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UPSCALING IN DYNAMICAL HEAT TRANSFER PROBLEMS IN BIOLOGICAL TISSUES

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The asymptotic behaviour of the solution of a nonlinear dynamical boundary-value problem describing the bio-heat transfer in microvascular tissues is analysed. Our domain Ω is an ε-periodic structure, consisting of two parts: a solid tissue part Ω^ε and small regions of blood $\Omega \setminus \overline{\Omega^\varepsilon}$ of a certain temperature, ε representing a small parameter related to the characteristic size of the blood regions. In such a domain, we shall consider a heat equation, with nonlinear sink and source terms and with a dynamical condition imposed on the boundaries of the blood zones. The limit equation, as $\varepsilon \to 0$, is a new heat equation, with extra-terms coming from the influence of the non-homogeneous dynamical boundary condition.

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1. Introduction and setting of the problem

The aim of this paper is to study the asymptotic behaviour of the solution of a dynamical nonlinear boundary-value problem modelling thermoregulation phenomena in the human microvascular system. Such dynamical boundary-value problems, although not too widely considered in the literature, are very natural in many other mathematical models, such as partially saturated flows in porous media, heat transfer in a solid in contact with a moving fluid, diffusion phenomena in porous media (see [2, 4, 16] and [17] and the references therein).

The bio-heat transport in living tissues is a complex process involving multiple mechanisms, such as conduction, convection, radiation, metabolism, *etc.* Bio-heat transfer models have significant applications in many clinical and environmental sciences. In particular, the heat transfer mechanism in biological tissues is important for therapeutic practices, such as cancer hyperthermia, burn injury, brain hypothermia resuscitation, disease diagnostics, cryosurgery, *etc.*

Let Ω be an open bounded set in $\mathbb{R}^n$ $(n \geq 2)$. For the model we intent to analyse, *i.e.* the problem of bio-heat transfer in microvascular tissues, we can consider that Ω is an ε-periodic structure, consisting of two parts: a solid tissue part Ω^ε of temperature u^ε and small regions of blood $\Omega \setminus \overline{\Omega^\varepsilon}$ of a certain temperature. ε represents a small parameter related to the characteristic size of the blood regions.

The nonlinear problem studied in this paper concerns the non-stationary heat transfer in the solid tissue part, in contact with the blood regions. We shall assume that we have some external thermal sources f inside Ω^ε and some nonlinear sink term describing heat loss (cell-destruction energy, generated, possible, by special chemical reactions), given by a nonlinear function β. Also, due to the fact that this complicated microstructure is dynamically evolving, we shall impose a dynamical nonlinear boundary condition on the boundaries of the blood zones.

If we denote by $(0,T)$ the time interval of interest, we shall analyse the asymptotic behaviour, as $\varepsilon \to 0$, of the solution of the following problem:

$$\begin{aligned}
\rho c_p \frac{\partial u^\varepsilon}{\partial t} - D_0 \Delta u^\varepsilon + \beta(u^\varepsilon) &= f(t,x)\,, && \text{in } \Omega^\varepsilon \times (0,T)\,, && (1)\\
D_0 \frac{\partial u^\varepsilon}{\partial \nu} + \alpha\varepsilon \frac{\partial u^\varepsilon}{\partial t} &= \varepsilon a(u_b^\varepsilon - u^\varepsilon)\,, && \text{on } S^\varepsilon \times (0,T)\,, && (2)\\
u^\varepsilon(0,x) &= u^0(x)\,, && \text{in } \Omega^\varepsilon\,, && (3)\\
u^\varepsilon(0,x) &= v^0(x)\,, && \text{on } S^\varepsilon\,, && (4)\\
u^\varepsilon &= 0\,, && \text{on } \partial\Omega \times (0,T)\,. && (5)
\end{aligned}$$

Here, ν is the exterior unit normal to Ω^ε, $f \in L^2(0,T;L^2(\Omega))$, $u^0 \in H_0^1(\Omega)$, $v^0 \in L^2(S^\varepsilon)$, $a > 0$, $D_0 > 0$, $c_p > 0$, $\rho > 0$, $\alpha > 0$, $u_b^\varepsilon \in H^1(\Omega)$ and S^ε is the boundary of the blood regions. Notice that on S^ε we assume that the temperature $v^0(x)$ is equal to the trace of $u^0(x)$. Also, we shall assume that the nonlinear function β is given (see Section 2).

The existence and uniqueness of a weak solution of problem (1)–(5) can be settled by using the theory of nonlinear monotone problems (see Section 2).

Let us notice that the scaling in front of the blood perfusion term in the boundary condition (2) is the unique one that enables us to preserve at the macroscale such an effect and to get the classical Helmholtz term in the macroscopic bio-heat equation. If we scale the dynamic term in (2) by ε^2 as in [29], we lose at the macroscale the extra term capturing the dynamical character of this complicated microstructure.

Also, let us mention that we can get similar results if the term $-D_0\Delta u^\varepsilon$ in (1) is replaced by a general strong elliptic operator $-\mathrm{div}(A^\varepsilon \nabla u^\varepsilon)$, where A^ε is Y-periodic and satisfies strong ellipticity conditions. The positive parameter ε will also define a length scale measuring how densely the inhomogeneities are distributed in Ω.

In fact, this means to assume that we are dealing with heterogeneous tissues. From a mathematical point of view, we shall consider the case of a general medium, having discontinuous properties, represented by a coercive periodic matrix with rapidly oscillating coefficients. Let $A \in L^\infty_\#(\Omega)^{n\times n}$ be a symmetric matrix whose entries are Y-periodic, bounded and measurable real functions. We use the symbol # to denote periodicity properties. Let us assume that for some $0 < \alpha < \beta$,

$$\alpha\,|\xi|^2 \le A(y)\xi\,\xi \le \beta\,|\xi|^2 \quad \forall\xi,\ y \in \mathbb{R}^n\,.$$

We shall denote by $A^\varepsilon(x)$ the value of $A(y)$ at the point $y = x/\varepsilon$, *i.e.*

$$A^\varepsilon(x) = A\left(\frac{x}{\varepsilon}\right)\,.$$

Since the period of the structure is small compared to the dimension of Ω, or in other words, since the non-homogeneities are small compared to the global dimension of the structure, an asymptotic analysis becomes necessary. Two scales are important for a suitable description of the given structure: one which is comparable with the dimension of the period, called the *microscopic scale* and denoted by $y = x/\varepsilon$, and another one which is of the same order of magnitude as the global dimension of our system, called the *macroscopic scale* and denoted by x.

The main goal of the homogenisation method is to pass from the microscopic scale to the macroscopic one; more precisely, using the homogenisation method, we try to describe the macroscopic properties of our non-homogeneous system in terms of the properties of its microscopic structure. Intuitively, the non-homogeneous real system, having a very complicated microstructure, is replaced by a fictitious homogeneous one, whose global characteristics represent a good approximation of the initial system. Hence, the homogenisation method provides a general framework for obtaining these macroscale properties, eliminating the difficulties related to the explicit determination of a solution of the problem at the microscale and offering a less detailed description, but one which is applicable to much more complex systems.

Also, from the point of view of numerical computation, the homogenised equation, defined on a fixed domain Ω and describing the effective behaviour of our system, will have constant coefficients, called effective or homogenised coefficients (see Section 2), and, hence, it will be easier to be solved numerically than the original equation, which was an equation with rapidly oscillating coefficients defined on a perforated domain and satisfying nonlinear conditions on the boundaries. The dependence on the real microstructure is given through the homogenised coefficients.

Hence, we shall be interested in getting the asymptotic behaviour, when $\varepsilon \to 0$, of the solution of problem (1)–(5). Using a classical homogenisation method, *i.e.* Tartar's method of oscillating test functions, coupled with monotonicity methods and results from the theory of semilinear problems, we can prove that the solution of problem (1)–(5), properly extended to the whole of Ω, converges to the unique solution of a new nonlinear problem, defined all over the domain Ω, given by a new operator and containing extra zero order terms, capturing the effect of the blood perfusion and of the influence of the non-homogeneous dynamical condition imposed on the boundaries of the blood regions (see Section 2).

We can treat in the same manner the case in which we consider variable metabolic heat generation, given by suitable nonlinear functions $f(t, x, u)$.

Other nonlinear problems modelling various physical phenomena arising in radiophysics, filtration theory, rheology, elasticity theory, in the theory of composites, polycrystals and smart materials and in other domains of mechanics, physics and technology can benefit from a similar effective medium approach (see, for instance, the monograph [20] and [7]).

The results of this paper constitute a generalisation of some of the results obtained in [15], by considering non-stationary processes, dynamical conditions on the boundaries of the blood regions and a nonlinear sink term acting inside the solid tissue, modelling cell-destruction energy, which can be of huge importance, for instance, in destroying malignant cells by hyperthermia (see [24]).

In 1948, based on experimental observation, Pennes (see [22]) proposed a simple linear mathematical model for describing the thermal interaction between human tissues and perfused blood, taking also into account the effects of the metabolism. Later on, alternative models for describing the heat exchange between tissues and blood have been developed (see [8, 15, 24] and the references therein).

From a mathematical point of view, problems closed to this one have been considered by Cioranescu and Donato [9], Cioranescu, Donato and Ene [11], Conca and Donato [14], Conca, Díaz and Timofte [13], Timofte [26–28], Ene and Polisevski [18], Timofte [25], Bourgeat and Pankratov [5], Pankratov, Piatnitskii and Rybalko [21] for the deterministic case, Wang and Duan [29] for the stochastic one.

The plan of the paper is as follows: in Section 2 we introduce some useful notations and assumptions and we give the main convergence result of this paper. For obtaining it, we need some preliminary results, which are given in Section 3. The last section is devoted to the proof of the convergence result.

2. Notation and assumptions

Let Ω be a bounded connected open subset of $\mathbb{R}^n$ $(n \geq 2)$, with $\partial\Omega$ of class C^2 and let $[0,T]$ be the time interval of interest. Let $Y = [0, l_1[\times ...[0, l_n[$ be the representative cell in $\mathbb{R}^n$ and F an open subset of Y with boundary ∂F of class C^2, such that $\overline{F} \subset Y$.

We shall denote by $F(\varepsilon, \boldsymbol{k})$ the translated image of εF by the vector $\varepsilon \boldsymbol{kl}$, $\boldsymbol{k} \in \mathbb{Z}^n$, $\boldsymbol{kl} = (k_1 l_1, ..., k_n l_n)$:

$$F(\varepsilon, \boldsymbol{k}) = \varepsilon(\boldsymbol{kl} + F)\,.$$

Also, we shall denote by F^ε the set of all the holes contained in Ω. So

$$F^\varepsilon = \bigcup_{\boldsymbol{k} \in \mathbb{Z}^n} \{F(\varepsilon, \boldsymbol{k}) \mid \overline{F}(\varepsilon, \boldsymbol{k}) \subset \Omega\}\,.$$

Let $\Omega^\varepsilon = \Omega \setminus \overline{F^\varepsilon}$. Hence, Ω^ε is a periodically perforated domain with holes of the same size as the period. Let us remark that the holes do not intersect the boundary $\partial\Omega$.

We shall use the following notations:

$$Y^* = \frac{Y}{\overline{F}}\,, \tag{6}$$

$$\theta = \frac{|Y^*|}{|Y|}\,. \tag{7}$$

Also, we shall denote by χ^ε the characteristic function of the domain Ω^ε and throughout the paper, by C we shall denote a generic fixed strictly positive constant, whose value can change from line to line.

As already mentioned, we are interested in studying the asymptotic behaviour, as $\varepsilon \to 0$, of the solution of the parabolic problem (1)–(5).

We shall consider that the function β in (1) is continuous, monotonously non-decreasing and such that $\beta(0) = 0$. Moreover, we shall assume that there exist $C \geq 0$ and an exponent q such that

$$|\beta(v)| \leq C(1 + |v|^q)\,, \tag{8}$$

with $0 \leq q < n/(n-2)$ if $n \geq 3$ and $0 \leq q < +\infty$ if $n = 2$.

For the blood temperature u_b^ε we shall assume that $u_b^\varepsilon \in H^1(\Omega)$ and $\|u_b^\varepsilon\|_{H^1(\Omega)} \leq C$.

Remark 1. The results of this paper will be obtained for the case $n \geq 3$. All of them are still valid, under our assumptions, in the case in which $n = 2$. Of course, for this case, $n/(n-2)$ has to be replaced by $+\infty$.

Let us notice that due to the compactness injection theorems in Sobolev spaces, it would be enough, with the same reasoning as in the paper, to assume that β satisfies, for $n \geq 3$, the growth condition (8) for some $0 \leq q < (n+2)/(n-2)$. For $n = 2$, $(n+2)/(n-2)$ have to be replaced by $+\infty$. ▪

The existence and uniqueness of a weak solution of (1)–(5) can be settled by using the classical theory of semilinear monotone problems (see [1,3,5,6, 25,29]). As a result, we know that there exists a unique weak solution

$$u^\varepsilon \in C\left([0,T]; H^1_{\partial\Omega}(\Omega^\varepsilon)\right) \bigcap L^2\left(0,T; Y_1(\Omega^\varepsilon)\right),$$

with

$$\frac{\partial u^\varepsilon}{\partial t} \in L^2(0,T; L^2(\Omega^\varepsilon))$$

and

$$\frac{\partial \gamma(u^\varepsilon)}{\partial t} \in L^2\left(0,T; L^2(S^\varepsilon)\right).$$

Here, $H^1_{\partial\Omega}(\Omega^\varepsilon)$ is the space of elements of $H^1(\Omega^\varepsilon)$ which vanish (in the sense of traces) on $\partial\Omega$,

$$Y_1(\Omega^\varepsilon) = \left\{ v \in H^1_{\partial\Omega}(\Omega^\varepsilon) \mid -\Delta v \in L^2(\Omega^\varepsilon), R\frac{\partial v}{\partial n} \in L^2_{\partial\Omega}(\partial\Omega^\varepsilon) \right\}$$

and $\gamma : H^1(\Omega^\varepsilon) \to L^2(S^\varepsilon)$ is the trace operator with respect to S^ε, which is continuous. Moreover, for a function φ defined on $\partial\Omega^\varepsilon$, $R\varphi$ denotes its restriction to S^ε.

The main convergence result of this paper is given by the following theorem:

Theorem 1. One can construct an extension $P^\varepsilon u^\varepsilon$ of the solution u^ε of the problem (1)–(5) such that $P^\varepsilon u^\varepsilon \rightharpoonup u$ weakly in $L^2(0,T; H^1_0(\Omega))$, where u is the unique solution of the following nonlinear problem:

$$\begin{cases} \rho c_p (1+\delta)\dfrac{\partial u}{\partial t} - \displaystyle\sum_{i,j=1}^n q_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \beta(u) \\ +a\dfrac{|\partial F|}{|Y^*|}(u - u_b) = f\,, & x \in \Omega\,,\ t \in (0,T)\,, \\ u = 0\,, & x \in \partial\Omega\,,\ t \in (0,T)\,, \\ u(0,x) = u_0(x)\,, & x \in \Omega\,. \end{cases} \tag{9}$$

Here,

$$\delta = \frac{\alpha}{\rho c_p} \frac{|\partial F|}{|Y^*|}$$

and $Q = ((q_{ij}))$ is the homogenised matrix, whose entries are defined by:

$$q_{ij} = D_0 \left(\delta_{ij} + \frac{1}{|Y^*|} \int_{Y^*} \frac{\partial \chi_j}{\partial y_i} dy \right), \tag{10}$$

in terms of the functions χ_i, $i = 1, ..., n$, solutions of the cell problems

$$\begin{cases} -\Delta \chi_i = 0 \ \text{ in } \ Y^*, \\ \dfrac{\partial(\chi_i + y_i)}{\partial \nu} = 0 \ \text{ on } \ \partial F, \\ \chi_i \ \ Y- \ \text{periodic.} \end{cases}$$ ▪

Thus, in the limit, when $\varepsilon \to 0$, we get a constant coefficient heat equation, with a Dirichlet boundary condition and with a constant (due to the periodicity) extra-term in front of the time derivative, coming from the well-balanced contribution of the dynamical part of our boundary condition on the surface of the blood regions. Also, the effect of the blood perfusion, described by the linear Newton's cooling law, is captured in the limit equation.

Remark 2. There exists a unique solution of the macromodel problem (9). ▪

Remark 3. As mentioned in the Introduction, in the general case of an heterogeneous medium, it is not difficult to see that the limit problem is the following one:

$$\begin{cases} \rho c_p (1+\delta) \dfrac{\partial u}{\partial t} - \operatorname{div}(A^0 \nabla u) + \beta(u) \\ + a \dfrac{|\partial F|}{|Y^*|}(u - u_b) = f\,, & x \in \Omega\,,\ t \in (0,T)\,, \\ u = 0\,, & x \in \partial\Omega\,,\ t \in (0,T)\,, \\ u(0,x) = u_0(x)\,, & x \in \Omega\,. \end{cases} \tag{11}$$

Here, $A^0 = (a^0_{ij})$ is the classical homogenized matrix, whose entries are defined as follows:

$$a^0_{ij} = \frac{1}{|Y^*|} \int_{Y^*} \left(a_{ij}(y) + a_{ik}(y) \frac{\partial \chi_j}{\partial y_k} \right) dy\,,$$

in terms of the functions χ_j, $j = 1, ..., n$, solutions of the cell problems

$$\begin{cases} -\mathrm{div}_y A(y)(D_y \chi_j + \boldsymbol{e}_j) = 0 \ \text{ in } \ Y^*\,, \\ A(y)(D\chi_j + \boldsymbol{e}_j) \cdot \nu = 0 \ \text{ on } \ \partial F\,, \\ \chi_j \in H^1_{\#Y}(Y^\star)\,, \qquad \displaystyle\int_{Y^\star} \chi_j = 0\,, \end{cases}$$

where $\boldsymbol{e}_i$, $1 \le i \le n$, are the elements of the canonical basis in $\mathbb{R}^n$. The constant matrix A^0 is symmetric and positive-definite.

In this limit problem, the periodic heterogeneous structure of our medium is reflected by the presence of the homogenized matrix A^0. ▪

3. Preliminary results

As already mentioned, there exists a unique solution for the nonlinear problem (1)–(5),

$$u^\varepsilon \in C([0,T]; H^1_{\partial\Omega}(\Omega^\varepsilon)) \bigcap L^2\left(0,T; Y_1(\Omega^\varepsilon)\right)\,,$$

with

$$\frac{\partial u^\varepsilon}{\partial t} \in L^2\left(0,T; L^2(\Omega^\varepsilon)\right)$$

and

$$\frac{\partial \gamma(u^\varepsilon)}{\partial t} \in L^2\left(0,T; L^2(S^\varepsilon)\right)\,.$$

In order to extend it to the whole of Ω, let us recall the following result (see [12]):

Lemma 1. There exists a linear continuous extension operator $P^\varepsilon \in \mathcal{L}(L^2(\Omega^\varepsilon); L^2(\Omega)) \cap \mathcal{L}(V^\varepsilon; H^1_0(\Omega))$ and a positive constant C, independent of ε, such that, for any $v \in V^\varepsilon$,

$$\|P^\varepsilon v\|_{L^2(\Omega)} \le C\,\|v\|_{L^2(\Omega^\varepsilon)}$$

and

$$\|\nabla P^\varepsilon v\|_{L^2(\Omega)} \le C\,\|\nabla v\|_{L^2(\Omega^\varepsilon)}\,.$$

Here,

$$V^\varepsilon = \{v \in H^1(\Omega^\varepsilon) \mid v = 0 \text{ on } \partial\Omega\}\,. \quad ▪$$

For getting the effective behaviour of our solution u^ε, we have to pass to the limit in the variational formulation of problem (1)–(5) (see (15)). To this end, let us introduce, for any $h \in L^{s'}(\partial F)$, $1 \le s' \le \infty$, the linear form μ_h^ε on $W_0^{1,s}(\Omega)$ defined by

$$\langle \mu_h^\varepsilon, \varphi \rangle = \varepsilon \int\limits_{S^\varepsilon} h\left(\frac{x}{\varepsilon}\right) \varphi d\sigma \quad \forall \varphi \in W_0^{1,s}(\Omega)\,,$$

with $1/s + 1/s' = 1$. It is proven in [9] that

$$\mu_h^\varepsilon \to \mu_h \quad \text{strongly in} \quad \left(W_0^{1,s}(\Omega)\right)'\,, \tag{12}$$

where

$$\langle \mu_h, \varphi \rangle = \mu_h \int\limits_{\Omega} \varphi dx\,,$$

with

$$\mu_h = \frac{1}{|Y|} \int\limits_{\partial F} h(y) d\sigma\,.$$

If $h \in L^\infty(\partial F)$ or even if h is constant, we have (see [11])

$$\mu_h^\varepsilon \to \mu_h \quad \text{strongly in} \quad W^{-1,\infty}(\Omega)\,. \tag{13}$$

We denote by μ^ε the above introduced measure in the case in which $h = 1$.

Also, for obtaining the limit behaviour of our homogenisation problem, let us recall another result from [13].

Let H be a continuously differentiable function, monotonously non-decreasing and such that $H(v) = 0$ if and only if $v = 0$. We shall suppose that there exists a positive constant C and an exponent q, with $0 \le q < n/(n-2)$, such that $|H| \le C(1 + |v|^q)$. If we denote by $\overline{q} = (2n)/(q(n-2)+n)$, one can prove (see [13]) that for any $z^\varepsilon \rightharpoonup z$ weakly in $H_0^1(\Omega)$, we get

$$H(z^\varepsilon) \rightharpoonup H(z) \quad \text{weakly in} \quad W_0^{1,\overline{q}}(\Omega)\,. \tag{14}$$

4. Proof of the main result

Let us consider the variational formulation of problem (1)–(5):

$$\begin{aligned} &\rho c_p \int\limits_0^T \int\limits_{\Omega^\varepsilon} \dot{u}^\varepsilon \varphi dx dt + D_0 \int\limits_0^T \int\limits_{\Omega^\varepsilon} \nabla u^\varepsilon \cdot \nabla \varphi dx dt + \int\limits_0^T \int\limits_{\Omega^\varepsilon} \beta \varphi dx dt \\ &+\alpha\varepsilon \int\limits_0^T \int\limits_{S^\varepsilon} \dot{u}^\varepsilon \varphi dx dt + a\varepsilon \int\limits_0^T \int\limits_{S^\varepsilon} (u^\varepsilon - u_b^\varepsilon) \varphi dx dt = \int\limits_0^T \int\limits_{\Omega^\varepsilon} f \varphi dx dt\,, \end{aligned} \tag{15}$$

for any $\varphi \in C_0^\infty([0,T]\times\Omega^\varepsilon)$. Here, we have denoted by $\dot{}$ the partial derivative with respect to the time.

By classical existence and uniqueness results, we know that there exists a unique weak solution of (15). Taking it as a test function in (15) and using our assumptions on the data and Cauchy–Schwartz, Poincaré's and Young's inequalities, we can obtain suitable energy estimates, independent of ε, for our solution (see [4, 5, 13, 23, 25, 29]).

Denoting by $P^\varepsilon u^\varepsilon$ the extension of u^ε given by Lemma 1, one can see that $P^\varepsilon u^\varepsilon$ is bounded in $L^2(0,T;H_0^1(\Omega))$ and $(\partial P^\varepsilon u^\varepsilon)/\partial t$ is bounded in $L^2(0,T;L^2(\Omega))$ (see, for details, [13, 25, 29]). So, by passing to a subsequence, we have $P^\varepsilon u^\varepsilon \rightharpoonup u$ weakly in $L^2(0,T;H_0^1(\Omega))$ and strongly in $L^2(0,T;L^2(\Omega))$ and $(\partial P^\varepsilon u^\varepsilon)/\partial t \rightharpoonup \partial u/\partial t$ weakly in $L^2(0,T;L^2(\Omega))$.

It is well-known by now how to pass to the limit, with $\varepsilon \to 0$, in the linear terms of (15) defined on Ω^ε (see, for instance [13], [25] and [29]). Also, recall that θ is the weak-$\star$ limit in $L^\infty(\Omega)$ of χ^ε. Thus, we get:

$$\int_0^T\int_{\Omega^\varepsilon} \dot{u}^\varepsilon \varphi dxdt \;\to\; \int_0^T\int_{\Omega} \dot{u}\theta\varphi dxdt\,, \tag{16}$$

$$D_0\int_0^T\int_{\Omega^\varepsilon} \nabla u^\varepsilon\cdot\nabla\varphi dxdt \;\to\; \int_0^T\int_{\Omega} \theta Q\nabla u\cdot\nabla\varphi dxdt\,, \tag{17}$$

$$\int_0^T\int_{\Omega^\varepsilon} f\varphi dxdt \;\to\; \int_0^T\int_{\Omega} \theta f\varphi dxdt\,. \tag{18}$$

Let us see now how we can pass to the limit in the nonlinear terms in (15). For the third term in the left-hand side of (15), let us notice that, exactly like in [13] (see (14)), one can prove that for any $z^\varepsilon \rightharpoonup z$ weakly in $H_0^1(\Omega)$, we see that $\beta(z^\varepsilon) \to \beta(z)$ strongly in $L^{\overline{q}}(\Omega)$, where $\overline{q} = (2n)/(q(n-2)+n)$. Therefore, we have

$$\int_0^T\int_{\Omega^\varepsilon} \beta(u^\varepsilon)\varphi dxdt \to \int_0^T\int_{\Omega} \beta(u)\theta\varphi dxdt\,. \tag{19}$$

For the other nonlinear term in (15), using the convergence (13) written for $h = 1$, we obtain that

$$\varepsilon\int_{S^\varepsilon} u^\varepsilon\varphi dx = \langle \mu^\varepsilon, P^\varepsilon u^\varepsilon\varphi\rangle \to \frac{|\partial F|}{|Y|}\int_{\Omega} u\varphi dx\,.$$

Since $u_b^\varepsilon \in H^1(\Omega)$ and $\|u_b^\varepsilon\|_{H^1(\Omega)} \leq C$, then, up to a subsequence, we get

$$u_b^\varepsilon \rightharpoonup u_b \quad \text{weakly in } H^1(\Omega)\,.$$

Hence, integrating in time and using Lebesgue's convergence theorem, it is not difficult to see that

$$a\varepsilon \int_0^T \int_{S^\varepsilon} (u_b^\varepsilon - u^\varepsilon)\varphi dx dt \to a\frac{|\partial F|}{|Y|} \int_0^T \int_\Omega (u_b - u)\varphi dx dt\,. \tag{20}$$

Also, we have

$$\alpha\varepsilon \int_0^T \int_{S^\varepsilon} \dot{u}^\varepsilon \varphi dx dt \to \alpha\frac{|\partial F|}{|Y|} \int_0^T \int_\Omega \dot{u}\varphi dx dt\,. \tag{21}$$

Putting together (16)–(21), we can pass to the limit in all the terms in (15) and we obtain exactly the variational formulation of the limit problem (9). As u is uniquely determined, the whole sequence $P^\varepsilon u^\varepsilon$ converges to u and this completes the proof of Theorem 1. ∎

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