}{\partial\bar{t}}\,, \tag{3.20}$$

$$L_{i4} = \frac{x^i}{\sqrt{1+\bar{r}^2}}\cos\bar{t}\frac{\partial}{\partial\bar{t}} + \sqrt{1+\bar{r}^2}\sin\bar{t}\frac{\partial}{\partial x^i}\,, \tag{3.21}$$

$$L_{i0} = -\frac{x^i}{\sqrt{1+\bar{r}^2}}\sin\bar{t}\frac{\partial}{\partial\bar{t}} + \sqrt{1+\bar{r}^2}\cos\bar{t}\frac{\partial}{\partial x^i}\,, \tag{3.22}$$

$$L_{ij} = x^i\frac{\partial}{\partial x^j} - x^j\frac{\partial}{\partial x^i}\,. \tag{3.23}$$

It is easy to notice that those fields are tangent to $\tilde{M}$ and, therefore, their restrictions to the submanifold are Killing fields of the induced metric. The fields defined on N as well as their restrictions to $\tilde{M}$ will be denoted by the same symbol L_{AB}. Restricting the fields L_{AB} to $\tilde{M}$ we get 10 linearly independent Killing fields of the metric $\tilde{g}$. This is the maximum number of the independent Killing fields the four-dimensional metric can have, so L_{AB} span the space of the Killing fields of the metric $\tilde{g}$.

The formula (3.15) shows that the metric η in the coordinates $(l, \bar{t}, \bar{r}, \theta, \phi)$ has the form (2.1) which implies that the CYK tensors of the metric η restricted to the surface $l =$ const. are the CYK tensors of the induced metric. In this way from the CYK tensors in N we obtain the CYK tensors in $\tilde{M}$. Let us consider 10 linearly independent tensors $dy^A \wedge dy^B$ defined in N. Obviously they are Yano tensors of the metric η. Their restriction to the submanifold $\tilde{M}$ gives us 10 linearly independent CYK tensors of $\tilde{g}$. None of them is Yano tensor. We have[4]:

[4] Tensors restricted to $\tilde{M}$ will be denoted by the same symbols as the tensors defined on N. Hence y^A can be treated as functions on $\tilde{M}$ defined by the formulae (3.5)–(3.9) or (3.10)–(3.14), where l is a constant.

$$\xi \quad \text{for} \quad dy^0 \wedge dy^i \quad \text{equals} \quad s\frac{3}{l^2}L_{0i}\,, \tag{3.24}$$

$$\xi \quad \text{for} \quad dy^0 \wedge dy^4 \quad \text{equals} \quad -s\frac{3}{l^2}L_{04}\,, \tag{3.25}$$

$$\xi \quad \text{for} \quad dy^i \wedge dy^j \quad \text{equals} \quad -s\frac{3}{l^2}L_{ij}\,, \tag{3.26}$$

$$\xi \quad \text{for} \quad dy^i \wedge dy^4 \quad \text{equals} \quad s\frac{3}{l^2}L_{i4} \tag{3.27}$$

(remember that according to the previous notation ξ for a CYK tensor Q is defined by the formula $\xi^\nu := Q^{\mu\nu}{}_{;\mu}$). It turns out that all the tensors of the form $*(\,dy^A \wedge dy^B)$ (where $*$ denotes Hodge duality related to the metric $\tilde{g}$) are Yano tensors. Tensors $dy^A \wedge dy^B$ and $*(\,dy^A \wedge dy^B)$ are linearly independent and there are twenty of them, therefore, they span the space of all solutions of the CYK equation for the (anti-)de Sitter metric.

At the end, we consider the correspondence between CYK tensors in Minkowski space-time and the solutions of CYK equation in (anti-)de Sitter space-time — tensors $dy^A \wedge dy^B$ and $*(\,dy^A \wedge dy^B)$. To be more precise, we examine the behaviour of the coefficients of the latter when we pass to the limit $l \to \infty$ (as we know, in this limit the metric $\tilde{g}$ becomes the flat Minkowski metric). There is, however, a crucial issue we have to mention. Any CYK tensor can be multiplied by a constant, but on $\tilde{M}$ the function l is constant. Therefore, in order to obtain finite, non-zero limit we have to multiply each CYK tensor by a proper power of l. Finally we get

$$\begin{aligned}
\lim_{l\to\infty}(\,dy^i \wedge dy^j) &= (\mathcal{T}_i \wedge \mathcal{T}_j)\,,\\
\lim_{l\to\infty} *(\,dy^i \wedge dy^j) &= *(\mathcal{T}_i \wedge \mathcal{T}_j)\,,\\
\lim_{l\to\infty}(\,dy^i \wedge dy^4) &= (\mathcal{T}_0 \wedge \mathcal{T}_i)\,,\\
\lim_{l\to\infty} *(\,dy^i \wedge dy^4) &= *(\mathcal{T}_0 \wedge \mathcal{T}_i)\,,\\
\lim_{l\to\infty}(l\,dy^0 \wedge dy^i) &= -s(\mathcal{D} \wedge \mathcal{T}_i)\,,\\
\lim_{l\to\infty} *(l\,dy^0 \wedge dy^i) &= -s*(\mathcal{D} \wedge \mathcal{T}_i)\,,\\
\lim_{l\to\infty}(l\,dy^0 \wedge dy^4) &= s(\mathcal{D} \wedge \mathcal{T}_0)\,,\\
\lim_{l\to\infty} *(l\,dy^0 \wedge dy^4) &= s*(\mathcal{D} \wedge \mathcal{T}_0)\,,
\end{aligned} \tag{3.28}$$

where the space of Killing fields is spanned by the fields

$$\mathcal{T}_\mu := \frac{\partial}{\partial x^\mu}\,, \qquad \mathcal{L}_{\mu\nu} := x_\mu\frac{\partial}{\partial x^\nu} - x_\nu\frac{\partial}{\partial x^\mu}\,, \tag{3.29}$$

(here (x^μ) are Cartesian coordinates, $x_\mu = \eta_{\mu\nu}x^\nu$, $\eta_{\mu\nu} := \mathrm{diag}(-1,1,1,1)$) and

$$\mathcal{D} := x^\mu \frac{\partial}{\partial x^\mu} \tag{3.30}$$

is a *dilation vector field.*

Remark: The formulae (3) imply that different CYK tensors in the (anti-) de Sitter metric may converge to the same tensor in Minkowski space-time, *e.g.* $*(dy^1 \wedge dy^2)$ and $dy^4 \wedge dy^3$ go to $*(\mathcal{T}_1 \wedge \mathcal{T}_2) = \mathcal{T}_3 \wedge \mathcal{T}_0$, although $*(dy^1 \wedge dy^2)$ differs from $dy^4 \wedge dy^3$.

Moreover, we obtain the rest of CYK tensors in the Minkowski space-time as follows:

$$\begin{aligned}
\lim_{l\to\infty} l^2(dy^1 \wedge dy^2 - *dy^3 \wedge dy^4) &= -s\widetilde{\mathcal{L}}_{12}\,,\\
\lim_{l\to\infty} l^2(dy^1 \wedge dy^3 - *dy^4 \wedge dy^2) &= -s\widetilde{\mathcal{L}}_{13}\,,\\
\lim_{l\to\infty} l^2(dy^2 \wedge dy^3 - *dy^1 \wedge dy^4) &= -s\widetilde{\mathcal{L}}_{23}\,,\\
\lim_{l\to\infty} l^2(dy^4 \wedge dy^1 - *dy^2 \wedge dy^3) &= s\widetilde{\mathcal{L}}_{01}\,,\\
\lim_{l\to\infty} l^2(dy^4 \wedge dy^2 - *dy^3 \wedge dy^1) &= s\widetilde{\mathcal{L}}_{02}\,,\\
\lim_{l\to\infty} l^2(dy^4 \wedge dy^3 - *dy^1 \wedge dy^2) &= s\widetilde{\mathcal{L}}_{03}\,,
\end{aligned} \tag{3.31}$$

where

$$\widetilde{\mathcal{L}}_{\mu\nu} := \mathcal{D} \wedge \mathcal{L}_{\mu\nu} - \frac{1}{2}\eta(\mathcal{D},\mathcal{D})\mathcal{T}_\mu \wedge \mathcal{T}_\nu \tag{3.32}$$

(and $s = 1$ for de Sitter, $s = -1$ for anti-de Sitter, respectively). The above formulae show how to obtain all CYK tensors in Minkowski space-time from the solutions of CYK equation in (anti-)de Sitter space-time.

4. Asymptotic anti-de Sitter space-time

For asymptotic analysis let us change the radial coordinate in the anti-de Sitter metric (3.1) as follows

$$z := \frac{l}{r + \sqrt{r^2 + l^2}}\,, \qquad \bar{r} = \frac{r}{l} = \frac{1 - z^2}{2z}\,,$$

which implies that

$$\tilde{g}_{\mathrm{AdS}} = \frac{l^2}{z^2}\left[dz^2 - \left(\frac{1+z^2}{2}\right)^2 d\bar{t}^2 + \left(\frac{1-z^2}{2}\right)^2 d\Omega_2\right]. \tag{4.1}$$

The above particular form of $\tilde{g}_{\mathrm{AdS}}$ is well adopted to the so-called conformal compactification (see *e.g.* [14, 18]). More precisely, the metric g on the interior $\tilde{M}$ of a compact manifold M with boundary ∂M is said to be conformally compact if $g \equiv \Omega^2 \tilde{g}$ extends continuously (or with some degree of smoothness) as a metric to M, where Ω is a defining function for the scri $\mathscr{I} = \partial M$, *i.e.* $\Omega > 0$ on $\tilde{M}$ and $\Omega = 0,\ d\Omega \neq 0$ on ∂M. In the case of AdS metric (4.1) we have

$$g_{\mathrm{AdS}} = \Omega^2 \tilde{g}_{\mathrm{AdS}}, \quad \text{where} \quad \Omega := \frac{z}{l}\,.$$

According to [13, 18] and [34], our four-dimensional asymptotic AdS space-time metric $\tilde{g}$ assumes in canonical coordinates[5] the following form:

$$\tilde{g} = \tilde{g}_{\mu\nu}\, dz^\mu \otimes dz^\nu = \frac{l^2}{z^2} \left(dz \otimes dz + h_{ab}\, dz^a \otimes dz^b \right) \tag{4.2}$$

and the three-metric h obeys the following asymptotic condition:

$$h = h_{ab}\, dz^a \otimes dz^b = \overset{(0)}{h} + z^2 \overset{(2)}{h} + z^3 \chi + O(z^4)\,. \tag{4.3}$$

Let us observe that the term χ vanishes for the pure AdS given by (4.1). Moreover, the terms $\overset{(0)}{h}$ and $\overset{(2)}{h}$ have the standard form

$$\overset{(0)}{h} = \frac{1}{4} \left(d\Omega_2 - d\vec{t}^2 \right), \tag{4.4}$$

$$\overset{(2)}{h} = -\frac{1}{2} \left(d\Omega_2 + d\vec{t}^2 \right). \tag{4.5}$$

For generalized (asymptotically locally) anti-de Sitter space-times tensors $\overset{(0)}{h}$ and $\overset{(2)}{h}$ need not to be conformally "trivial", *i.e.* in the form (4.4) and (4.5), respectively. Such more general situation has been considered *e.g.* by Anderson, Chruściel [2], Graham [13], Skenderis [35]. Let us stress that in the general case only the induced metric $\overset{(0)}{h}$ may be changed freely beyond the conformal class, $\overset{(2)}{h}$ is always given by (4.81). Moreover, $\overset{(0)}{h}$ and χ form a symplectic structure on conformal boundary (*cf.* Section 5).

[5] Sometimes it is called Fefferman–Graham coordinate system.

However, we assume the standard asymptotic AdS: The induced metric h on $\mathscr{I}$ is in the conformal class of the "Einstein static universe", *i.e.*

$$\overset{(0)}{h} = \exp(\omega)(d\Omega_2 - d\bar{t}^2) \tag{4.6}$$

for some smooth function ω. This implies that our $\mathscr{I}$ is a time-like boundary.

We use the following convention: Greek indices $\mu, \nu, \ldots$ label space-time coordinates in $\tilde{M}$ and run from 0 to 3; Latin indices $a, b, \ldots$ label coordinates on a tube $S := \{z = \text{const.}\}$ and run from 0 to 2.

Functions y^A given by equations (3.10)–(3.14) and restricted to $\tilde{M}$ can be expressed in coordinate system $(z^\mu) \equiv (z^0, z^1, z^2, z^3) \equiv (\bar{t}, \theta, \phi, z)$ as follows

$$y^0 = \Omega^{-1}\frac{1+z^2}{2}\cos\bar{t}\,, \tag{4.7}$$

$$y^k = \Omega^{-1}\frac{1-z^2}{2}n^k\,, \tag{4.8}$$

$$y^4 = \Omega^{-1}\frac{1+z^2}{2}\sin\bar{t}\,, \tag{4.9}$$

where $k = 1, 2, 3$, and

$$n := \begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix}$$

is a radial unit normal in Euclidean three-space (identified with a point on a unit sphere parameterized by coordinates (θ, ϕ)).

Let us denote a CYK tensor in AdS space-time by ${}^{[AB]}\tilde{Q} := l\, dy^A \wedge dy^B$. Coordinates y^A restricted to $\tilde{M}$, given by equations (4.7–4.9), lead to the following explicit formulae for two-forms ${}^{[AB]}\tilde{Q}$:

$${}^{[04]}\tilde{Q} = \frac{1}{4}\Omega^{-3}(1 - z^4)\, d\bar{t} \wedge dz\,, \tag{4.10}$$

$${}^{[jk]}\tilde{Q} = \frac{1}{4}\Omega^{-3}\Big[(1-z^4)(n^j\, dn^k - n^k\, dn^j) \wedge d + zz(1-z^2)^2 dn^j \wedge dn^k\Big], \tag{4.11}$$

$$\begin{aligned} {}^{[0k]}\tilde{Q} = \frac{1}{4}\Omega^{-3}\Big[&(1 - z^2)^2 \cos\bar{t}\, dn^k \wedge dz + n^k(1 + z^2)^2 \sin\bar{t}\, d\bar{t} \wedge dz \\ &+ z(1 - z^4) \sin\bar{t}\, dn^k \wedge d\bar{t}\Big]\,, \end{aligned} \tag{4.12}$$

$$\begin{aligned} {}^{[4k]}\tilde{Q} = \frac{1}{4}\Omega^{-3}\Big[&(1 - z^2)^2 \sin\bar{t}\, dn^k \wedge dz - n^k(1 + z^2)^2 \cos\bar{t}\, d\bar{t} \wedge dz \\ &- z(1 - z^4) \cos\bar{t}\, dn^k \wedge d\bar{t}\Big]\,. \end{aligned} \tag{4.13}$$

The other ten solutions we get applying Hodge star isomorphism. More precisely, an orthonormal frame $e^0 := \Omega^{-1}\left(\frac{1+z^2}{2}\right) d\bar{t}$, $e^1 := \Omega^{-1}\left(\frac{1-z^2}{2}\right) d\theta$, $e^2 := \Omega^{-1}\left(\frac{1-z^2}{2}\right) \sin\theta\, d\phi$, $e^3 := \Omega^{-1}\, dz$ for the metric tensor (4.1), *i.e.*

$$\tilde{g}_{\mathrm{AdS}} = -e^0 \otimes e^0 + \sum_{k=1}^{3} e^k \otimes e^k \,,$$

enables one to calculate Hodge dual in a simple way, *i.e.*

$$\begin{aligned} *(e^0 \wedge e^1) &= -e^2 \wedge e^3 \,, & *(e^0 \wedge e^2) &= -e^3 \wedge e^1 \,, \\ *(e^0 \wedge e^3) &= -e^1 \wedge e^2 \,, & *(e^1 \wedge e^2) &= e^0 \wedge e^3 \,, \\ *(e^2 \wedge e^3) &= e^0 \wedge e^1 \,, & *(e^3 \wedge e^1) &= e^0 \wedge e^2 \,. \end{aligned}$$

Moreover,

$$\begin{aligned} dn &= \begin{bmatrix} \cos\theta\cos\phi\, d\theta - \sin\theta\sin\phi\, d\phi \\ \cos\theta\sin\phi\, d\theta + \sin\theta\cos\phi\, d\phi \\ -\sin\theta\, d\theta \end{bmatrix} \\ &= \frac{2\Omega}{1-z^2} \begin{bmatrix} \cos\theta\cos\phi\, e^1 - \sin\phi\, e^2 \\ \cos\theta\sin\phi\, e^1 + \cos\phi\, e^2 \\ -\sin\theta\, e^1 \end{bmatrix} . \end{aligned}$$

Finally, for the dual two-forms $*\tilde{Q}$ we have

$$*^{[04]}\tilde{Q} = \left(\frac{1-z^2}{2\Omega}\right)^3 \sin\theta\, d\theta \wedge d\phi \,, \tag{4.14}$$

$$*^{[jk]}\tilde{Q} = \frac{1+z^2}{2\Omega^3}\, d\bar{t} \wedge \left[z n^l\, dz - \frac{1-z^4}{4}\, dn^l \right] \epsilon_{jkl} \,, \tag{4.15}$$

$$\begin{aligned} *^{[0i]}\tilde{Q} = \frac{1-z^2}{2\Omega^3} \Bigg[&\left(\frac{1-z^4}{4} \cos\bar{t}\, d\bar{t} + z\sin\bar{t}\, dz \right) \wedge n^j\, dn^k \\ &- \frac{1-z^4}{8} \sin\bar{t}\, dn^j \wedge dn^k \Bigg] \epsilon_{ijk} \,, \end{aligned} \tag{4.16}$$

$$\begin{aligned} *^{[4i]}\tilde{Q} = \frac{1-z^2}{2\Omega^3} \Bigg[&z\cos\bar{t}\, n^j\, dn^k \wedge dz - \sin\bar{t} \left(\frac{1-z^4}{4} \right) n^j\, dn^k \wedge d\bar{t} \\ &+ \cos\bar{t} \left(\frac{1-z^4}{8} \right) dn^j \wedge dn^k \Bigg] \epsilon_{ijk} \,, \end{aligned} \tag{4.17}$$

where

$$\epsilon_{ijk} := \begin{cases} +1 & \text{if } ijk \text{ is an even permutation of } 1,2,3 \\ -1 & \text{if } ijk \text{ is an odd permutation of } 1,2,3 \\ 0 & \text{in any other cases}. \end{cases}$$

According to Theorem 2 for conformally rescaled metric $g_{\rm AdS}$ we get conformally related CYK tensors $Q := \Omega^{-3}\tilde{Q}$. Their boundary values at conformal infinity $\mathscr{I} := \{z = 0\}$ take the following form:

$$\begin{aligned}
{}^{[04]}Q\big|_{z=0} &= \tfrac{1}{4}\, d\bar{t} \wedge dz\,, && (4.18)\\
{}^{[jk]}Q\big|_{z=0} &= \tfrac{1}{4}(n^j\, dn^k - n^k\, dn^j) \wedge dz\,, && (4.19)\\
{}^{[0k]}Q\big|_{z=0} &= \tfrac{1}{4}\left(\cos\bar{t}\, dn^k \wedge dz + n^k \sin\bar{t}\, d\bar{t} \wedge dz\right)\,, && (4.20)\\
{}^{[4k]}Q\big|_{z=0} &= \tfrac{1}{4}\left(\sin\bar{t}\, dn^k \wedge dz - n^k \cos\bar{t}\, d\bar{t} \wedge dz\right)\,. && (4.21)
\end{aligned}$$

In Section 6, when we define charges associated with CYK tensors, it will be clear that (4.18) corresponds to the total energy and (4.19) to the angular momentum. From this point of view CYK tensors (4.20)–(4.21) correspond to the linear momentum and static moment. Similarly, for dual conformally related CYK tensors $*Q := \Omega^{-3} * \tilde{Q}$ we obtain the following boundary values at conformal infinity:

$$\begin{aligned}
*{}^{[04]}Q\big|_{z=0} &= \tfrac{1}{8}\sin\theta\, d\theta \wedge d\phi\,, && (4.22)\\
*{}^{[jk]}Q\big|_{z=0} &= \tfrac{1}{8}\epsilon_{jki}\, dn^i \wedge d\bar{t}\,, && (4.23)\\
*{}^{[0i]}Q\big|_{z=0} &= \tfrac{1}{8}\epsilon_{ijk}\left[\cos\bar{t}\, d\bar{t} \wedge n^j\, dn^k - \tfrac{1}{2}\sin\bar{t}\, dn^j \wedge dn^k\right]\,, && (4.24)\\
*{}^{[4i]}Q\big|_{z=0} &= \tfrac{1}{8}\epsilon_{ijk}\left[\tfrac{1}{2}\cos\bar{t}\, dn^j \wedge dn^k - \sin\bar{t}\, n^j\, dn^k \wedge d\bar{t}\right]\,. && (4.25)
\end{aligned}$$

Let us notice that the "rotated in time" boundary values for $*{}^{[0i]}Q$, $*{}^{[4i]}Q$

$$\begin{aligned}
\left(*{}^{[0i]}Q\cos\bar{t} + *{}^{[4i]}Q\sin\bar{t}\right)\big|_{z=0} &= \tfrac{1}{8}\epsilon_{ijk}\, d\bar{t} \wedge n^j\, dn^k\,, && (4.26)\\
\left(*{}^{[4i]}Q\cos\bar{t} - *{}^{[0i]}Q\sin\bar{t}\right)\big|_{z=0} &= \tfrac{1}{16}\epsilon_{ijk}\, dn^j \wedge dn^k && (4.27)
\end{aligned}$$

and, respectively, for ${}^{[4i]}Q$, ${}^{[0i]}Q$

$$\begin{aligned}
\left({}^{[0k]}Q\cos\bar{t} + {}^{[4k]}Q\sin\bar{t}\right)\big|_{z=0} &= \tfrac{1}{4}\, dn^k \wedge dz\,, && (4.28)\\
\left({}^{[4k]}Q\cos\bar{t} - {}^{[0k]}Q\sin\bar{t}\right)\big|_{z=0} &= \tfrac{1}{4}n^k\, dz \wedge d\bar{t} && (4.29)
\end{aligned}$$

significantly simplify.

We denote by (z^M) the coordinates on a unit sphere ($M = 1, 2$, $z^1 = \theta$, $z^2 = \phi$) and by γ_{MN} the round metric on a unit sphere:

$$d\Omega_2 = \gamma_{MN}\, dz^M\, dz^N = d\theta^2 + \sin^2\theta\, d\phi^2\,.$$

Let us also denote by ε^{MN} a two-dimensional skew-symmetric tensor on S^2 such that $\sin\theta\varepsilon^{\theta\phi} = 1$. Boundary values for Killing vector fields L_{AB} at $\mathscr{I}$ are:

$$L_{40}\big|_{z=0} = \frac{\partial}{\partial \bar t}\,, \tag{4.30}$$

$$L_{jk}\big|_{z=0} = \epsilon_{jkl}\varepsilon^{NM} n^l_{,M}\frac{\partial}{\partial z^N}\,, \quad L_{12}\big|_{z=0} = \frac{\partial}{\partial\phi}\,, \tag{4.31}$$

$$L_{i0}\big|_{z=0} = \cos\bar t\gamma^{-1}(\,dn^i) - \sin\bar t n^i\frac{\partial}{\partial \bar t}\,, \tag{4.32}$$

$$L_{i4}\big|_{z=0} = \sin\bar t\gamma^{-1}(\,dn^i) + \cos\bar t n^i\frac{\partial}{\partial \bar t}\,. \tag{4.33}$$

Together with (4.18)–(4.21) and (4.4) they lead to the following universal formula:

$$^{[AB]}Q = \overset{(0)}{h}(L^{AB}) \wedge\, dz\,, \tag{4.34}$$

where $L^{AB} := \eta^{AC}\eta^{BD}L_{CD}$. Similarly,

$$*^{[AB]}Q = L^{AB}\rfloor\mathrm{vol}(\overset{(0)}{h})\,, \tag{4.35}$$

where $\mathrm{vol}(\overset{(0)}{h}) := \sqrt{-\det\overset{(0)}{h}}\, d\bar t\wedge\, d\theta\wedge\, d\phi$ is a canonical volume three-form on $\mathscr{I}$.

Moreover,

$$\begin{aligned} *^{[34]}\tilde Q \;=\; & \Omega^{-3}\Bigg[\sin\bar t\sin^2\theta\left(\frac{1+z^2}{2}\right)\left(\frac{1-z^2}{2}\right)^2 d\phi\wedge\, d\bar t \\ & -\,\cos\bar t\cos\theta\sin\theta\left(\frac{1+z^2}{2}\right)\left(\frac{1-z^2}{2}\right)^2 d\theta\wedge\, d\phi \\ & - z\cos\bar t\sin^2\theta\left(\frac{1-z^2}{2}\right) d\phi\wedge\, dz\Bigg] \end{aligned} \tag{4.36}$$

$$\begin{aligned} \;=\; & \Omega^{-3}\Bigg[\sin\bar t\left(\frac{1+z^2}{2}\right)\left(\frac{1-z^2}{2}\right)^2 (n^1\, dn^2 - n^2\, dn^1)\wedge\, d\bar t \\ & -\,\cos\bar t\left(\frac{1+z^2}{2}\right)\left(\frac{1-z^2}{2}\right)^2 dn^1\wedge\, dn^2 \\ & - z\cos\bar t\left(\frac{1-z^2}{2}\right)(n^1\, dn^2 - n^2\, dn^1)\wedge\, dz\Bigg] \end{aligned} \tag{4.37}$$

and

$$
\begin{aligned}
{}^{[12]}\tilde{Q} \;=\; & \Omega^{-3}\left[z\left(\frac{1-z^2}{2}\right)^2 dn^1 \wedge dn^2\right. \\
& \left. + \left(\frac{1-z^2}{2}\right)\left(\frac{1+z^2}{2}\right)(n^1\, dn^2 - n^2\, dn^1) \wedge dz\right] . \qquad (4.38)
\end{aligned}
$$

Formula (3.31) suggests that CYK tensor ${}^{[12]}Q - *{}^{[34]}Q$ should correspond to $\widetilde{\mathcal{L}}_{12}$ in Minkowski space-time hence the third (z-th) component of angular momentum may correspond to ${}^{[12]}Q - *{}^{[34]}Q$ instead of ${}^{[12]}Q$ (see Section 6).

4.1. Examples

Schwarzschild–AdS solution (*cf.* [22]):

$$
ds^2 = -\left(\frac{r^2}{l^2} + 1 - \frac{2m}{r}\right) dt^2 + \left(\frac{r^2}{l^2} + 1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2\, d\Omega_2 \qquad (4.39)
$$

may be transformed into the canonical form (4.2) with the help of the coordinate z defined by the following elliptic integral:

$$
z = \exp\left(\int \frac{dw}{w\sqrt{1+w^2-bw^3}}\right) ,
$$

where $b := 2m/l$ and $w := l/r$. For the function F implicitly defined by the following conditions:

$$
\begin{aligned}
F(b,0) \;&=\; 0\,, \qquad F(0,w) = -\frac{w}{1+w^2} + \operatorname{arsinh} w\,, \\
\frac{\partial F}{\partial w} \;&=\; \frac{w^2}{(\sqrt{1+w^2}+\sqrt{1+w^2-bw^3})\sqrt{1+w^2}\sqrt{1+w^2-bw^3}}\,,
\end{aligned}
$$

we have

$$
z = \frac{w}{1+\sqrt{1+w^2}} \exp\Big(bF(b,w)\Big) . \qquad (4.40)
$$

Let us change a temporal coordinate in M to $\bar{t} = t/l$. On surface S the three-metric h can be expressed as follows:

$$
h = \left(\frac{\exp(bF)}{1+\sqrt{1+w^2}}\right)^2 \left[d\Omega_2 - \left(1+w^2-bw^3\right) d\bar{t}^2\right]
$$

with the components given only in an implicit form[6]. Moreover, the asymptotics of F:

$$
F(b,w) = \frac{1}{6}w^3 - \frac{3}{20}w^5 + \frac{b}{16}w^6 + O(w^7)
$$

[6] In order to have it explicitly we should express variable w in terms of z, *i.e.* we have to find the inverse function $w(z)$ for $z(w)$ given by (4.40).

enable one to derive the asymptotic form (4.3) for the three-metric h_{ab} in the Schwarzschild–AdS space-time. More precisely,

$$h = \mathrm{e}^{2bF} \left\{ \left(\frac{1 - z^2 \mathrm{e}^{-2bF}}{2} \right)^2 d\Omega_2 - \left[\left(\frac{1 + z^2 \mathrm{e}^{-2bF}}{2} \right)^2 - \frac{2bz^3 \mathrm{e}^{-3bF}}{1 - z^2 \mathrm{e}^{-2bF}} \right] d\bar{t}^2 \right\}$$

and

$$F = \frac{4}{3} z^3 \Big(1 + O(z^2) \Big) ,$$

give

$$\overset{(0)}{h} = \frac{1}{4} (\, d\Omega_2 - \, d\bar{t}^2) \, , \tag{4.41}$$

$$\overset{(2)}{h} = - \frac{1}{2} (\, d\Omega_2 + \, d\bar{t}^2) \, , \tag{4.42}$$

$$\chi = \frac{4m}{3 \, l} (\, d\Omega_2 + 2 \, d\bar{t}^2) \, . \tag{4.43}$$

The solution of Einstein equations with mass, angular momentum and negative cosmological constant is explicitly given by

$$\tilde{g}_{\mathrm{Kerr-AdS}} = \tilde{g}_{tt} \, dt^2 + 2 \tilde{g}_{t\phi} \, dt \, d\phi + \tilde{g}_{rr} \, dr^2 + \tilde{g}_{\theta\theta} \, d\theta^2 + \tilde{g}_{\phi\phi} \, d\phi^2 \, , \tag{4.44}$$

where

$$\tilde{g}_{tt} = -1 + \frac{2mr}{\rho^2} - \frac{r^2 + a^2 \sin^2 \theta}{l^2} \, , \quad \tilde{g}_{t\phi} = -a \sin^2 \theta \left(\frac{2mr}{\rho^2} - \frac{r^2 + a^2}{l^2} \right) ,$$

$$\tilde{g}_{rr} = \frac{\rho^2}{\triangle + (r^2 + a^2) \frac{r^2}{l^2}} \, , \qquad \tilde{g}_{\theta\theta} = \frac{\rho^2}{1 - \frac{a^2 \cos^2 \theta}{l^2}} \, ,$$

$$\tilde{g}_{\phi\phi} = \sin^2 \theta \left[(r^2 + a^2) \left(1 - \frac{a^2}{l^2} \right) + \frac{2mra^2 \sin^2 \theta}{\rho^2} \right] , \tag{4.45}$$

with ρ^2 and $\triangle$ defined as follows:

$$\rho^2 := r^2 + a^2 \cos^2 \theta \qquad \text{and} \qquad \triangle := r^2 - 2mr + a^2 \, . \tag{4.46}$$

Asymptotic behavior of h_{ab} for Kerr–AdS is analyzed in Appendix A and gives the following result:

$$
\begin{aligned}
\overset{(0)}{h} &= \frac{1}{4}\left(\bar a \sin\bar\theta + \sqrt{1-\bar a^2\cos^2\bar\theta}\right)^2 \\
&\quad \times \left[\frac{1}{1-\bar a^2\cos^2\bar\theta}\, d\bar\theta^2 + 2\bar a\sin^2\bar\theta\, d\bar t\, d\phi + \sin^2\bar\theta(1-\bar a^2)\, d\phi^2 - d\bar t^2\right], \qquad (4.47)\\
\overset{(0)}{h} &= \frac{1-\bar a^2}{4\,(1-\bar a\sin\Theta)^2}\left[d\Theta^2 + \sin^2\Theta\, d\Phi^2 - d\bar t^2\right], \qquad (4.48)\\
\chi &= \frac{b\omega}{3}\Big\{2\, d\bar t^2 - 4\bar a\sin^2\bar\theta\, d\bar t\, d\phi + \left(1-\bar a^2+3\bar a^2\sin^2\bar\theta\right)\sin^2\bar\theta\, d\phi^2 \\
&\quad + \frac{d\bar\theta^2}{1-\bar a^2\cos^2\bar\theta}\Big\}, \qquad (4.49)
\end{aligned}
$$

where the canonical coordinate systems $(\bar t, \bar\theta, \phi, z)$ is precisely defined in Appendix A. Moreover, another system of coordinates $(\bar t, \Theta, \Phi, R)$ given by (A.19)–(A.21) enables one to check explicitly conformal flatness of $\overset{(0)}{h}$ (see formula (4.48) above). Parameters $\bar a$, b are the rescaled constants a and m, respectively, $\bar a := a/l$, $b := 2m/l$.

According to [19], the Plebański–Demiański metric for the case of black-hole space-times becomes

$$
\begin{aligned}
ds^2 &= \frac{1}{\Omega^2}\Bigg\{\frac{Q}{\rho^2}\left[dt - \left(a\sin^2\theta + 4\boldsymbol{l}\sin^2\tfrac{\theta}{2}\right)d\phi\right]^2 - \frac{\rho^2}{Q}\, dr^2 \\
&\quad - \frac{\tilde P}{\rho^2}\left[a\, dt - \left(r^2 + (a+\boldsymbol{l})^2\right)d\phi\right]^2 - \frac{\rho^2}{\tilde P}\sin^2\theta\, d\theta^2\Bigg\}, \qquad (4.50)
\end{aligned}
$$

where

$$
\begin{aligned}
\Omega &= 1 - \frac{\alpha}{\omega}(\boldsymbol{l} + a\cos\theta)\, r, \\
\rho^2 &= r^2 + (\boldsymbol{l} + a\cos\theta)^2, \\
\tilde P &= \sin^2\theta\,(1 - a_3\cos\theta - a_4\cos^2\theta), \\
Q &= (\omega^2 k + e^2 + g^2) - 2mr + \epsilon r^2 - 2\alpha\frac{n}{\omega}r^3 - \left(\alpha^2 k + \frac{\Lambda}{3}\right) r^4 \qquad (4.51)
\end{aligned}
$$

and

$$
\begin{aligned}
a_3 &= 2\alpha\frac{a}{\omega}m - 4\alpha^2\frac{a\,\boldsymbol{l}}{\omega^2}(\omega^2 k + e^2 + g^2) - 4\frac{\Lambda}{3}a\,\boldsymbol{l}, \\
a_4 &= -\alpha^2\frac{a^2}{\omega^2}(\omega^2 k + e^2 + g^2) - \frac{\Lambda}{3}a^2, \qquad (4.52)
\end{aligned}
$$

with ϵ, n and k given by the following formulae:

$$\epsilon = \frac{\omega^2 k}{a^2 - \boldsymbol{l}^2} + 4\alpha\frac{\boldsymbol{l}}{\omega}\, m - (a^2 + 3\boldsymbol{l}^2)\left[\frac{\alpha^2}{\omega^2}(\omega^2 k + e^2 + g^2) + \frac{\Lambda}{3}\right], \tag{4.53}$$

$$n = \frac{\omega^2 k\boldsymbol{l}}{a^2 - \boldsymbol{l}^2} - \alpha\frac{(a^2 - \boldsymbol{l}^2)}{\omega}\, m + (a^2 - \boldsymbol{l}^2)\boldsymbol{l}\left[\frac{\alpha^2}{\omega^2}(\omega^2 k + e^2 + g^2) + \frac{\Lambda}{3}\right], \tag{4.54}$$

$$\left(\frac{\omega^2}{a^2 - \boldsymbol{l}^2} + 3\alpha^2\boldsymbol{l}^2\right) k = 1 + 2\alpha\frac{\boldsymbol{l}}{\omega}\, m - 3\alpha^2\frac{\boldsymbol{l}^2}{\omega^2}(e^2 + g^2) - \boldsymbol{l}^2\Lambda\,. \tag{4.55}$$

It is also assumed that $|a_3|$ and $|a_4|$ are sufficiently small that $\tilde{P}$ has no additional roots with $\theta \in [0, \pi]$. This solution contains eight arbitrary parameters m, e, g, a, $\boldsymbol{l}$, α, Λ and ω. Of these, the first seven can be varied independently, and ω can be set to any convenient value if a or $\boldsymbol{l}$ are not both zero.

When $\alpha = 0$, (4.55) becomes $\omega^2 k = (1 - \boldsymbol{l}^2\Lambda)(a^2 - \boldsymbol{l}^2)$ and hence (4.53) and (4.54) become

$$\epsilon = 1 - \left(\frac{1}{3}a^2 + 2\boldsymbol{l}^2\right)\Lambda\,, \qquad n = \boldsymbol{l} + \frac{1}{3}\left(a^2 - 4\boldsymbol{l}^2\right)\boldsymbol{l}\Lambda\,.$$

The metric is then given by (4.50) with

$$\begin{aligned}
\Omega &= 1\,,\\
\rho^2 &= r^2 + (\boldsymbol{l} + a\cos\theta)^2\,,\\
\tilde{P} &= \sin^2\theta\left(1 + \frac{4}{3}\Lambda a\boldsymbol{l}\cos\theta + \frac{1}{3}\Lambda a^2\cos^2\theta\right),\\
Q &= \left(a^2 - \boldsymbol{l}^2 + e^2 + g^2\right) - 2mr + r^2 - \Lambda\left[(a^2 - \boldsymbol{l}^2)\boldsymbol{l}^2 + \left(\frac{1}{3}a^2 + 2\boldsymbol{l}^2\right) r^2 + \frac{1}{3}r^4\right].
\end{aligned}$$

This is exactly the Kerr–Newman–NUT–de Sitter solution in the form which is regular on the half-axis $\theta = 0$. It represents a non-accelerating black hole with mass m, electric and magnetic charges e and g, a rotation parameter a and a NUT parameter $\boldsymbol{l}$ in a de Sitter or anti-de Sitter background. It reduces to known forms when $\boldsymbol{l} = 0$ or $a = 0$ or $\Lambda = 0$.

It would be nice to understand in what sense this solution is asymptotically AdS when we choose Λ negative. We start the analysis of this question with a simplest nontrivial extension of Schwarzschild–AdS, namely we assume the following form of the metric:

$$ds^2 = \frac{l^2}{w^2}(1 + \bar{l}^2 w^2)\left[d\Omega_2 + A^{-1}\, dw^2 - \frac{A}{(1 + \bar{l}^2 w^2)^2}\left(d\bar{t} - 4\bar{l}\sin^2\frac{\theta}{2}\, d\phi\right)^2\right], \tag{4.56}$$

which corresponds to the choice $\alpha = a = e = g = 0$, $\Lambda = -3/l^2$ in (4.50). Moreover,

$$A := 1 + w^2(1 + 6\bar{l}^2) - bw^3 - (1 + 3\bar{l}^2)\bar{l}^2 w^4 \,, \qquad \bar{l} := \frac{\boldsymbol{l}}{l} \,,$$

and b, w, $\bar{t}$ were defined previously ($b := 2m/l$, $w := l/r$, $\bar{t} := t/l$). The canonical coordinate z is defined by the following integral:

$$z = \exp\left(\int \frac{dw}{w}\sqrt{\frac{1+\bar{l}^2 w^2}{A(w)}}\right) \,,$$

which should be more deeply analyzed if we want to obtain χ. For $\overset{(0)}{h}$ the situation is much simpler because $\mathscr{I}$ corresponds to $z = w = 0$ and the induced metric at $\mathscr{I}$ takes the following form:

$$\overset{(0)}{h} = \frac{1}{4}\left[d\Omega_2 - \left(d\bar{t} - 4\bar{l}\sin^2\frac{\theta}{2}\,d\phi\right)^2\right] \qquad (4.57)$$

and its inverse

$$\overset{(0)}{h}{}^{ab}\partial_a\partial_b = 4\left[-\partial_{\bar{t}}^2 + \partial_\theta^2 + \frac{1}{\sin^2\theta}\left(\partial_\phi + 4\bar{l}\sin^2\frac{\theta}{2}\partial_{\bar{t}}\right)^2\right] . \qquad (4.58)$$

All the above calculations in this Section bring us closer (with the help of "successive approximation method") to the answer what is the asymptotics of h_{ab} in Kerr–NUT–AdS space-time. We know already the full asymptotics of Kerr–AdS, *i.e.* when $\boldsymbol{l} = 0$ (see Appendix A), and we derive $\overset{(0)}{h}$, $\overset{(2)}{h}$ for NUT–AdS, *i.e.* when $a = 0$ (*cf.* (4.57), (4.81) and Appendix B). This is enough to perform the analysis of asymptotic charges (see Section 6). We hope to extend our calculations for the full Kerr–Newman–NUT–AdS space-time in the future.

4.2. Conformal rescaling

The method exploits the properties of the asymptotic anti-de Sitter space-time under conformal transformations. Let us consider a metric $\tilde{g}$ related to g by a conformal rescaling:

$$\tilde{g}_{\mu\nu} = \Omega^{-2} g_{\mu\nu} \quad \Longleftrightarrow \quad \tilde{g}^{\mu\nu} = \Omega^2 g^{\mu\nu} \,. \qquad (4.59)$$

It is straightforward to derive

$$\tilde{\Gamma}^\mu{}_{\nu\kappa} \;=\; \Gamma^\mu{}_{\nu\kappa} + \delta^\mu{}_\kappa\partial_\nu U + \delta^\mu{}_\nu\partial_\kappa U - g_{\nu\kappa}\nabla^\mu U \,, \qquad (4.60)$$

where ∇ denotes the covariant derivative of g and $U := -\log \Omega$. Moreover, in the case of asymptotic AdS we can choose

$$\Omega := \frac{z}{l} \,.$$

Riemann tensor is defined as usual:

$$\tilde{R}^{\mu}{}_{\nu\alpha\beta} = -\tilde{\Gamma}^{\mu}{}_{\nu\alpha,\beta} + \tilde{\Gamma}^{\mu}{}_{\nu\beta,\alpha} + \tilde{\Gamma}^{\mu}{}_{\gamma\alpha}\tilde{\Gamma}^{\gamma}{}_{\beta\nu} - \tilde{\Gamma}^{\mu}{}_{\gamma\beta}\tilde{\Gamma}^{\gamma}{}_{\alpha\nu} \,.$$

To analyze Einstein equations we shall use the following formula (*cf.* Chapter 3.7 in [38]):

$$\tilde{R}_{\mu\nu} = R_{\mu\nu} - U^{;\alpha}{}_{;\alpha} g_{\mu\nu} + (n-2)\left(U_{;\mu}U_{;\nu} - U_{;\gamma}U^{;\gamma}g_{\mu\nu} - U_{;\mu\nu}\right) . \qquad (4.61)$$

A (3+1)-decomposition of the rescaled metric g in canonical coordinates takes a simple form:

$$g = g_{\mu\nu}\, dz^{\mu} \otimes dz^{\nu} = h_{ab}\, dz^{a} \otimes dz^{b} + dz \otimes dz \,,$$

where h is the induced metric on a tube S.

The extrinsic curvature

$$K_{ab} := \frac{1}{2}\, \partial_3 g_{ab}$$

of the surface S enables one to derive the following Christoffel symbols for the Levi–Civita connection of the metric $g_{\mu\nu}$:

$$\Gamma^{3}_{ab} = -K_{ab} \,, \qquad \Gamma^{a}_{3b} = K_{cb}h^{ca} \,.$$

Moreover, the rest of them are as follows:

$$\Gamma^{3}_{3\mu} = \Gamma^{\mu}_{33} = 0 \,, \qquad \Gamma^{a}_{bc} = \Gamma^{a}_{bc}(h) \,.$$

Ricci tensor of the four-metric g expresses in terms of initial data (h, K) on S as follows:

$$R_{33} = -\partial_3 K^{a}{}_{a} - K^{a}{}_{b}K^{b}{}_{a} \,, \qquad (4.62)$$
$$R_{ab} = \mathcal{R}_{ab}(h) - \partial_3 K_{ab} + 2K_{bc}K^{c}{}_{a} - K^{c}{}_{c}K_{ab} \,, \qquad (4.63)$$
$$R_{3a} = K^{b}{}_{a|b} - K^{b}{}_{b|a} \,, \qquad (4.64)$$

where by "|" we denote a covariant derivative with respect to the three-metric h and $\mathcal{R}_{ab}(h)$ is its Ricci tensor.

Riemann tensor (curvature of the metric g):

$$R^a{}_{bcd} = \mathcal{R}^a{}_{bcd}(h) + K_{bc}K^a{}_d - K_{bd}K^a{}_c\,, \tag{4.65}$$

$$R^3{}_{abc} = K_{ab|c} - K_{ac|b}\,, \tag{4.66}$$

$$R^3{}_{a3b} = -\partial_3 K_{ab} + K_{ac}K^c{}_b\,. \tag{4.67}$$

In particular,

$$R^0{}_{303} = -\partial_3 K^0{}_0 - K^0{}_a K^a{}_0\,.$$

The conformal Weyl tensor is defined as follows (*cf.* [15])

$$W^\mu{}_{\nu\lambda\rho} = R^\mu{}_{\nu\lambda\rho} + g_{\nu[\lambda}S_{\rho]}{}^\mu - \delta^\mu{}_{[\lambda}S_{\rho]\nu}\,, \tag{4.68}$$

where

$$S_{\mu\nu} = R_{\mu\nu} - \frac{1}{6}Rg_{\mu\nu}\,.$$

Gauss–Codazzi–Ricci equations (4.65)–(4.67) imply:

$$\begin{aligned} R_{acdb}h^{cd} &= -\mathcal{R}_{ab}(h) + K^c{}_c K_{ab} - K_{bc}K^c{}_a\,, \\ \varepsilon_a{}^{cd}R_{3bcd} &= \varepsilon_a{}^{cd}\left(K_{bd|c} - K_{bc|d}\right) = -2\varepsilon_a{}^{cd}K_{bc|d}\,, \end{aligned}$$

where ε^{abc} is the Levi–Civita antisymmetric tensor for the three-metric h such that $\sqrt{-\det h}\,\varepsilon^{012} = 1$.

Components of the Weyl tensor (with respect to initial data on S) one can nicely describe in terms of traceless symmetric tensors E and B, electric and magnetic parts of W, respectively. The electric part can be derived as follows:

$$E_{ab} := W_{a3b3} = W_{acdb}h^{cd} = R_{acdb}h^{cd} + \left(h_{c[d}S_{b]}{}^a - h_{a[d}S_{b]c}\right)h^{cd} \tag{4.69}$$

$$= -\mathcal{R}_{ab}(h) + K^c{}_c K_{ab} - K_{bc}K^c{}_a + \frac{1}{2}S_{ab} + \frac{1}{2}h_{ab}S_{cd}h^{cd}\,, \tag{4.70}$$

where

$$\begin{aligned} S_{ab} &= \mathcal{R}_{ab}(h) - \frac{1}{6}\mathcal{R}(h)h_{ab} + \frac{1}{6}h_{ab}\left(K^2 + 2\partial_3 K\right) \\ &\quad + 2K_{bc}K^c{}_a - \partial_3 K_{ab} - KK_{ab} \end{aligned} \tag{4.71}$$

as a consequence of (4.62)–(4.64) and $K := K^c{}_c$.

The magnetic part of W takes the following form:

$$B_{ab} := {}^*W_{a3b3} = -{}^*W_{a33b} \tag{4.72}$$

$$= -\frac{1}{2}\varepsilon_a{}^{cd}R_{3bcd} + \varepsilon_a{}^{cd}g_{c[d}S_{b]3} \tag{4.73}$$

$$= \varepsilon_a{}^{cd}K_{bc|d} + \frac{1}{2}\varepsilon_a{}^{cd}h_{bc}S_{d3} \tag{4.74}$$

$$= \frac{1}{2}\varepsilon_a{}^{cd}K_{bc|d} + \frac{1}{2}\varepsilon_b{}^{cd}K_{ac|d}\,, \tag{4.75}$$

where $S_{3a} = R_{3a}$ is given by (4.64), and the magnetic part (4.72) we derived as follows:

$$\begin{aligned} 2^*W_{a33b} &= \varepsilon_{a3\mu\nu}W^{\mu\nu}{}_{3b} = \varepsilon_{acd3}W^{cd}{}_{3b} = \varepsilon_{acd}W^{cd}{}_{3b} = \varepsilon_a{}^{cd}W_{3bcd} \\ &= \varepsilon_a{}^{cd}R_{3bcd} + \varepsilon_a{}^{cd}g_{b[c}S_{d]3} - g_{3[c}S_{d]b}\,. \end{aligned} \tag{4.76}$$

The Einstein equations with cosmological constant

$$\Lambda = -\frac{3}{l^2}$$

may be rewritten with the help of the conformal transformation (4.61) in the following form:

$$0 = \tilde{R}_{33} - \Lambda\tilde{g}_{33} = R_{33} + \frac{1}{z}g_{ab}K^c{}_c\,, \tag{4.77}$$

$$0 = \tilde{R}_{ab} - \Lambda\tilde{g}_{ab} = R_{ab} + \frac{2}{z}K_{ab} + \frac{1}{z}g_{ab}K^c{}_c\,, \tag{4.78}$$

$$0 = \tilde{R}_{3a} - \Lambda\tilde{g}_{3a} = R_{3a} = K^b{}_{a|b} - K^b{}_{b|a}\,, \tag{4.79}$$

where the last equality follows from (4.64). Equation (4.79) is the usual vector constraint. The scalar one is hidden in the term $R_{ab}h^{ab} - R_{33}$. More precisely, from (4.62), (4.63), (4.77) and (4.78) we obtain

$$\left[\mathcal{R}_{ab}(h) + K_{bc}K^c{}_a - K^c{}_cK_{ab}\right]h^{ab} + \frac{4}{z}K^c{}_c = 0\,.$$

The equations (4.78) imply the standard asymptotics (4.3) for the three-metric h. Moreover, for the extrinsic curvature K_{ab} we get:

$$K_{ab} = z\left(\overset{(2)}{h} + 3z\chi\right) + O(z^3)\,, \tag{4.80}$$

where

$$\overset{(2)}{h}{}_{ab} = \frac{1}{4}\overset{(0)}{h}{}_{ab}\mathcal{R}\left(\overset{(0)}{h}\right) - \mathcal{R}_{ab}\left(\overset{(0)}{h}\right) \tag{4.81}$$

(*cf.* [34]). In addition, equation (4.62) together with (4.77) rewritten in the form

$$\partial_3K^c{}_c - \frac{1}{z}K^c{}_c + K^a{}_bK^b{}_a = 0$$

imply

$$\chi_{ab}\overset{(0)}{h}{}^{ab} = 0\,. \tag{4.82}$$

Let us also notice that the leading order term in the vector constraint

$$K^b{}_{a|b} - K^b{}_{b|a} = 0$$

corresponding to $\overset{(2)}{h}$ in (4.80) is equivalent to the contracted Bianchi identity

$$\mathcal{R}^b{}_{a|b} - \frac{1}{2}\mathcal{R}^b{}_{b|a} = 0\,.$$

However, the next order term gives for χ the following constraint:

$$\nabla_b(\overset{(0)}{h})\chi^b{}_a = 0\,. \tag{4.83}$$

Equations (4.82) and (4.83) express the fact that the tensor χ which is not determined by asymptotic analysis is transverse traceless with respect to the metric $\overset{(0)}{h}$.

Finally,

$$E_{ab} = -3z\chi_{ab} + O(z^2)\,, \tag{4.84}$$

$$B_{ab} = \frac{z}{2}\left(\varepsilon_a{}^{cd}\overset{(2)}{h}{}_{bc|d} + \varepsilon_b{}^{cd}\overset{(2)}{h}{}_{ac|d}\right) + O(z^2) \tag{4.85}$$

$$= -\frac{z}{2}\left(\varepsilon_a{}^{cd}\mathcal{R}_{bc|d} + \varepsilon_b{}^{cd}\mathcal{R}_{ac|d}\right) + O(z^2)\,. \tag{4.86}$$

Let us denote

$$\beta_{ab} := \varepsilon_a{}^{cd}\mathcal{R}_{bc|d} + \varepsilon_b{}^{cd}\mathcal{R}_{ac|d} \tag{4.87}$$

the leading order term in B_{ab} which plays a similar role to the tensor χ in E_{ab}. Symmetric tensor β is equivalent to the Cotton tensor[7]

$$C_{abc} := \mathcal{R}_{ab|c} - \mathcal{R}_{ac|b} + \frac{1}{4}\left(h_{ac}\mathcal{R}_{|b} - h_{ab}\mathcal{R}_{|c}\right)\,, \tag{4.88}$$

via the following relation between them:

$$\beta_{ab} = \varepsilon_a{}^{cd}C_{bcd} = 2\varepsilon_a{}^{cd}\mathcal{R}_{bc|d} - \frac{1}{2}\varepsilon_{ab}{}^c\mathcal{R}_{|c}$$

which implies that for the usual asymptotic AdS space-time, *i.e.* when metric $\overset{(0)}{h}$ is conformally flat and its Cotton tensor vanishes, the tensor β has to vanish as well.

In general case, for non-trivial $\overset{(0)}{h}$, contracted Bianchi identities for Ricci tensor $\mathcal{R}_{ab}$ imply that the tensor β has the same properties as χ, *i.e.* is transverse traceless.

A generalization of some results presented in this Section to higher dimension of the space-time can be found in [5, 21].

[7] Three-dimensional counterpart of the Weyl tensor.

5. Symplectic structure on tube

In [29] the following theorem was proved:

Theorem 3. *Field dynamics in a four-dimensional region O is equivalent to*

$$\delta \int_{\mathcal{O}} L = -\frac{1}{16\pi} \int_{\partial\mathcal{O}} g_{kl}\, \delta \Pi^{kl} \,, \tag{5.1}$$

where g_{kl} is the three-dimensional metric induced on the boundary $\partial\mathcal{O}$ by $g_{\mu\nu}$ and Π is the extrinsic curvature (in A.D.M. densitized form) of $\partial\mathcal{O}$.

This theorem shows the universality of the symplectic structure:

$$\int_{\partial\mathcal{O}} \delta \Pi^{kl} \wedge \delta g_{kl} \,.$$

In our case a boundary data on S consists of the three-metric $\tilde{h}_{ab}$ and canonical A.D.M. momentum $\tilde{Q}^{ab}$ which is related with extrinsic curvature $\tilde{K}_{ab}$ in the usual way:

$$\tilde{Q}_{ab} = \sqrt{-\tilde{h}} \left(\tilde{K}_{ab} - \tilde{h}^{cd} \tilde{K}_{cd} \tilde{h}_{ab} \right) \,.$$

Conformal rescaling of the three-metric and extrinsic curvature

$$\tilde{h}_{ab} = \Omega^{-2} 2 h_{ab} \,, \qquad \tilde{K}_{ab} = -\frac{\tilde{\Gamma}^3_{ab}}{\sqrt{\tilde{h}^{33}}} = \Omega^{-1} \left(K_{ab} - \frac{1}{z} h_{ab} \right)$$

enables one to analyze the symplectic structure as follows:

$$\begin{aligned} \tilde{h}_{ab} \delta \tilde{Q}^{ab} &= \delta \left(\tilde{h}_{ab} \tilde{Q}^{ab} \right) - \tilde{Q}^{ab} \delta \tilde{h}_{ab} \\ &= \delta \left(\tilde{h}^{ab} \tilde{Q}_{ab} \right) + \tilde{Q}_{ab} \delta \tilde{h}^{ab} \,. \end{aligned} \tag{5.2}$$

In particular, (5.2) implies

$$\int_S \delta \tilde{h}_{ab} \wedge \delta \tilde{Q}^{ab} = \int_S \delta \tilde{Q}_{ab} \wedge \delta \tilde{h}^{ab} \,.$$

Moreover,

$$\tilde{Q}_{ab} \delta \tilde{h}^{ab} = \Omega^{-2} \sqrt{-h} \left(K_{ab} - \frac{2}{z} h_{ab} - K^c{}_c h_{ab} \right) \delta h^{ab} \tag{5.3}$$

$$= 4 \Omega^{-3} \frac{1}{l} \delta \sqrt{-h} + \Omega^{-2} \sqrt{-h} \left(K_{ab} - h^{cd} K_{cd} h_{ab} \right) \delta h^{ab} \,. \tag{5.4}$$

With the help of standard variational identities:

$$\delta\sqrt{-h} \;=\; \frac{1}{2}\sqrt{-h}h^{ab}\delta h_{ab}\,, \tag{5.5}$$

$$\delta R_{ab}(h) \;=\; \delta\Gamma^{c}{}_{ab|c} - \delta\Gamma^{c}{}_{ac|b}\,, \tag{5.6}$$

$$\begin{aligned}\delta\left(\sqrt{-h}R(h)\right) \;&=\; \sqrt{-h}\left(R_{ab}-\frac{1}{2}h_{ab}R\right)\delta h^{ab}\\ &\quad+\partial_c\left[\sqrt{-h}\left(h^{ab}\delta\Gamma^{c}{}_{ab}-h^{ac}\delta\Gamma^{b}{}_{ab}\right)\right]\,,\end{aligned} \tag{5.7}$$

we analyze the singular part of (5.3) as follows:

$$\begin{aligned}\mathrm{sing}(\tilde{Q}_{ab}\delta\tilde{h}^{ab}) \;&=\; \delta\left(4\Omega^{-3}\frac{1}{l}\sqrt{-h}\right)+\sqrt{-h}\Omega^{-2}z\left(\mathcal{R}_{ab}-\frac{1}{2}h_{ab}h^{cd}\mathcal{R}_{cd}\right)\delta h^{ab}\\ &=\; \frac{1}{l}\delta\left\{4\Omega^{-3}\sqrt{-h}+\Omega^{-2}\sqrt{-h}\mathcal{R}\right\}+\text{full divergence}\,,\end{aligned} \tag{5.8}$$

which is a full variation up to boundary terms. Finally

$$\begin{aligned}\lim_{\epsilon\to 0}\int_{S_\epsilon}\delta\tilde{Q}_{ab}\wedge\delta\tilde{h}^{ab} \;&=\; 3l^2\int_{\mathscr{I}}\delta\left[\sqrt{-\det\overset{(0)}{h}}\left(\chi_{ab}-\overset{(0)}{h}{}_{ab}\chi^{c}{}_{c}\right)\right]\wedge\delta\overset{(0)}{h}{}^{ab}\\ &=\; 3l^2\int_{\mathscr{I}}\delta\left[\sqrt{-\det\overset{(0)}{h}}\left(\chi^{c}{}_{c}\overset{(0)}{h}{}^{ab}-\chi^{ab}\right)\right]\wedge\delta\overset{(0)}{h}{}_{ab}\,,\end{aligned} \tag{5.9}$$

where $S_\epsilon := \{z = x^3 = \epsilon\}$ is a tube close to infinity. Symplectic structure (5.9) on conformal boundary consists of the metric $\overset{(0)}{h}{}_{ab}$ and canonically conjugated momenta $\pi^{ab} := 3l^2\sqrt{-\det\overset{(0)}{h}}\left(\chi^{c}{}_{c}\overset{(0)}{h}{}^{ab}-\chi^{ab}\right)$ in A.D.M. densitized form.

6. Asymptotic charges

Let $\tilde{F}_{\mu\nu} := \tilde{W}_{\mu\nu\rho\sigma}\tilde{Q}^{\rho\sigma}$ and $F_{\mu\nu} := W_{\mu\nu\rho\sigma}Q^{\rho\sigma}$, respectively. The conformal rescaling (4.59) and Theorem 2 (see [24]) imply a simple relation between $\tilde{F}$ and F:

$$\tilde{F}_{\mu\nu} = \Omega^{-1}F_{\mu\nu}\,. \tag{6.1}$$

According to [28], Hodge dual of the two-form $\tilde{F}$ represented by a bivector density defines an integral quantity at $\mathscr{I}$ as follows:

$$I(C) := \lim_{z\to 0^+}\int_{C_z}\tilde{F}^{\mu\nu}\,d\tilde{S}_{\mu\nu} = \int_{C}\left(\lim_{z\to 0^+}\Omega^{-1}F^{\mu\nu}\,dS_{\mu\nu}\right)\,,$$

where C_z is a family of spheres approaching C, sphere at infinity (cut of $\mathscr{I}$),

$$
\begin{aligned}
d\tilde{S}_{\mu\nu} &:= \sqrt{-\det \tilde{g}}\partial_\mu \wedge \partial_\nu \rfloor dz^0 \wedge \ldots \wedge dz^3 \,, \\
dS_{\mu\nu} &:= \sqrt{-\det g}\partial_\mu \wedge \partial_\nu \rfloor dz^0 \wedge \ldots \wedge dz^3 \,.
\end{aligned}
$$

Let us consider asymptotic CYK tensor as a two-form $\tilde{Q}$ such that a boundary value at $\mathscr{I}$ of the corresponding rescaled tensor Q is a linear combination of (4.34) and (4.35), *i.e.* its boundary value is the same as in the case of pure AdS space-time (4.1).

For asymptotic AdS space-time formulae (4.84) and (4.86) imply that $\lim_{z\to 0^+} \Omega^{-1}W$ is finite. Moreover, for a given ACYK tensor $\tilde{Q}$ we obtain the well defined expression $I(C)$ which depends only on asymptotic values at $\mathscr{I}$. Let us check that for a given value Q at $\mathscr{I}$ the quantity $I(C)$ does not depend on the choice of cut C, *i.e.* represents a conserved quantity. We have

$$
\int\limits_C \left(\lim_{z\to 0^+} \Omega^{-1} F^{\mu\nu}\, dS_{\mu\nu} \right) = \int\limits_C \left(\lim_{z\to 0^+} \Omega^{-1} F^{3a}\, dS_a \right) ,
$$

where $dS_a := \partial_a \rfloor \mathrm{vol}(\overset{(0)}{h})$. Moreover, for $Q(L) = \overset{(0)}{h}(L) \wedge dz$ (*cf.* (4.34))

$$
\lim_{z\to 0^+} \Omega^{-1} F^{3a} = \lim_{z\to 0^+} \Omega^{-1} W^{3a\mu\nu} Q_{\mu\nu} = 6l\chi^a{}_b L^b \,.
$$

Similarly, for $*Q(L) = L\rfloor \mathrm{vol}(\overset{(0)}{h})$ (*cf.* (4.35)) we get

$$
\lim_{z\to 0^+} \Omega^{-1} F^{3a} = \lim_{z\to 0^+} \Omega^{-1} W^{3a\mu\nu} *Q_{\mu\nu} = l\beta^a{}_b L^b \,,
$$

where $\beta^a{}_b$ is given by (4.87). If $L = L^b\partial_b$ is a conformal Killing vector field for the metric $\overset{(0)}{h}$ (which is true for ACYK tensors Q) the conservation law for $I(C)$ results from transverse traceless property of tensors χ and β. More precisely, for three-volume $V \subset \mathscr{I}$ such that $\partial V = C_1 \cup C_2$ we have

$$
\begin{aligned}
\int\limits_{C_1} \chi^a{}_b L^b\, dS_a - \int\limits_{C_2} \chi^a{}_b L^b\, dS_a &= \int\limits_{\partial V} \chi^a{}_b L^b\, dS_a = \int\limits_V \nabla_a (\chi^a{}_b L^b) \mathrm{vol}(\overset{(0)}{h}) \\
&= \int\limits_V \left[L^b \nabla_a \chi^a{}_b + \chi^{ab} L_{(a|b)} \right] \mathrm{vol}(\overset{(0)}{h}) = 0 \,.
\end{aligned}
$$

Let us define the following quantity:

$$
H(Q) := \frac{l}{32\pi} \int\limits_C \Omega^{-1} F^{\mu\nu}(Q)\, dS_{\mu\nu} \,. \tag{6.2}
$$

For ACYK tensor $\tilde{Q}$ in asymptotic AdS space-time the corresponding quantity $H(Q)$ is conserved, *i.e.* does not depend on the choice of spherical cut C. In particular, for the conformal Killing vector field L and $Q(L)$ given by (4.34) the conserved charge $H(Q(L))$ may be expressed in terms of electric part of Weyl tensor and takes the following form[8] proposed by Ashtekar [4, 5] (see also [21]):

$$H(Q(L)) = -\frac{l}{16\pi} \int\limits_C \Omega^{-1} E^a{}_b L^b \, dS_a \,. \tag{6.3}$$

In the Schwarzschild–AdS space-time (4.39) for the Killing vector

$$L = \frac{\partial}{\partial t} = l^{-1} \frac{\partial}{\partial \bar{t}} = l^{-1} \partial_0 \tag{6.4}$$

definition (6.2) gives (minus) mass:

$$H(Q(L)) = -\frac{1}{16\pi} \int\limits_C \Omega^{-1} E^a{}_0 \, dS_a = \frac{3l}{16\pi} \int\limits_C \chi^0{}_0 \sqrt{-\det \overset{(0)}{h}} \, d\theta \, d\phi = -m \,. \tag{6.5}$$

The last equality in the above formula follows from (4.41) and (4.43). Obviously, the same value $-m$ we obtain for Kerr–AdS metric (4.44). Moreover, in the Kerr–AdS space-time for $L = \frac{\partial}{\partial \phi}$ we obtain the angular momentum:

$$H(Q(L)) = -\frac{l}{16\pi} \int\limits_C \Omega^{-1} E^a{}_\phi \, dS_a = \frac{3l^2}{16\pi} \int\limits_C \chi^0{}_\phi \sqrt{-\det \overset{(0)}{h}} \, d\theta \, d\phi = ma \,. \tag{6.6}$$

The details of calculations for the Kerr–AdS space-time we present in the Appendix A. Let us observe that our conserved quantity $H(Q(L))$ in terms of the symplectic momenta π^{ab} at $\mathscr{I}$ takes the following form:

$$H(Q(L)) = -\frac{1}{16\pi} \int\limits_C \pi^0{}_b L^b \, d\theta \, d\phi \,, \tag{6.7}$$

which is in the same A.D.M. form as the usual linear or angular momentum at spatial infinity in asymptotically flat space-time (*cf.* [11] p. 80).

[8] The apparent incompatibility of factors between our integral $H(Q(L))$ and Ashtekar's definition is related to a different choice of conformal factor Ω, our choice is twice smaller.

<u>Remark:</u> In general case, when $\overset{(0)}{h}$ is not conformally flat, it may happen that one obtains asymptotic charge which is no longer conserved — Bondi-like phenomena (*cf.* [28]).

The "topological" charge one can try to define as follows:

$$H(*Q(L)) = \frac{l}{32\pi} \int\limits_{S^2} \Omega^{-1} F^{\mu\nu}(*Q)\, dS_{\mu\nu} = -\frac{l}{16\pi} \int\limits_{S^2} \Omega^{-1} B^a{}_b L^b\, dS_a\,.$$

We want to stress that, in general, we can meet problems with finding spherical cuts of $\mathscr{I}$. Hence the choice of a domain of integration for the corresponding two-form $\Omega^{-1}F^{\mu\nu}(*Q)\, dS_{\mu\nu}$ has to be carefully analyzed. In NUT–AdS space-time a conformal boundary $\mathscr{I}$ equipped with the metric (4.57) is a non-trivial bundle over S^2 – two-dimensional sphere. However, for L given by (6.4), when the above formula pretends to define "dual mass" charge, we have

$$\begin{aligned}
-\Omega^{-1} B^a{}_b L^b\, dS_a &= -\frac{1}{l}\Omega^{-1} B^a{}_0\, dS_a \\
&= \frac{1}{2}\sqrt{-\det \overset{(0)}{h}} \left[\beta^0{}_0\, d\theta \wedge d\phi + d\bar{t} \wedge (\beta^\phi{}_0\, d\theta - \beta^\theta{}_0\, d\phi)\right] \\
&= 2\bar{l} \sin\theta\, d\theta \wedge d\phi\,, \qquad (6.8)
\end{aligned}$$

where the last equality one can easily check using formulae from Appendix B. Let us notice that the resulting two-form projects uniquely on the base manifold which is a two-dimensional sphere. Finally we have

$$2H(*Q(L)) = \frac{l}{16\pi} \int\limits_{S^2} 4\bar{l} \sin\theta\, d\theta\, d\phi = l\bar{l} = \boldsymbol{l}\,,$$

which confirms that we can interpret the NUT parameter $\boldsymbol{l}$ as a dual mass charge.

7. Conclusions

We have constructed all solutions to CYK equation in AdS (and de Sitter) space-time via pullback technique from five-dimensional flat ambient space.

Analyzing three important examples: Schwarzschild–AdS, Kerr–AdS and NUT–AdS, we have shown how geometrically natural two-form F (built from Weyl tensor W and ACYK tensor Q), leading to the universal definition of a global charge (6.2), enables one to understand energy, angular momentum and dual mass in asymptotic AdS space-time. Definition (6.2) occurred to be equivalent to two other important formulae (6.3) and (6.7).

The relation between Killing vector fields L and CYK tensors Q has been examined. However, the relation (3.31) between AdS and Minkowski suggests some ambiguity in the definition of angular momentum when the three-metric $\overset{(0)}{h}$ is not conformally flat. More precisely, CYK tensor ${}^{[12]}Q - *{}^{[34]}Q$ corresponds to CYK tensor $\widetilde{\mathcal{L}}_{12}$ in Minkowski hence $H({}^{[12]}Q - *{}^{[34]}Q)$ should correspond to the third component of the angular momentum. Obviously, in the standard asymptotic AdS space-time the quantity $H(*{}^{[34]}Q)$ vanishes and the ambiguity disappears.

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Appendix A

Canonical coordinates for AdS-Kerr near $\mathscr{I}$

The solution of Einstein equations with mass, angular momentum and negative cosmological constant explicitly given by (4.44) can be rewritten near $\mathscr{I}$ as follows:

$$\begin{aligned}\tilde{g}_{\mathrm{Kerr-AdS}} &= \frac{l^2}{w^2}(1+\bar{a}^2 w^2 \cos^2\theta)\left(\frac{dw^2}{1+w^2(1+\bar{a}^2-bw+\bar{a}^2 w^2)}+\frac{d\theta^2}{1-\bar{a}^2\cos^2\theta}\right)\\ &\quad + \tilde{g}_{tt}\,dt^2 + 2\tilde{g}_{t\phi}\,dt\,d\phi + \tilde{g}_{\phi\phi}\,d\phi^2\,, \end{aligned} \tag{A.1}$$

where $\bar{a} := a/l$, $b := 2m/l$, $w := l/r$ and

$$\begin{aligned}\tilde{g}_{tt} &= -\frac{1}{w^2} - 1 - \bar{a}^2\sin^2\theta + \frac{bw}{1+\bar{a}^2 w^2\cos^2\theta}\,,\\ \tilde{g}_{t\phi} &= l\bar{a}\sin^2\theta\left(\frac{1}{w^2}+\bar{a}^2-\frac{bw}{1+\bar{a}^2 w^2\cos^2\theta}\right),\\ \tilde{g}_{\phi\phi} &= l^2\sin^2\theta\left[\left(\frac{1}{w^2}+\bar{a}^2\right)(1-\bar{a}^2)+\frac{bw\bar{a}^2\sin^2\theta}{1+\bar{a}^2 w^2\cos^2\theta}\right]. \end{aligned} \tag{A.2}$$

The canonical coordinate $z(w,\theta)$ fulfilling eikonal equation:

$$\left\|\frac{l\,dz}{z}\right\|^2 = 1 \tag{A.3}$$

can be found with the help of the following conditions:

$$\left(\frac{w}{z}\frac{\partial z}{\partial w}\right)^2 [1+w^2(1+\bar{a}^2-bw+\bar{a}^2 w^2)] = 1\,, \tag{A.4}$$

$$\left(\frac{1}{z}\frac{\partial z}{\partial \theta}\right)^2 (1-\bar{a}^2\cos^2\theta) = \bar{a}^2\cos^2\theta\,. \tag{A.5}$$

It is easy to check that each solution of (A.4)–(A.5) is simultaneously a solution of (A.3) for the metric (A.1). We are looking for a solution of (A.4)–(A.5) in the following form:

$$\ln z = A(w) + B(\theta)$$

which leads to the following ODE's:

$$w\frac{dA}{dw} = \frac{1}{\sqrt{1+w^2(1+\bar{a}^2-bw+\bar{a}^2w^2)}} \tag{A.6}$$

$$\frac{dB}{d\theta} = \frac{\bar{a}\cos\theta}{\sqrt{1-\bar{a}^2\cos^2\theta}}\,. \tag{A.7}$$

The solution of equation (A.6) expresses in terms of the elliptic integral

$$A = \int \frac{dw}{w\sqrt{1+w^2(1+\bar{a}^2-bw+\bar{a}^2w^2)}}$$

but (A.7) possesses the simple solution:

$$B = \ln\left(\bar{a}\sin\theta + \sqrt{1-\bar{a}^2\cos^2\theta}\right)\,.$$

Moreover, considerations (similar to (4.40) in the case of Schwarzschild–AdS metric) lead to the following formula:

$$A = \ln\left(\frac{w}{1+\sqrt{1+w^2(1+\bar{a}^2)}}\right) + \bar{F}(\bar{a},b,w)$$

with the function $\bar{F}$ implicitly defined by the following conditions:

$$\bar{F}(\bar{a},b,0) = 0\,, \qquad \bar{F}(0,b,w) = bF(b,w)\,,$$

$$w\frac{\partial \bar{F}}{\partial w} = \frac{1}{\sqrt{1+w^2(1+\bar{a}^2-bw+\bar{a}^2w^2)}} - \frac{1}{\sqrt{1+w^2(1+\bar{a}^2)}}\,. \tag{A.8}$$

Finally, we have

$$z(w,\theta) = \frac{w}{1+\sqrt{1+w^2(1+\bar{a}^2)}}\left(\bar{a}\sin\theta + \sqrt{1-\bar{a}^2\cos^2\theta}\right)\exp[\bar{F}(\bar{a},b,w)]\,. \tag{A.9}$$

Analyzing series expansion of the right-hand side of (A.8) we can easily produce an asymptotic form of $\bar{F}$:

$$\bar{F} = \frac{b}{6}w^3 - \frac{1}{8}\bar{a}^2 w^4 - \frac{3b}{20}(1+\bar{a}^2)w^5 + \frac{1}{16}\left[2\bar{a}^2(1+\bar{a}^2)+b^2\right]w^6 + O(w^7) \quad \text{(A.10)}$$

which enables one to analyze asymptotics at $\mathscr{I}$. However, to obtain the canonical form (4.2) for the metric $\tilde{g}_{\text{Kerr-AdS}}$ we should change a coordinate θ because $\tilde{g}(dz, d\theta)$ is not vanishing. We are looking for a new coordinate $\bar{\theta}(w,\theta)$ with the following properties:

$$\bar{\theta}(0,\theta) = \theta\,, \qquad \tilde{g}(\,dz,\, d\bar{\theta}) = 0\,.$$

Orthogonality of coordinates z and $\bar{\theta}$ leads to the following condition:

$$\frac{\partial\bar{\theta}}{\partial w} + \frac{\bar{a}w\cos\theta\sqrt{1-\bar{a}^2\cos^2\theta}}{\sqrt{1+w^2(1+\bar{a}^2-bw+\bar{a}^2w^2)}}\frac{\partial\bar{\theta}}{\partial\theta} = 0$$

which implies that the curve $\bar{\theta}(w,\theta) = \text{const.}$ obeys the following ODE:

$$\frac{d\theta}{dw} = -\frac{\dfrac{\partial\bar{\theta}}{\partial w}}{\dfrac{\partial\bar{\theta}}{\partial\theta}} = \frac{\bar{a}w\cos\theta\sqrt{1-\bar{a}^2\cos^2\theta}}{\sqrt{1+w^2(1+\bar{a}^2-bw+\bar{a}^2w^2)}}\,. \quad \text{(A.11)}$$

It is easy to verify that ODE (A.11) leads again to elliptic integral. More precisely, let us define

$$\delta(\tau) := \bar{a}\int_0^\tau \frac{w\,dw}{\sqrt{1+w^2(1+\bar{a}^2-bw+\bar{a}^2w^2)}}\,,$$

then the solution of (A.11) takes the following form:

$$\begin{aligned}\delta(w) &= \operatorname{artanh}\frac{\sin\theta}{\sqrt{1-\bar{a}^2\cos^2\theta}} - \operatorname{artanh}\frac{\sin\bar{\theta}}{\sqrt{1-\bar{a}^2\cos^2\bar{\theta}}} && \text{(A.12)}\\ &= \bar{a}\left[\frac{1}{2}w^2 - \frac{1}{8}(1+\bar{a}^2)w^4 + \frac{b}{10}w^5 + O(w^6)\right]\,. && \text{(A.13)}\end{aligned}$$

Now, we are ready to calculate induced three-metric h on the surface $S = \{z = \text{const.}\}$. We have

$$0 = dA + dB = \frac{dw}{w\sqrt{1+w^2(1+\bar{a}^2-bw+\bar{a}^2w^2)}} + \frac{\bar{a}\cos\theta}{\sqrt{1-\bar{a}^2\cos^2\theta}}\,d\theta$$

and (from (A.12))

$$\frac{d\theta}{\cos\theta\sqrt{1-\bar{a}^2\cos^2\theta}} - \frac{d\bar{\theta}}{\cos\bar{\theta}\sqrt{1-\bar{a}^2\cos^2\bar{\theta}}} = \frac{\bar{a}w\,dw}{\sqrt{1+w^2(1+\bar{a}^2-bw+\bar{a}^2w^2)}}$$

which implies

$$\frac{d\theta}{\cos\theta\sqrt{1-\bar{a}^2\cos^2\theta}}\,(1+\bar{a}^2w^2\cos^2\theta) = \frac{d\bar{\theta}}{\cos\bar{\theta}\sqrt{1-\bar{a}^2\cos^2\bar{\theta}}}\,. \tag{A.14}$$

Let us define $\bar{t} := t/l$. Formulae (A.12) and (A.14) together with (A.1) enable one to derive implicitly the induced three-metric h with respect to coordinates $\bar{t}, \bar{\theta}, \phi$:

$$\begin{aligned} \frac{z^2}{l^2}\tilde{g}_{\text{Kerr–AdS}}\Big|_S = \frac{z^2}{w^2}\Bigg\{ & \frac{\cos^2\theta}{\cos^2\bar{\theta}(1-\bar{a}^2\cos^2\bar{\theta})}\,d\bar{\theta}^2 \\ & + 2\bar{a}\sin^2\theta\left(1+\bar{a}^2w^2-\frac{bw^3}{1+\bar{a}^2w^2\cos^2\theta}\right)d\bar{t}\,d\phi \\ & + \sin^2\theta\left[(1+\bar{a}^2w^2)(1-\bar{a}^2)+\frac{bw^3\bar{a}^2\sin^2\theta}{1+\bar{a}^2w^2\cos^2\theta}\right]d\phi^2 \\ & - \left(1+\bar{a}^2w^2\sin^2\theta+w^2-\frac{bw^3}{1+\bar{a}^2w^2\cos^2\theta}\right)d\bar{t}^2\Bigg\}. \end{aligned} \tag{A.15}$$

Let us observe that the conformal factor

$$\frac{z}{w} = \frac{\bar{a}\sin\theta+\sqrt{1-\bar{a}^2\cos^2\theta}}{1+\sqrt{1+w^2(1+\bar{a}^2)}}\,\exp\bar{F} \tag{A.16}$$

is regular at $\mathscr{I}$ (corresponding to surface $\{w=0\}$). To finish derivation of asymptotics of h we have to notice that (A.12) written in equivalent form as follows:

$$\frac{\sin\bar{\theta}}{\sqrt{1-\bar{a}^2\cos^2\bar{\theta}}} = \frac{\sin\theta-\tanh\delta(w)\sqrt{1-\bar{a}^2\cos^2\theta}}{\sqrt{1-\bar{a}^2\cos^2\theta}-\tanh\delta(w)\sin\theta} \tag{A.17}$$

defines (implicitly) the function $\bar{\theta}(\theta, w)$. In particular

$$\cos^2\theta-\cos^2\bar{\theta} = -2\delta\cos^2\bar{\theta}\sin\bar{\theta}\sqrt{1-\bar{a}^2\cos^2\bar{\theta}}+O(\delta^2)$$

which is a straightforward consequence of (A.17).

Eqs. (A.9) and (A.17) define the mapping $(w,\theta) \mapsto (z,\bar{\theta})$. To obtain the explicit form of h this mapping should be inverted in the neighborhood of $z = w = 0$ corresponding to $\mathscr{I}$ and applied to (A.15).

If we put $w = 0$ in (A.15), we obtain the asymptotic value of h at $\mathscr{I}$:

$$\overset{(0)}{h} = \frac{1}{4}\left(\bar{a}\sin\bar{\theta} + \sqrt{1-\bar{a}^2\cos^2\bar{\theta}}\right)^2 \times\left[\frac{1}{1-\bar{a}^2\cos^2\bar{\theta}}\,d\bar{\theta}^2 + 2\bar{a}\sin^2\bar{\theta}\,d\bar{t}\,d\phi + \sin^2\bar{\theta}(1-\bar{a}^2)\,d\phi^2 - \,d\bar{t}^2\right]. \quad \text{(A.18)}$$

<u>Remark</u>: Let us notice that sometimes the coordinates $(\bar{t}, w, \theta, \phi)$ are not convenient in the asymptotic region. According to [20] one can introduce new spatial coordinates (ρ, Θ, Φ) defined as follows:

$$\Phi := (1-\bar{a}^2)\phi + \bar{a}\bar{t}\,, \quad \text{(A.19)}$$

$$\rho^{-1}\cos\Theta = w^{-1}\cos\theta\,, \quad \text{(A.20)}$$

$$(1-\bar{a}^2)\rho^{-2} = w^{-2} + \bar{a}^2\sin^2\theta - \bar{a}^2 w^{-2}\cos^2\theta\,. \quad \text{(A.21)}$$

Some useful formulae describing coordinate transformation $(w,\theta,\phi) \leftrightarrow (\rho,\Theta,\Phi)$ are given in Appendix A. In particular, they enable one to prove that in new coordinates the induced metric h at $\mathscr{I}$ takes the following form:

$$\overset{(0)}{h} = \frac{1-\bar{a}^2}{4\,(1-\bar{a}\sin\Theta)^2}\left[d\Theta^2 + \sin^2\Theta\,d\Phi^2 - \,d\bar{t}^2\right], \quad \text{(A.22)}$$

which explicitly shows that metric $\overset{(0)}{h}$ (for $\bar{a} < 1$) is in the conformal class of the Einstein static universe (*cf.* (4.6)).

Let us denote by ω the following function:

$$\omega(\bar{\theta}) := \frac{2}{\bar{a}\sin\bar{\theta} + \sqrt{1-\bar{a}^2\cos^2\bar{\theta}}}\,,$$

which comes from conformal factor in metric tensor $\overset{(0)}{h}$ given by (A.18). To derive higher order terms in h_{ab} for Kerr–AdS we have to check the following formulae:

$$\frac{\omega(\bar{\theta})}{\omega(\theta)} = \frac{\bar{a}\sin\theta + \sqrt{1-\bar{a}^2\cos^2\theta}}{\bar{a}\sin\bar{\theta} + \sqrt{1-\bar{a}^2\cos^2\bar{\theta}}} = 1 + \bar{a}\cos^2\bar{\theta}\,\delta + O(\delta^2)\,,$$

$$\frac{\cos^2\theta}{\cos^2\bar{\theta}} = 1 - 2\delta\sin\bar{\theta}\sqrt{1-\bar{a}^2\cos^2\bar{\theta}} + O(\delta^2)$$

$$= 1 - \bar{a}\sin\bar{\theta}\sqrt{1-\bar{a}^2\cos^2\bar{\theta}}\,w^2 + O(w^4)\,, \quad \text{(A.23)}$$

$$\frac{\sin^2\theta}{\sin^2\bar{\theta}} = 1 + \bar{a}\frac{\cos^2\bar{\theta}}{\sin\bar{\theta}}\sqrt{1-\bar{a}^2\cos^2\bar{\theta}}\,w^2 + O(w^4)\,, \quad \text{(A.24)}$$

which together with (A.16) and (A.10) imply

$$\left(\frac{z}{w}\right)^2 = \omega(\bar\theta)^{-2}\left[1-(1+\bar a^2\sin^2\bar\theta)w^2+\frac{b}{3}w^3+O(w^4)\right]. \tag{A.25}$$

Now we are ready to derive higher order terms in (A.15). We obtain the following non-vanishing components of three-metric h:

$$\begin{aligned}
h_{\bar\theta\bar\theta} &= \frac{z^2}{w^2}\frac{\cos^2\theta}{\cos^2\bar\theta}\frac{1}{(1-\bar a^2\cos^2\bar\theta)} \\
&= \frac{1-(1+\bar a^2\sin^2\bar\theta)w^2-\bar a\sin\bar\theta\sqrt{1-\bar a^2\cos^2\bar\theta}w^2+\frac{b}{3}w^3+O(w^4)}{(1-\bar a^2\cos^2\bar\theta)\omega(\bar\theta)^2}, \quad \text{(A.26)}\\
h_{\bar t\bar t} &= -\frac{z^2}{w^2}\left(1+\bar a^2w^2\sin^2\theta+w^2-\frac{bw^3}{1+\bar a^2w^2\cos^2\theta}\right)\\
&= -\omega(\bar\theta)^{-2}\left[1-\frac{2b}{3}w^3+O(w^4)\right], \quad \text{(A.27)}\\
h_{\bar t\phi} &= \bar a\sin^2\theta\frac{z^2}{w^2}\left(1+\bar a^2w^2-\frac{bw^3}{1+\bar a^2w^2\cos^2\theta}\right) = \bar a\sin^2\bar\theta\omega(\bar\theta)^{-2}\\
&\quad\times\left[1+\bar a\frac{\cos^2\bar\theta}{\sin\bar\theta}\sqrt{1-\bar a^2\cos^2\bar\theta}w^2-(1-\bar a^2\cos^2\bar\theta)w^2-\frac{2b}{3}w^3+O(w^4)\right], \quad \text{(A.28)}\\
h_{\phi\phi} &= \sin^2\theta\frac{z^2}{w^2}\left[(1+\bar a^2w^2)(1-\bar a^2)+\frac{bw^3\bar a^2\sin^2\theta}{1+\bar a^2w^2\cos^2\theta}\right]\\
&= \sin^2\bar\theta\omega(\bar\theta)^{-2}(1-\bar a^2)\left[1+\frac{\bar a^2\sin^2\bar\theta}{1-\bar a^2}bw^3+O(w^4)\right]\\
&\quad\times\left[1+\bar a\frac{\cos^2\bar\theta}{\sin\bar\theta}\sqrt{1-\bar a^2\cos^2\bar\theta}w^2-(1-\bar a^2\cos^2\bar\theta)w^2+\frac{b}{3}w^3+O(w^4)\right]. \quad \text{(A.29)}
\end{aligned}$$

Let us observe that asymptotics at $\mathscr{I}$ with respect to coordinate w with coefficients depending on $\bar\theta$ can be easily transformed into demanded asymptotics with respect to coordinate z. More precisely, from (A.25) we have $z\omega(\bar\theta) = w(1+O(w^2))$ which implies

$$a_0(\bar\theta)+a_2(\bar\theta)w^2+a_3(\bar\theta)w^3+O(w^4) = a_0(\bar\theta)+a_2(\bar\theta)\omega(\bar\theta)^2z^2+a_3(\bar\theta)\omega(\bar\theta)^3z^3+O(z^4),$$

which means that the coefficients in the third degree polynomial do not mix each other when we pass from w to z.

Hence, we get

$$
\begin{aligned}
h \;=\; & \frac{\omega^{-2}-(1+\bar{a}^2\sin^2\bar{\theta})z^2-\bar{a}\sin\bar{\theta}\sqrt{1-\bar{a}^2\cos^2\bar{\theta}}\,z^2+\frac{b}{3}\omega z^3}{(1-\bar{a}^2\cos^2\bar{\theta})}\,d\bar{\theta}^2+2\bar{a}\sin^2\bar{\theta} \\
& \times\left[\frac{1}{\omega^2}+\bar{a}\frac{\cos^2\bar{\theta}}{\sin\bar{\theta}}\sqrt{1-\bar{a}^2\cos^2\bar{\theta}}\,z^2-(1-\bar{a}^2\cos^2\bar{\theta})z^2-\frac{2b}{3}\omega z^3\right]d\bar{t}\,d\phi \\
& +\sin^2\bar{\theta}(1-\bar{a}^2)\left[1+\frac{\bar{a}^2\sin^2\bar{\theta}}{1-\bar{a}^2}b\omega^3z^3\right] \\
& \times\left[\omega^{-2}+\bar{a}\frac{\cos^2\bar{\theta}}{\sin\bar{\theta}}\sqrt{1-\bar{a}^2\cos^2\bar{\theta}}\,z^2-(1-\bar{a}^2\cos^2\bar{\theta})z^2+\frac{b}{3}\omega z^3\right]d\phi^2 \\
& -\left[\omega^{-2}-\frac{2b}{3}\omega z^3\right]d\bar{t}^2+O(z^4)\,, \qquad \text{(A.30)}
\end{aligned}
$$

$$
\begin{aligned}
\overset{(2)}{h} \;=\; & -\frac{(1+\bar{a}^2\sin^2\bar{\theta})+\bar{a}\sin\bar{\theta}\sqrt{1-\bar{a}^2\cos^2\bar{\theta}}}{(1-\bar{a}^2\cos^2\bar{\theta})}\,d\bar{\theta}^2 \\
& +2\bar{a}\sin^2\bar{\theta}\left[\bar{a}\frac{\cos^2\bar{\theta}}{\sin\bar{\theta}}\sqrt{1-\bar{a}^2\cos^2\bar{\theta}}-(1-\bar{a}^2\cos^2\bar{\theta})\right]d\bar{t}\,d\phi \\
& +\sin^2\bar{\theta}(1-\bar{a}^2)\left[\bar{a}\frac{\cos^2\bar{\theta}}{\sin\bar{\theta}}\sqrt{1-\bar{a}^2\cos^2\bar{\theta}}-(1-\bar{a}^2\cos^2\bar{\theta})\right]d\phi^2\,, \qquad \text{(A.31)}
\end{aligned}
$$

$$
\begin{aligned}
\chi \;=\; & \frac{b\omega}{3}\Bigg\{2\,d\bar{t}^2-4\bar{a}\sin^2\bar{\theta}\,d\bar{t}\,d\phi+\left(1-\bar{a}^2+3\bar{a}^2\sin^2\bar{\theta}\right)\sin^2\bar{\theta}\,d\phi^2 \\
& +\frac{d\bar{\theta}^2}{1-\bar{a}^2\cos^2\bar{\theta}}\Bigg\}\,. \qquad \text{(A.32)}
\end{aligned}
$$

Mass and angular momentum of Kerr–AdS

From (A.18) we have

$$\sqrt{-\det\overset{(0)}{h}}=\omega^{-3}\sin\theta\,.$$

Moreover, (A.32) implies

$$\chi^0{}_\phi=b\bar{a}\omega^3\sin^2\theta=\frac{2ma}{l^2}\omega^3\sin^2\theta\,.$$

Hence for cut $C := \{t = \text{const.}\} \subset \mathscr{I}$ we obtain

$$\frac{3l^2}{16\pi}\int_C \chi^0{}_\phi \sqrt{-\det \overset{(0)}{h}}\, d\theta\, d\phi = \frac{3ma}{4}\int_0^\pi \sin^3\theta\, d\theta = ma$$

which gives (6.6). To obtain (6.5) we observe that

$$\chi^0{}_0 = -\frac{2}{3}b\omega^3 = -\frac{4m}{3l}\omega^3$$

which implies

$$\frac{3l}{16\pi}\int_C \chi^0{}_0 \sqrt{-\det \overset{(0)}{h}}\, d\theta\, d\phi = -\frac{m}{2}\int_0^\pi \sin\theta\, d\theta = -m\,.$$

"Henneaux–Teitelboim" coordinates ρ, Θ, Φ in Kerr–AdS

The following formulae:

$$\rho^2 = \frac{(1-\bar{a}^2)w^2}{1-\bar{a}^2\cos^2\theta + \bar{a}^2 w^2 \sin^2\theta}\,, \tag{A.33}$$

$$\cos^2\Theta = \frac{(1-\bar{a}^2)\cos^2\theta}{1-\bar{a}^2\cos^2\theta + \bar{a}^2 w^2 \sin^2\theta}\,, \tag{A.34}$$

describe the coordinate transformation $(w, \theta) \mapsto (\rho, \Theta)$. Moreover, we have

$$\cos^2\theta = \frac{(1+\bar{a}^2 w^2)\cos^2\Theta}{1-\bar{a}^2\sin^2\Theta + \bar{a}^2 w^2\cos^2\Theta}\,,$$

$$\sin^2\theta = \frac{(1-\bar{a}^2)\sin^2\Theta}{1-\bar{a}^2\sin^2\Theta + \bar{a}^2 w^2\cos^2\Theta}\,,$$

$$\rho^2 = \frac{1-\bar{a}^2\sin^2\Theta + \bar{a}^2 w^2\cos^2\Theta}{1+\bar{a}^2 w^2}\, w^2\,,$$

$$\phi = \frac{\Phi - \bar{a}\bar{t}}{1-\bar{a}^2}\,, \qquad d\phi = \frac{d\Phi - \bar{a}\, d\bar{t}}{1-\bar{a}^2}\,,$$

$$w^2 = 2\rho^2\left[1-\bar{a}^2\sin^2\Theta - \bar{a}^2\rho^2 + \sqrt{(1-\bar{a}^2\sin^2\Theta - \bar{a}^2\rho^2)^2 + 4\bar{a}^2\rho^2\cos^2\Theta}\right]^{-1}\,,$$

$$d\theta = \frac{\sqrt{(1-\bar{a}^2)(1+\bar{a}^2 w^2)}}{1-\bar{a}^2\sin^2\Theta + \bar{a}^2 w^2\cos^2\Theta}\left(d\Theta - \frac{\bar{a}^2 w\sin\Theta\cos\Theta}{1+\bar{a}^2 w^2}\, dw\right)\,.$$

Appendix B

Curvature of conformal boundary in NUT–AdS space-time

For the metric (4.57) we have the following non-vanishing Christoffel symbols:

$$\Gamma^0{}_{\theta\phi} = -\bar{l}\sin\theta(1+4\bar{l}^2)\tan^2\frac{\theta}{2}\,, \tag{B.1}$$

$$\Gamma^0{}_{\theta 0} = 2\bar{l}^2\tan\frac{\theta}{2}\,, \tag{B.2}$$

$$\Gamma^\theta{}_{\phi\phi} = \sin^2\theta\left(4\bar{l}^2\tan\frac{\theta}{2} - \cot\theta\right)\,, \tag{B.3}$$

$$\Gamma^\theta{}_{0\phi} = -\bar{l}\sin\theta\,, \tag{B.4}$$

$$\Gamma^\phi{}_{\theta 0} = \frac{\bar{l}}{\sin\theta}\,, \tag{B.5}$$

$$\Gamma^\phi{}_{\theta\phi} = -2\bar{l}^2\tan\frac{\theta}{2} + \cot\theta\,. \tag{B.6}$$

For the Ricci tensor

$$\mathcal{R}_{ab} = \partial_c\Gamma^c{}_{ab} - \partial_b\Gamma^c{}_{ac} + \Gamma^c{}_{ab}\Gamma^d{}_{cd} - \Gamma^d{}_{ac}\Gamma^c{}_{db}$$

we get the following components:

$$\mathcal{R}_{\theta\theta} = 1+2\bar{l}^2\,, \tag{B.7}$$

$$\mathcal{R}_{00} = 2\bar{l}^2\,, \tag{B.8}$$

$$\mathcal{R}_{\phi\phi} = \sin^2\theta\left(1+2\bar{l}^2+8\bar{l}^4\tan^2\frac{\theta}{2}\right)\,, \tag{B.9}$$

$$\mathcal{R}_{0\phi} = -4\bar{l}^3\sin\theta\tan\frac{\theta}{2}\,, \tag{B.10}$$

$$\mathcal{R}_{\theta 0} = 0\,, \tag{B.11}$$

$$\mathcal{R}_{\theta\phi} = 0\,, \tag{B.12}$$

together with the scalar curvature

$$\mathcal{R} = h^{ab}\mathcal{R}_{ab} = 8(1+\bar{l}^2)\,.$$

The above formulae for the Ricci tensor imply

$$\mathcal{R}_{0\theta|\phi} - \mathcal{R}_{0\phi|\theta} = 2\bar{l}\sin\theta\,.$$

Moreover,

$$\sqrt{-\det \overset{(0)}{h}}\,\beta^0{}_0 = 2C_{0\theta\phi} = 2\mathcal{R}_{0\theta|\phi} - 2\mathcal{R}_{0\phi|\theta}$$

because $\mathcal{R}_{,a} = 0$. Similarly we have

$$C_{0\theta 0} = 0 = C_{0\phi 0}$$

which implies

$$\beta^{\phi}{}_{0} = 0 = \beta^{\theta}{}_{0}\,.$$

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