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SCATTERING AMPLITUDES AT WEAK AND STRONG COUPLING IN $\mathcal{N} = 4$ SUPER-YANG–MILLS THEORY*

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In these lectures we discuss methods for computing scattering amplitudes in $\mathcal{N} = 4$ super-Yang–Mills theory in both weak and strong coupling expansions.

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1. Introduction

There are two approaches to understanding and solving $\mathcal{N} = 4$ Yang–Mills (SYM): on the one hand, being a conformal field theory, it is uniquely specified by the spectrum of (anomalous) dimensions of gauge-invariant operators and their three-point correlation functions, while, on the other hand, like any other quantum field theory, it is completely specified by its scattering matrix[1]. The remarkable properties of $\mathcal{N} = 4$ SYM theory in the planar limit, in particular its high degree of symmetry, allowed important progress on both fronts: on the one hand, the integrability of the generator of scale transformations allows the evaluation of the anomalous dimensions of infinitely long operators through a Bethe ansatz [1–3] while on the other

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[1] In the presence of a regulator, the definition of the scattering matrix in a conformal field theory is no different than in any massive quantum field theory.

hand the theory is sufficiently symmetric and with sufficiently good high energy behavior to allow high order perturbative calculations of its scattering matrix (see *e.g.* [4]).

The strong coupling regime of the theory is directly accessible through the AdS/CFT duality [5–7] (see [8] for a review), which provides a description of $\mathcal{N} = 4$ SYM theory solely in terms of colorless, gauge invariant quantities. It casts the analysis of the strongly coupled planar theory in terms of the weakly-coupled worldsheet theory for superstrings in $AdS_5 \times S^5$. Being in one to one correspondence with closed string states, local gauge invariant operators have a natural place in the AdS/CFT duality. This fact played a major role in our understanding of the spectrum of operators of the $\mathcal{N} = 4$ SYM theory (as well as in many other contexts).

Scattering amplitudes describe the scattering of on-shell states of the theory. As such, they carry color charge and thus it is not immediately clear whether they can be described directly by the closed string theory dual. It is however possible to extend the closed string theory in $AdS_5 \times S^5$ by an open string sector. Depending on the precise physical problem, they are described either by semiclassical worldsheet configurations (*e.g.* when they describe the expectation value of Wilson loops) or by vertex operators (*e.g.* when they capture the scattering amplitudes of open string states). Appropriately integrated, the correlation functions of open string vertex operators are what one might define as the gauge theory scattering amplitudes. Vertex operators carry Chan–Paton factors and the correlation functions of vertex operators decompose, in a natural way, into a sum of terms, each of which exhibits a clean separation of the color degrees of freedom and the dependence on particle momenta. The factors carrying the kinematic dependence are known as partial amplitudes. This decomposition mirrors closely the color decomposition of gauge theory scattering amplitudes which we will discuss in Section 2. While non-local quantities, partial amplitudes carry no color charge and thus could in principle be described by the closed string theory dual to $\mathcal{N} = 4$ SYM theory.

Strong coupling information extracted along these lines, combined with weak coupling higher-loop calculations lead us to hope that, at least in some sectors, the scattering matrix of planar $\mathcal{N} = 4$ SYM theory can be found exactly. The four- and five-gluon amplitudes, which are currently known to all orders in perturbation theory (up to a set of undetermined constants), provide a proof of principle in this direction.

In these lectures we have presented a largely self-contained account of some of the recent developments and techniques for the evaluation of the scattering amplitudes of planar $\mathcal{N} = 4$ SYM theory at both weak and strong coupling. Related reviews of these topics may be found in references [9, 10]. These lectures are organized as follows. Section 2 summarizes the properties of scattering amplitudes and is devoted to their calculation in weakly coupled perturbation theory. After setting up the notation and describing some of their general properties, we proceed to outline techniques for tree- and loop-level high-multiplicity calculations. While the discussion is kept general at times, the main focus is planar $\mathcal{N} = 4$ SYM theory. The generalized unitarity-based method is the technique of choice for loop-level calculations, as it combines in a natural way, order by order in perturbation theory, the consequences of global symmetries and of gauge invariance.

A common feature of all on-shell scattering amplitudes in massless gauge theories in four dimensions is the presence of infrared divergences, originating from low energy virtual particles as well as from virtual momenta almost parallel to external ones. We will discuss their structure captured by the soft/collinear factorization theorem. A surprising feature of certain planar amplitudes of $\mathcal{N} = 4$ SYM theory, noticed in explicit calculations, is that the exponential structure of the infrared divergences extends also to the finite part of certain amplitudes. We will describe the conjectured iteration relations of Anastasiou, Bern, Dixon and Kosower (ABDK) and of Bern, Dixon and Smirnov (BDS) based on these observations, which suggest that any maximally helicity violating loop amplitude may be written in terms of the corresponding one loop amplitude. We end Section 2 with an outline of potential departures from these relations and the current state of the art in testing them.

For a variety of reasons, the identification and evaluation of the strong coupling counterpart of the partial amplitudes described in Section 2 is not entirely straightforward. In Section 3 we describe how the AdS/CFT duality can be used for this purpose. The main result is that, at strong coupling, partial amplitudes are closely related to a special class of polygonal, light-like Wilson loops. Thus, they may be evaluated as the area of certain minimal surfaces with boundary conditions fixed by the momenta of the massless particles participating in the scattering process[2]. The strong coupling calculations exhibit features analogous to their weak coupling counterparts, such as the presence of long distance/low energy divergences. Thus, in analogy with the weak coupling situation, the very definition of scattering amplitudes requires the presence of a regulator. Finding gauge-invariant regulators is not completely obvious in weakly-coupled gauge theories; by contrast, any regulator which may be realized on the string theory side of the AdS/CFT correspondence without direct reference to the color degrees of freedom of the open string sector is manifestly gauge-invariant. To set-up

[2] Certain features of partial amplitudes, such as the polarization of the scattered particles, are however not captured directly by their Wilson loop interpretation. This information is probably best captured in the vertex operator picture for the scattering process.

the computation we begin by introducing a D-brane as an infrared regulator. Actual computations are, however, carried out using a string theory analogue of dimensional regularization, obtained by taking the near horizon limit of $D(3-2\varepsilon)$ branes. While not yet clear how to extend the calculations beyond leading order, this regularization scheme has the advantage of being analogous to dimensional regularization as used in gauge theory calculations and thus of allowing a direct comparison of results.

We describe the calculation of the four-gluon scattering amplitude both in the strong coupling version of dimensional regularization as well as using an infrared cut-off which removes, in a gauge-invariant way, all dangerous low energy modes. This cut-off scheme is particularly appropriate for understanding the conformal properties of the amplitudes at strong coupling.

Partial amplitudes and (null polygonal) Wilson loops are *a priori* unrelated quantities. It is remarkable that a relation such as the one reviewed here can exist at all. This strong coupling observation led to the conjecture that MHV amplitudes and null polygonal Wilson loops are equal, order by order in weakly coupled perturbation theory as well. Developments in this direction will be reviewed elsewhere in this volume. The origins and full implications of such relations remain to be uncovered; it is however clear that they point to the existence of deep and powerful structures governing the dynamics of $\mathcal{N} = 4$ super-Yang–Mills theory and perhaps other four-dimensional gauge theories.

2. Scattering amplitudes at weak coupling

On-shell scattering amplitudes are perhaps the most basic quantities computed in any quantum field theory. The standard textbook approaches proceed to relate them through the LSZ reduction to Green's functions which are in turn computed in terms of Feynman diagrams. Each diagram evaluated separately is typically more complicated that the complete amplitude; the reason may be traced to Feynman diagrams not exhibiting and taking advantage of the symmetries of the theory — neither local nor global. The first instance where this shows up is for tree level amplitudes, where one notices major simplifications as all diagrams are added together.

Indeed, besides the scattering of physical polarizations, off-shell scattering amplitudes also describe the scattering of (unphysical) longitudinal polarizations of vector fields. On-shell, the equations of motion (or, more generally, Ward identities) guarantee the decoupling of such states. One may expose this decoupling at the Lagrangian level by choosing a physical gauge. The resulting gauge-fixed action does not, however, have a transparent use at the quantum level. As usual, in an off-shell covariant and renormalizable approach to loop corrections to scattering amplitudes, Faddeev–Popov ghosts are needed to cancel the contribution of unphysical fields propagating in loops.

The (generalized) unitarity-based method provides means of eliminating the appearance of unphysical degrees of freedom, while preserving all on-shell symmetries of the theory and avoiding the proliferation of Feynman diagrams. It allows the analytic construction of loop amplitudes in terms of tree-level amplitudes. Thus, it manifestly incorporates most (if not all) simplifying consequences of gauge invariance and symmetries. Simplicity of loop level amplitudes is to a large extent a consequence of simplicity of tree-level amplitudes.

In addition to the use of Feynman diagrams, there are several methods for computing tree-level scattering amplitudes: the Berends–Giele (off-shell) recursion relations [12], MHV vertex rules for gluons [13] and other fields [14, 15][3] and the BCFW recursion relations [17, 18]. We will review these methods in Section 2.2, referring the reader to the original literature and existing reviews [19–21] for further developments. After setting up the convenient notation and describing some of the general properties of scattering amplitudes, we will review the factorization of infrared divergences, discuss the unitarity method and illustrate it with several examples. We will then describe the BDS conjecture for the all-loop resummation of n-point MHV amplitudes, the potential corrections and the fact that such corrections indeed appear starting with the six-point two-loop amplitude. We will also describe the emergence of dual conformal invariance from the explicit expressions of amplitudes.

2.1. Organization, presentation and general properties

A good notation as well as an efficient organization of the calculation and result are indispensable ingredients for the calculation of scattering amplitudes, whether with Feynman diagrams or by other means. They are provided, respectively, by the spinor helicity method (for massless theories) and by color ordering, which we now review. These methods allow the decomposition of amplitudes in smaller, gauge-invariant pieces with transparent properties. An enlightening discussion of these topics may be found in [20].

2.1.1. Spinor helicity and color ordering

In a massless theory, solutions of the chiral Dirac equation provide [22–27] an excellent parametrization of momenta and polarization vectors which allows, among other things, the construction of physical polarization vectors without fixing noncovariant gauges. The main observation is that the sum over polarizations of a direct product of a Dirac spinor and its conjugate is

$$\sum_{s=\pm} u_s(k)\bar{u}_s(k) = -\not{k}\,. \tag{2.1}$$

[3] The MHV vertex rules have been successfully extended to one loop level in [16].

Upon projecting onto the chiral components one immediately finds that

$$u_-(k)\bar{u}_-(k) = -k_\mu \bar{\sigma}^\mu \,, \tag{2.2}$$

where as usual $\bar{\sigma} = (\mathbf{1}, -\boldsymbol{\sigma})$ are the Pauli matrices. The decomposition of a massless four-dimensional vector as a direct product of two 2-component commuting "spinors" follows also more formally from the fact that $p^2 = \det(p_\mu \bar{\sigma}^\mu)$, implying that the mass-shell condition requires that $p_\mu \bar{\sigma}^\mu$ has unit rank, *i.e.*

$$(k_\mu \bar{\sigma}^\mu)_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}} \,, \qquad \lambda \equiv u_-(k) \,, \qquad \tilde{\lambda} = \bar{u}_-(k) \,, \tag{2.3}$$

the multiplication of spinors follows from Lorentz invariance:

$$\langle ij \rangle = \varepsilon^{\alpha\beta} \lambda_{i\alpha} \lambda_{j\beta} \,, \qquad [ij] = -\varepsilon^{\dot{\alpha}\dot{\beta}} \tilde{\lambda}_{i\dot{\alpha}} \tilde{\lambda}_{j\dot{\beta}} \,. \tag{2.4}$$

In Minkowski signature λ and $\tilde{\lambda}$ are complex conjugate of each other. It is useful to promote momenta to (holomorphic) complex variables and the Lorentz group to $\mathrm{SL}(2,\boldsymbol{C})\times \mathrm{SL}(2,\boldsymbol{C})$. Then, λ and $\tilde{\lambda}$ are independent complex variables and the decomposition (2.3) exhibits a scaling invariance

$$\lambda \mapsto S\lambda \,, \qquad \tilde{\lambda} \mapsto \frac{1}{S}\tilde{\lambda} \,, \tag{2.5}$$

where S is an arbitrary constant. We will shortly see that scattering amplitudes have definite scaling properties under this transformation[4].

This parametrization of four-dimensional momenta allows the construction of simple expressions for the physical polarizations of massless vector fields. In general, gauge invariance requires that they be transverse, and that shifts by the momentum of the corresponding field should not change their form and properties. Moreover, in the frame in which the vector field propagates along a specified axis, they should take the standard form of circular polarization vectors.

A solution to these constraints can be constructed in terms of an arbitrary null (reference) vector ξ ($\xi_\mu \sigma^\mu_{\alpha\dot{\alpha}} = \xi_\alpha \tilde{\xi}_{\dot{\alpha}}$)

$$\begin{aligned}
\varepsilon^+_\mu(k,\xi) &= \frac{\langle \xi|\gamma_\mu|k]}{\sqrt{2}\langle \xi k\rangle} \,, & \varepsilon^+_{\alpha\dot{\alpha}}(k,\xi) &= \sqrt{2}\,\frac{\xi_\alpha \tilde{\lambda}_{\dot{\alpha}}}{\langle \xi k\rangle} \,, \\
\varepsilon^-_\mu(k,\xi) &= -\frac{[\xi|\gamma_\mu|k\rangle}{\sqrt{2}[\xi k]} \,, & \varepsilon^-_{\alpha\dot{\alpha}}(k,\xi) &= -\sqrt{2}\,\frac{\lambda_\alpha \tilde{\xi}_{\dot{\alpha}}}{[\xi k]} \,.
\end{aligned} \tag{2.6}$$

[4] For a Minkowski signature metric S is a pure phase.

The reference vector may be changed by a gauge transformation. Indeed, the transformation $\varepsilon(p) \mapsto \varepsilon(p) + A\,k$ for some A can be realized as a change of the reference vector:

$$\xi_\alpha \mapsto \xi_\alpha + A\,\langle \xi k\rangle \lambda_\alpha \tilde{\xi}_{\dot\alpha} \mapsto \tilde{\xi}_{\dot\alpha} - A\,[\xi k]\tilde{\lambda}_{\dot\alpha}\,. \tag{2.7}$$

This freedom of choosing independently the reference vector for each of the gluons participating in the scattering process is a very convenient tool for simplifying (somewhat effortlessly) the expressions for (tree-level) scattering amplitudes.

A clean organization of scattering amplitudes is a second useful ingredient in the calculation of scattering amplitudes at any fixed loop order L. Besides the organization following the helicity of external states implied by spinor helicity, at each loop order l an organization following the color structure is also possible and desirable, if only because amplitudes are separated in at least $(n-1)!$ gauge invariant pieces (here n is the number of external legs). For an SU(N) gauge theory with gauge group generators denoted by T^a, it is possible to show that any L-loop amplitude may be decomposed as follows:

$$\begin{aligned} A^{(L)} &= N^L \sum_{\rho\in S_n/\boldsymbol{Z}_n} \mathrm{Tr}\,[T^{a_{\rho(1)}}\dots T^{a_{\rho(n)}}] A^{(L)}(k_{\rho(1)}\dots k_{\rho(n)}, N) \\ &\quad + \text{multi}-\text{traces}\ , \end{aligned} \tag{2.8}$$

where the sum extends over all non-cyclic permutations ρ of $(1\dots n)$. This is equivalent to fixing one leg, say the first, and summing over all permutations of the other legs. The coefficients $A(k_{\rho(1)}\dots k_{\rho(n)}, N)$ are called color-ordered amplitudes. The multi-trace terms left unspecified in the equation above do not appear in the planar (large N) limit, which will be our main focus. We shall therefore ignore them in the following. In the same limit the N dependence of the partial amplitudes drops out

$$A(k_{\rho(1)}\dots k_{\rho(n)}, N) \stackrel{N\to\infty}{\longrightarrow} A(k_{\rho(1)}\dots k_{\rho(n)})\,. \tag{2.9}$$

The result of this limit are the planar partial amplitudes.

It is possible to argue for this presentation of amplitudes by inspecting the Feynman rules and noting that their color dependence separates from their momentum dependence. Perhaps the cleanest argument however is in terms of string theory diagrams [28]. Indeed, in string theory, gluon scattering amplitudes are computed in terms of Riemann surfaces with boundaries. Open string vertex operators, carrying Chan–Paton factors, are inserted on their boundaries, with color indices contracted along boundaries (see Fig. 1). As one integrates over the insertion points one sweeps over all possible orders of inserting the operators. The cyclic permutations, however, are naturally excluded because the boundaries in question are closed curves. The boundaries carrying no vertex operators contribute the explicit factors of N in equation (2.8).

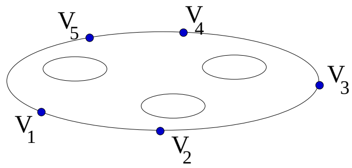


Fig. 1. The planar three-loop open string diagram contributing to the five-gluon scattering. The single-trace structure is manifest.

2.1.2. General properties of color ordered amplitudes

The general properties of color-ordered amplitudes follow from their construction in terms of Feynman diagrams (or string diagrams). The results of other constructions must obey the same properties. Some of them — such as the analytic structure — impose powerful constraints and in some cases uniquely determine the (tree-level) amplitudes. We collect here some of the more important properties [29]:

- cyclicity (this is a consequence of the cyclic symmetry of traces)

$$A(1,\ldots n) = A(2,\ldots,n,1)\,, \tag{2.10}$$

- reflection (this is a consequence of the fact that 3-point vertices pick up a sign under such a reflection and that an amplitude with n external legs has $(n+2L-2)$ three-point vertices)

$$A(1,\ldots n) = (-)^n A(n\ldots 1)\,, \tag{2.11}$$

- photon decoupling: In a theory with only adjoint fields, the diagonal U(1) does not interact with anyone. Thus, all amplitudes involving this field identically vanish. At tree-level this property may be captured by a Ward identity: fixing one of the external legs (n below) and summing over cyclic permutations $C(1,\ldots,n-1)$ of the remaining $(n-1)$ legs leads to a vanishing result:

$$\sum_{C(1,\ldots,n-1)} A(1,2,3,\ldots,n) = 0\,. \tag{2.12}$$

  In string theory language this is a consequence of the structure of the operator product expansion of vertex operators. At loop level this identity is modified and relates planar and non planar partial amplitudes [28].

- parity invariance (a color-ordered amplitude containing all choices of helicities of external legs is invariant if all helicities are reversed and simultaneously all spinors λ are replaced by the spinors $\tilde{\lambda}$ and vice-versa). This operation may be expressed as a fermionic Fourier-transform [30]

$$A(\lambda_i, \tilde{\lambda}_i, \eta_{iA}) = \int d^{4n}\psi \; \exp\left[i \sum_{i=1}^{n} \eta_{iA}\psi_i^A \right] A(\tilde{\lambda}_i, \lambda_i, \psi_i^A) \,, \quad (2.13)$$

- soft (momentum) limit: in the limit in which one momentum becomes soft the amplitude universally factorizes as follows

$$A^{\text{tree}}(1^+, 2, \dots, n) \longrightarrow \frac{\langle n\, 2\rangle}{\langle n\, 1\rangle\langle 1\, 2\rangle} A^{\text{tree}}(2, \dots, n) \,, \quad (2.14)$$

- collinear limit: in the limit in which two adjacent momenta become collinear $k_{n-1} \cdot k_n \to 0$ an L-loop amplitude factorizes as

$$A_n^{(L)}(1 \dots (n-1)^{h_{n-1}}, n^{h_n}) \;\mapsto\; \sum_{l=0}^{L} \sum_{h} A_{n-1}^{(L-l)}(1 \dots k^h) \times \text{Split}_{-h}^{(l)}\left((n-1)^{h_{n-1}}, n^{h_n} \right) \,, \quad (2.15)$$

where h_i denotes the helicity of the i-th gluon. For a given gauge theory, the l-loop splitting amplitudes $\text{Split}_{-h}^{(l)}((n-1)^{h_{n-1}}, n^{h_n})$ are universal functions [31] of the helicities of the collinear particles, the helicity of the external leg of the resulting amplitude and of the momentum fraction z defined as

$$z = \frac{\xi \cdot k_{n-1}}{\xi \cdot (k_{n-1} + k_n)} \,. \quad (2.16)$$

In the strict collinear limit one may also use $k_{n-1} \to zk$ and $k_n \to (1-z)k$ with $k^2 = (k_{n-1} + k_n)^2 = 0$. For example, the tree-level splitting amplitudes are:

$$\begin{aligned} \text{Split}_{-}^{(0)}(1^+, 2^+) &= \frac{1}{\sqrt{z(1-z)}} \frac{1}{\langle 1\, 2\rangle} \,, \\ \text{Split}_{-}^{(0)}(1^+, 2^-) &= \frac{z^2}{\sqrt{z(1-z)}} \frac{1}{[1\, 2]} \,, \\ \text{Split}_{+}^{(0)}(1^+, 2^-) &= \frac{(1-z)^2}{\sqrt{z(1-z)}} \frac{1}{\langle 1\, 2\rangle} \,. \end{aligned} \quad (2.17)$$

In $\mathcal{N}=4$ SYM theory Ward identities imply that all splitting amplitudes rescaled by their tree-level expressions are the same. Scattering amplitudes have similar factorization properties when more than two adjacent momenta become simultaneously collinear [31].

- multi-particle factorization: color ordered amplitudes exhibit poles if the square of the sum of some adjacent momenta vanishes. At tree-level this pole corresponds to some propagator going on-shell. At higher loops, the amplitude decomposes into a completely factorized part given by the sum of products of lower loop amplitudes and a non-factorized part, given in terms of additional universal functions. At one-loop level and in the limit $k_{1,m}^2 \equiv (k_1+\cdots+k_m)^2 \to 0$ one finds [32]

$$\begin{aligned}
&A_n^{1\,\text{loop}}(1,\ldots,n) \longrightarrow \\
&\sum_{h_p=\pm}\Big[A_{m+1}^{\text{tree}}(1,\ldots,m,k^{h_k})\frac{i}{k_{1,m}^2}A_{n-m+1}^{1\,\text{loop}}((-k)^{-h_k},m+1,\ldots,n) \\
&+A_{m+1}^{1\,\text{loop}}(1,\ldots,m,k^{h_k})\frac{i}{k_{1,m}^2}A_{n-m+1}^{\text{tree}}((-k)^{-h_k},m+1,\ldots,n) \\
&+A_{m+1}^{\text{tree}}(1,\ldots,m,k^{h_k})\frac{i\mathcal{F}(1\ldots n)}{k_{1,m}^2}A_{n-m+1}^{\text{tree}}((-k)^{-h_k},m+1,\ldots,n)\Big].
\end{aligned} \tag{2.18}$$

While color ordering (2.8) in the planar theory implies that complete amplitudes may be reconstructed from $(n-1)!$ gauge invariant partial amplitudes, the first four properties listed above imply that only a much smaller number is in fact necessary.

2.1.3. Some simple examples

Besides color ordering, scattering amplitudes can be organized following the number of negative helicity gluons. One can easily see that the amplitude with only positive helicity gluons as well as the amplitude with a single negative helicity gluons vanish identically at tree level in any gauge theory. This is realized by choosing the same reference vectors for all gluons with the same helicity and equal to the momentum of the negative helicity gluon. In absence of supersymmetry, quantum corrections spoil this conclusion. In the presence of supersymmetry, its Ward identities imply that this vanishing result is protected to all orders in perturbation theory. Indeed, the supersymmetry transformation rules are

$$\begin{aligned}
&[Q^a(\eta),\, g^{\pm}(k)] = \mp\Gamma^{\pm}(k,\eta)\lambda^{a\pm}(k)\,, \\
&[Q^b(\eta),\, \lambda^{b\pm}(k)] \;=\; \mp\Gamma^{\pm}(k,\eta)g^{\pm}(k)\delta^{ab} \mp i\Gamma^{\pm}(k,\eta)\phi_{\pm}^{ab}\varepsilon^{ab}\,, \\
&\Gamma(k,\eta)^{+} \;=\; \theta[\eta,\,k]\,, \qquad \Gamma(k,\eta)^{-} = \theta\langle\eta,\,k\rangle\,,
\end{aligned} \tag{2.19}$$

where η is a reference spinor. Acting with them on the vanishing matrix element $\langle 0|\lambda^{a+} g^{\pm} g^{+} \ldots g^{+}|0\rangle$ and using the fact that fermions have only helicity-conserving interactions, it immediately follows that the all-plus amplitude vanishes. Similarly, using the vanishing of $\langle 0|\lambda^{a+} g^{-} g^{+} \ldots g^{+}|0\rangle$ and making judicious choices for the reference spinor leads to the vanishing of the amplitude with a single negative helicity gluon [33]

$$A^{\rm tree}(g^{+}\ldots g^{+}) = 0\,, \qquad A^{\rm tree}(g^{-}g^{+}\ldots g^{+}) = 0\,. \tag{2.20}$$

In the following we will focus mainly on $\mathcal{N}=4$ SYM.

The first nonvanishing amplitude, having two negative helicity gluons, takes the form [34, 35]

$$A^{\rm tree}_{\rm MHV}(1^{+}\ldots i^{-}\ldots j^{-}\ldots n) = \frac{\langle ij\rangle^4}{\prod_{k=1}^{n}\langle k, k+1\rangle}\,, \tag{2.21}$$

where k is a cyclic index (*i.e.* $n+1\equiv 1$) and i and j are the labels of the negative helicity gluons. The fact that in $\mathcal{N}=4$ SYM the two gluon helicity states are related by supersymmetry makes it possible to show [89] that, to all loop orders, n-point MHV amplitudes are cyclicly symmetric, up to an overall factor of $\langle ij\rangle^4$ where i and j label, as above, the negative helicity gluons. Indeed, using supersymmetric Ward identities it is possible to relate the n-gluon amplitude to the two scalar, $(n-2)$-gluon amplitude. After interchanging the position of the two scalars, which does not affect the amplitude, one may use the same identities to obtain an amplitude with one of the two negative helicity gluons displaced to any position. It thus follows that, to any loop order L,

$$A^{(L)}_{\rm MHV} = A^{\rm tree}_{\rm MHV}\, \mathcal{M}^{(L)}(s_{i,i+1}, s_{i\ldots i+2}, \ldots)\,, \tag{2.22}$$

where $\mathcal{M}^{(L)}(s_{i,i+1}, s_{i\ldots i+2}, \ldots)$ is a cyclicly symmetric function of momenta and $s_{i\,\ldots\,j} = (k_i + k_{i+1} + \ldots + k_j)^2$. This factorization of the tree-level amplitude also holds for the infrared-singular terms of all amplitudes in all massless gauge theories. A similar expression holds in $\mathcal{N}=4$ SYM also for collinear splitting amplitudes introduced in (2.15):

$$\mathrm{Split}^{(L)}_{\lambda}(a^{h_a}, b^{h_b}) = \mathrm{Split}^{\rm tree}_{\lambda}(a^{h_a}, b^{h_b})\; r^{(L)}_{S}(z, s_{ab})\,, \tag{2.23}$$

where the momentum fraction z is defined in equation (2.16). A direct argument follows closely the one for MHV amplitudes. Alternatively, one may extract it by simply comparing the collinear limit of (2.22) and the expected behavior (2.15).

2.2. On-shell methods for tree-level amplitudes

The existence of MHV vertex rules (or CSW rules) was initially recognized in [13] as a consequence of the existence of a string theory [90] which captures the tree-level scattering amplitudes of Yang–Mills theories.

Perhaps the main observation [90], based on an earlier construction of Nair [91], is that MHV amplitudes are localized on complex lines in a half-position–half-momentum space called twistor space. Indeed, consider such amplitudes (2.21) together with their momentum conservation constraint written in terms of the spinors λ_i and $\tilde{\lambda}_i$:

$$A^{\rm tree}_{\rm MHV}(1^+ \dots i^- \dots j^- \dots n) = \frac{\langle ij\rangle^4}{\prod_{k=1}^n \langle k, k+1\rangle}\delta^4\left(\sum_{i=1}^n \lambda_{i\alpha}\tilde{\lambda}_{i\dot{\alpha}}\right) . \quad (2.24)$$

Let us, moreover, choose a signature such that λ_i and $\tilde{\lambda}_i$ may be treated independently and Fourier-transform these amplitudes with respect to all spinors $\tilde{\lambda}$. The calculation is simplified upon choosing an integral representation for the momentum conservation constraint:

$$\delta^4\left(\sum_{i=1}^n \lambda_{i\alpha}\tilde{\lambda}_{i\dot{\alpha}}\right) = \frac{1}{(2\pi)^4}\int d^4x_{\alpha\dot{\alpha}} \exp\left[ix^{\alpha\dot{\alpha}}\sum_{i=1}^n \lambda_{i\alpha}\tilde{\lambda}_{i\dot{\alpha}}\right] . \quad (2.25)$$

Since $\tilde{\lambda}$ enters the expression of the amplitude only though the momentum conservation constraint the Fourier-transform may be easily evaluated with the result

$$\begin{aligned} &\int \prod_i \frac{d^2\tilde{\lambda}_{i\dot{\alpha}}}{(2\pi)^2} e^{i[\mu_i,\tilde{\lambda}_i]} A^{\rm tree}_{\rm MHV}(1^+ \dots i^- \dots j^- \dots n) \\ &= \int \frac{d^4x_{\alpha\dot{\alpha}}}{(2\pi)^4}\prod_{i=1}^n \delta^2(\mu_{\dot{\alpha}} + \lambda^\alpha x_{\alpha\dot{\alpha}})\frac{\langle ij\rangle^4}{\prod_{k=1}^n\langle k, k+1\rangle} . \end{aligned} \quad (2.26)$$

The constraints imposed by the Dirac delta-functions imply that, in the space parametrized by the coordinates (λ, μ), all points characterizing the external momenta lie on the same complex line of slope $x_{\alpha\dot{\alpha}}$ — *i.e.* they are all collinear.

Such complete collinearity properties are characteristics of MHV amplitudes. Analysis of the available examples of non-MHV amplitudes as well as considerations of the twistor string led to the observation that amplitudes with n negative helicity gluons lie on a collection of $(n-1)$ intersecting lines. Such a conclusion may be reached by identifying triplets of external momenta which, when viewed from twistor space, are collinear. A useful tool for this purpose are differential operators which annihilates such triplets. An operator with this property is:

$$\tilde{F}_{ijk;\dot{\alpha}} = \mu_{i\dot{\alpha}}\langle jk\rangle + \mu_{j\dot{\alpha}}\langle ki\rangle + \mu_{k\dot{\alpha}}\langle ij\rangle\,. \qquad (2.27)$$

The fact that $\tilde{F}_{ijk;\dot{\alpha}}$ vanishes when the three vectors $(\lambda,\mu)_{i,j,k}$ are collinear follows from the fact that, on the one hand, $\tilde{F}_{ijk;\dot{\alpha}}$ is totally antisymmetric while on the other collinearity implies that one of the three vectors may be written as a linear combination of the other two.

When transformed to momentum space the collinear (multiplicative) operator $\tilde{F}$ becomes a first order differential operators F:

$$F_{ijk;\dot{\alpha}} = \langle ij\rangle\frac{\partial}{\partial\tilde{\lambda}_k^{\dot{\alpha}}} + \langle jk\rangle\frac{\partial}{\partial\tilde{\lambda}_i^{\dot{\alpha}}} + \langle ki\rangle\frac{\partial}{\partial\tilde{\lambda}_j^{\dot{\alpha}}}\,. \qquad (2.28)$$

Testing for collinear properties of the external momenta of an N^{n-1}MHV amplitude implies that the $(n-1)$ intersecting complex lines building it are such that each one of them has exactly two negative helicity gluons attached to it, including the point of intersection with another line which counts as a pair $(+-)$ with each member of the pair sitting on one of the two line. For example, it is possible to check that

$$[\tilde{\zeta}, F_{612}][\tilde{\zeta}, F_{234}][\tilde{\zeta}, F_{345}][\tilde{\zeta}, F_{561}]\, A(1^-2^-3^-4^+5^+6^+) = 0\,. \qquad (2.29)$$

Here, for convenience, we contracted the free index of the collinear operators with some arbitrary antiholomorphic spinor $\tilde{\zeta}$. This implies that $A(1^-2^-3^-4^+5^+6^+)$ may be represented as a sum of products of two MHV amplitudes, at least one of the two factors in each such product being annihilated by one of the four collinear operators appearing in equation (2.29). It was suggested in [90] that such a decomposition holds for all amplitudes; it was moreover suggested in [13] that this observation may be used to determine systematically all tree-level gluon scattering amplitudes through a set of Feynman-like rules which treat MHV amplitudes as effective vertices.

The tree-level rules necessary to carry out such a construction of tree-level have been layed out in [13] and extended to one-loop level in [16]:

- for an amplitude with n negative helicity gluons one uses $(n-1)$ MHV vertices. Similarly to the construction of amplitudes in terms of Feynman diagrams, one sums over all tree diagrams containing this number of MHV vertices which are consistent with the cyclic ordering of the external legs and the requirement that each vertex has an MHV helicity configuration. The same graph may appear several times with a different assignment of external legs and of helicities for internal lines.

- vertices are glued together using standard Feynman propagators[5]

$$\Delta(p) = \frac{i}{p^2 + i\varepsilon} \,. \tag{2.30}$$

  The $i\varepsilon$ prescription, while not crucial at tree-level, becomes extremely important at loop level [16].
- the momentum of an internal leg is not massless and thus, the corresponding holomorphic spinor does not exist. The corresponding spinor entering the vertices defined to be

$$P_\alpha \equiv P_{\alpha\dot{\alpha}}\eta^{\dot{\alpha}} \,, \tag{2.31}$$

  where $\tilde{\eta}$ is an arbitrary antiholomorphic commuting spinor. Then, the relevant factors entering the MHV amplitudes are

$$\langle k_i P\rangle = \langle k_i|P|\tilde{\eta}] = \lambda^\alpha_{k_i} P_{\alpha\dot{\alpha}}\tilde{\eta}^{\dot{\alpha}} \,. \tag{2.32}$$

As an example of the use of the MHV vertex rules let us consider the the NMHV amplitude in split-helicity configuration $A(1^-2^-3^-4^+\ldots n^+)$, initially found in [100] using off-shell recursion relations. The relevant two-vertex diagrams which satisfy the requirement that each vertex has a maximally helicity violating configuration are shown in Fig. 2. While a three-point $(--+)$ should be included both in Fig. 2(a) (corresponding to $i=3$) and in Fig. 2(b) (corresponding to $i=n$), no $(++-)$ exists[6]:

$$\begin{aligned} &A(1^-2^-3^-4^+\ldots n^+) \\ &= \sum_{i=3}^{n-1}\left[\frac{\langle 1P_i\rangle^3}{\langle P_i, i+1\rangle\langle i+1,i+2\rangle\ldots\langle n1\rangle}\right]\frac{1}{P_i^2}\left[\frac{\langle 23\rangle^3}{\langle P_i 2\rangle\ldots\langle iP_i\rangle}\right] \\ &\quad + \sum_{i=4}^{n}\left[\frac{\langle 12\rangle^3}{\langle 2P_i\rangle\langle P_i, i+1\rangle\ldots\langle n1\rangle}\right]\frac{1}{P_i^2}\left[\frac{\langle 34\rangle^3}{\langle P_i 2\rangle\ldots\langle iP_i\rangle}\right] \,. \end{aligned} \tag{2.33}$$

Here all products involving the momentum P_i carried by the internal line are defined as in (2.32) using the arbitrary spinor η.

---

[5] While this may appear unjustified given only physical polarizations propagate between vertices, the use of the Feynman propagator may be justified both based on the structure of poles of tree-level amplitudes as well as by the fact that the amplitudes used as vertices may be thought of as carrying a projector onto physical states for each of their external lines.

[6] This is indeed so either by invoking the fact that $(++-)$ is not an MHV helicity configuration or by choosing (complex) momenta such that all $\overline{\text{MHV}}$ three-point amplitudes vanish identically.

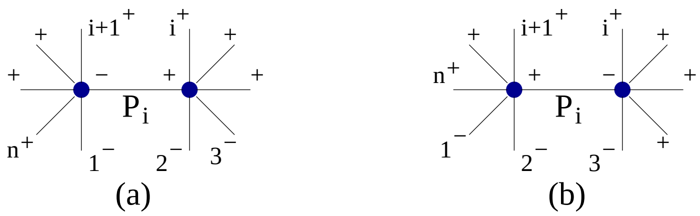


Fig. 2. The MHV diagrams contributing to the NMHV amplitude in the split-helicity configuration. In figure (a) the label i takes all values between 3 and $(n-1)$ while in figure (b) the label i takes all values between 4 and n.

Hints and arguments for the correctness of this approach to the calculation of tree-level scattering amplitudes may be constructed on a variety of basis. For example, one may check that all properties of scattering amplitudes discussed previously are satisfied by the MHV vertex construction. Some of these properties are manifest — such as the correct multi-particle factorization, where all propagators which may become singular in such limits appear explicitly in the expression of the amplitude. The collinear limits in which the number of negative helicity gluons does not change follows equally easily from the properties of MHV amplitudes; it is moreover possible to show that, if the number of negative helicity gluons changes under the collinear limit, the entire contribution comes from diagrams in which the collinear legs belong to the same three-point vertex[7]. Last but not least, the explicit appearance of the arbitrary spinor $\tilde{\eta}$ suggests that the expression of the amplitude breaks Lorentz invariance; this is however not the case; it was argued in [13] that the dependence on $\tilde{\eta}$ drops out of the amplitude, as it should.

Apparent violation of Lorentz invariance occurs also in standard Feynman diagram calculations in axial gauges $n \cdot A = 0$; there as well, the dependence on the fixed vector n cancels out in all gauge-invariant quantities. In fact, these two instances of breaking of Lorentz invariance at intermediate stages of calculations are not unrelated. Their relation emerged in the construction of the Lagrangian proof of the MHV vertex rules [92, 93]. The starting point is the Yang–Mills Lagrangian in light-cone gauge defined by $n \cdot A = 0$ with $n^2 = 0$, *i.e.* $n_{\alpha\dot{\alpha}} = \eta_\alpha \tilde{\eta}_{\dot{\alpha}}$. It is well-known that, after fixing this gauge and integrating out the component of the gauge field which is auxiliary one finds that

$$S_{\mathrm{LC\,SYM}} = \frac{1}{2g^2}\int d^4x \,\mathrm{Tr}[F^2] = \frac{1}{2g^2}\int dx^0 (L^{-+} + L^{-++} + L^{--+} + L^{--++}), \quad (2.34)$$

[7] This is indeed so because collinear limits in which a negative helicity gluon is replaced by a positive helicity gluon are trivial for MHV amplitudes.

where $L^{-^n+^m}$ contains n negative and m positive helicity gluons. The only term which departs from the MHV helicity structure is the cubic term L^{-++}. It was shown in [92, 93] that it is possible to construct a canonical transformation which maps $L^{-+} + L^{-++}$ to a quadratic Lagrangian L^{-+} for the transformed field. This transformation also generates additional infinitely many terms with MHV helicity configuration. As this transformation is carried out at the level of the Lagrangian, the momenta carried by the various fields are not null; nevertheless, the explicit momentum dependence of the generated terms comes in the form of MHV amplitudes with the spinors corresponding to off-shell momenta defined using the antiholomorphic spinor $\tilde{\eta}$ of the constant null vector used to fix the light-cone gauge. This construction puts on very firm ground the MHV vertex rules; it was also extended to include fermions in the fundamental representation of the gauge group [93]. Other representations may be accounted for without difficulty, by making use of the fact that the MHV vertices which are generated by the canonical transformation have the color dependence stripped off.

While the MHV vertex rules reduce substantially the number of contributions to any one amplitude, there exists nevertheless reason to look beyond them. One possibility, discussed in detail in [94], is to use non-MHV amplitudes are vertices and thus build a more recursive construction of higher-point amplitudes from lower-point ones. Such an approach leads to a further reduction of the number of diagrams contributing to amplitudes with given number of legs, assuming that all amplitudes with fewer legs have been computed. While a direct Lagrangian derivation seems difficult, this approach may nevertheless be justified by a variety of means. As with the MHV rules, it is possible to show that the result obeys all properties requires of tree-level scattering amplitudes in Yang–Mills theories. A further justification may be provided, through the use of MHV vertices, by showing that the use of non-MHV amplitudes as effective vertices amounts to a reorganization of the MHV diagrams.

Advances in the techniques for one-loop calculations in $\mathcal{N} = 4$ super-Yang–Mills theory [84], which will be discussed in the next section, triggered [95] the construction of the so-called on-shell or BCFW recursion relations [17]. Indeed, as we will discuss in the next section, the infrared singularities of loop amplitudes are, on the one hand, proportional to tree-level amplitudes and on the other they are determined as products of on-shell lower-point tree-level amplitudes. This observation may be turned into a recursion relation which relates higher-point and lower-point tree-level amplitudes. While initially justified using the expressions of one-loop amplitudes in $\mathcal{N} = 4$ SYM, this resulting recursion relation was proven in [18] by making use of only the factorization properties of scattering amplitudes and of complex analysis. This proof emphasizes the generality and power of on-shell recursion relations; they may be derived, with only minimal conceptual differences, for both massless and massive theories. In the following we will focus on massless theories, commenting only briefly on other cases.

The key observation that is at the basis of the on-shell recursion relations is the that, from the standpoint of scattering amplitudes momenta are just parameters. The fact that they are real is only a consequence of the fact that eventually they are interpreted as momenta of physical particles. Thus, from the perspective of constructing a function which has all the properties of amplitudes it is of course legal to treat momenta as complex. This may be interpreted as analytic continuation. The result can then be analytically continued back to real momenta. To make use of this observation, following [18], we single out two momenta p_i and p_j (the choice of momenta is, to a large extent, arbitrary; we will discuss shortly the origin of constraints on the choice of i and j) and shift them as

$$p_i \to p_i(z) = p_i + z\hat{\eta}_{ij}\,, \qquad p_j \to p_j(z) = p_j - z\hat{\eta}_{ij}\,, \tag{2.35}$$

such that overall momentum conservation is satisfied. Moreover, the vector $\hat{\eta}_{ij}$ is chosen such that the shifted momenta continue to be massless, *i.e.*

$$z^2\hat{\eta}_{ij}^2 + 2z\hat{\eta}_{ij}\cdot p_i = 0\,, \qquad z^2\hat{\eta}_{ij}^2 - 2z\hat{\eta}_{ij}\cdot p_j = 0\,. \tag{2.36}$$

A solution, which explicitly assumes that the momenta p_i and p_j are massless, is given by

$$(\eta_{ij})_{\alpha\dot{\alpha}} = \lambda_{i\alpha}\tilde{\lambda}_{j\dot{\alpha}}\,. \tag{2.37}$$

The original amplitude, evaluated on the unshifted momenta p_i and p_j, may be recovered from the shifted amplitude by a simple contour integral around the origin

$$A(1\dots n) = \oint\limits_{C_0} \frac{dz}{z} A(1\dots n; z)\,. \tag{2.38}$$

An important assumption in writing this equation is the further observation that all poles of the shifted amplitude lie at non-zero distance from the origin. This may be easily justified by considering the amplitudes expressed in terms of Feynman rules. Indeed, each vertex that appears in some Feynman diagram contributes a factor with polynomial dependence on momenta and thus cannot produce any poles in the vicinity of the origin. The only poles of the amplitude arise from the z dependence of the propagators whose structure implies that, apart from singular configurations of external momenta, all of the poles which may arise in tree-level amplitudes are located away from the origin of the complex z plane[8].

[8] Poles on the z-plane may drift close to the origin only in multi-particle factorization limits *of the unshifted amplitude.*

This argument also leads to another expression for the integral (2.38). Indeed, instead of taking the integral around the origin, one may interpret it as an integral on a contour around the point at infinity. As such, it will receive contributions from all poles at finite distance from the origin as well as from the pole at infinity (if any is present). As described above, all poles of the amplitude that exist at finite distance from the origin arise from propagators becoming singular for special values of z

$$P^2_{i,\dots,i+k} \mapsto P^2_{i,\dots,i+k}(z) = P^2_{i,\dots,i+k} + 2z\,\hat{\eta}_{ij} \cdot P_{i,\dots,i+k}\,. \qquad (2.39)$$

It is clear that this is also a simple pole. The multi-particle factorization properties of (the shifted) amplitudes determine for us the residue of this pole: it is given by the product of two tree-level amplitudes each having, besides the internal line going on-shell, the set of external lines that appear on each side of the singular propagator.

For definiteness and ease of notation, consider shifting the external momenta p_1 and p_n, denoted below by $\hat{1}$ and $\hat{n}$. Then, the amplitude is given by

$$\begin{aligned} A(1\dots n) &= \frac{1}{2\pi i}\oint\limits_{C_0} \frac{dz}{z} A(\hat{1},2\dots\hat{n};z) = -\frac{1}{2\pi i}\oint\limits_{C_\infty} \frac{dz}{z} A(\hat{1},2\dots\hat{n};z) \\ &= -\sum_{l,h}\left(-\frac{2\,\hat{\eta}_{1n}\cdot P_{1,\dots,l}}{P^2_{1,\dots,l}}\right) \\ &\quad\times \frac{A_{\mathrm{L}}(\hat{1},2\dots l,\hat{q}^h;z_{0l})A_{\mathrm{R}}(-\hat{q}^{-h},(l+1),\dots\hat{n};z_{0l})}{2\,\hat{\eta}_{1n}\cdot P_{1,\dots,l}} + \mathcal{C}_\infty \\ &= \sum_{l,h} A_{\mathrm{L}}(\hat{1},2\dots l,\hat{q}^h;z_{0l})\frac{1}{P^2_{1,\dots,l}}A_{\mathrm{R}}(-\hat{q}^{-h},(l{+}1),\dots\hat{n};z_{0l}) + \mathcal{C}_\infty\,. \end{aligned} \qquad (2.40)$$

The momentum $\hat{q}$ of the internal line is determined by momentum conservation and it depends on z. Here one sums over all possible helicity assignments for the propagator carrying momentum q, as required in the multi-particle factorization limit, and one also evaluates each term for the value of $z = z_{0l}$ which renders the propagator singular:

$$z_{0l} = \frac{P^2_{1,\dots,l}}{2\,\hat{\eta}_{1n}\cdot P_{1,\dots,l}}\,. \qquad (2.41)$$

The term denoted by $\mathcal{C}_\infty$ represents the contribution of the pole at $z=\infty$. It is possible to argue [18] using either Feynman diagrammatics or the MHV vertex rules that this contribution is absent if the shifted momenta are carried by gluons of like helicities or if p_j is carried by a gluon of positive helicity and p_i is carried by a gluon of negative helicity.

A natural contribution to the sum in equation (2.40) involves three-particle amplitudes. The use of complex momenta is crucial for interpreting these contributions which identically vanish in Minkowski signature since $\lambda = (\tilde{\lambda})^*$. Complex momenta allows one to choose, for any three momenta $p_{i,j,k}$, either the MHV or the $\overline{\text{MHV}}$ three-particle amplitude to be nonvanishing. It is the non-vanishing three-point amplitude which appears in the BCFW recursion relation.

Let us consider a simple example, the six-point amplitude in split helicity configuration $A(1^-2^-3^-4^+5^+6^+)$, and consider shifting p_3 and p_4. The diagrams representing the terms in equation (2.40) are shown in Fig. 3.

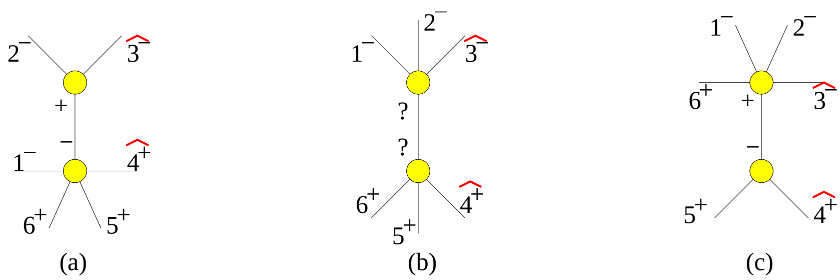


Fig. 3. The diagrammatic presentation of the terms in equation (2.40) for $A(1^-2^-3^-4^+5^+6^+)$.

$$
\begin{aligned}
T_1 &= \frac{\langle 2\hat{3}\,\rangle^3}{\langle \hat{3}\widehat{p_{23}}\rangle\langle \widehat{p_{23}}2\rangle}\,\frac{1}{p_{23}^2}\,\frac{\langle 1\widehat{p_{23}}\rangle^3}{\langle \widehat{p_{23}}\hat{4}\rangle\langle \hat{4}5\rangle\langle 56\rangle\langle 61\rangle}\,, \qquad z_{01} = \frac{p_{23}^2}{\langle 4|P_{23}|3]}\,,\\
T_2 &= 0\,,\\
T_3 &= \frac{[\widehat{p_{45}}6]^3}{[\widehat{p_{23}}6][61][12][2\hat{3}][\hat{3}\widehat{p_{23}}]}\,\frac{1}{p_{45}^2}\,\frac{[\hat{4}5]^3}{[5\widehat{p_{45}}][\widehat{p_{45}}\hat{4}]}\,, \qquad z_{03} = \frac{p_{45}^2}{\langle 4|p_{45}|3]}\,, \qquad (2.42)
\end{aligned}
$$

$$
\begin{aligned}
A_{1^-2^-3^-4^+5^+6^+} &= \frac{1}{\langle 5|p_{34}|2]}\\
&\quad\times\left[\frac{\langle 1|p_{23}|4]^3}{[23][34]\langle 56\rangle\langle 61\rangle p_{234}^2} + \frac{\langle 3|p_{45}|6]^3}{[61][12]\langle 34\rangle\langle 45\rangle p_{345}^2}\right]. \qquad (2.43)
\end{aligned}
$$

This is indeed the correct answer for the six-point split-helicity tree-level gluon amplitude, as may be verified by direct comparison with the results of [29].

The on-shell recursion relations have been extended to amplitudes involving fermions [98]. The manipulations are not different from those discussed above and we will not repeat them. The BCFW recursion relations have been extended to scattering amplitudes of massive particles as well [96, 97, 99]. The main complication comes from the fact that, for massive momenta, the constraints on the shift vector $\hat{\eta}_{ij}$ can no longer be solved in a simple way. For processes involving both massive and massless particles it is possible to choose the shifted momenta to be massless, case in which the only additional ingredient is the presence of some massive propagators:

$$\frac{1}{P^2_{l\ldots j\ldots l+m} + M^2_{l\ldots m}} \mapsto \frac{1}{P^2_{l\ldots j\ldots l+m}(z) + M^2_{l\ldots m}}\,. \tag{2.44}$$

The same manipulations in the complex z plane lead to the conclusion that an amplitude for any type of particles maybe be written as a sum of bilinears in lower-point amplitudes evaluated on momenta which are shifted from the desired values. We will not review the details here, but rather refer the reader to the original literature.

2.3. Loop amplitudes; generalized unitarity-based method

Having discussed general properties of scattering amplitudes, we now proceed to describe methods for their construction at loop level. The goal will be to use only on-shell information for this purpose. we will be assuming (quite accurately) that tree-level amplitudes are known. As we will see, the fact that Feynman diagramatics underlies the calculation of scattering amplitudes is a very important and useful guide. The properties of color ordered amplitudes discussed previously will serve as a useful guide for the completeness of the result. While most arguments apply to any (supersymmetric) gauge theory, we will be having in mind applications to $\mathcal{N} = 4$ SYM.

The idea that one can use only on-shell information to construct loop-level scattering amplitudes is of course very appealing. For starters, one would use complete lower-loop amplitudes as building blocks of higher amplitudes and, as such, one would build in the calculations simplifications due to symmetries and gauge invariance.

There is a long history associated with on-shell methods going back to the time of the analytic S-matrix theory. The idea is that, given the discontinuity of the amplitude in some channel, or a cut, one could use a dispersion integral to reconstruct the complete amplitude. In turn, the discontinuity of amplitudes is determined by the unitarity condition of the scattering matrix. Indeed, separating the interaction part of the scattering matrix

$$S = 1 + iT \tag{2.45}$$

and requiring that S is unitary $S^\dagger S = 1$ implies that

$$i(T^\dagger - T) = 2\,\Im T = T^\dagger T\,. \tag{2.46}$$

The right hand side is the product of lower loop on-shell amplitudes; this may be interpreted as a higher loop amplitude with some number of Feynman propagators replaced by on-shell (or "cut") propagators

$$\frac{1}{l^2 + i\varepsilon} \mapsto -2\pi i\theta(l^0)\delta(l^2)\ . \tag{2.47}$$

The difference on the left-hand side of equation (2.46) is interpreted as the discontinuity in the multi-particle invariant obtained by squaring the sum of the momenta of the cut propagators. This interpretation is a consequence of the $i\varepsilon$ prescription. Thus, this discontinuity at L-loops is determined in terms of products of lower-loop amplitudes. There are two types of cuts: singlet and non-singlet. In the former only one type of field crosses the cut. In the latter several types of particles (such as a complete multiplet in a supersymmetric theory) cross the cut. For the one-loop four-gluon amplitude this is illustrated in Fig. 4; in Fig. 4(a) the tree-level amplitudes require that only gluons can propagate along the cut propagators while in Fig. 4(b) fields with any helicity h can cross the cut, *i.e.* $h = \pm 1, \pm 1/2, 0$.

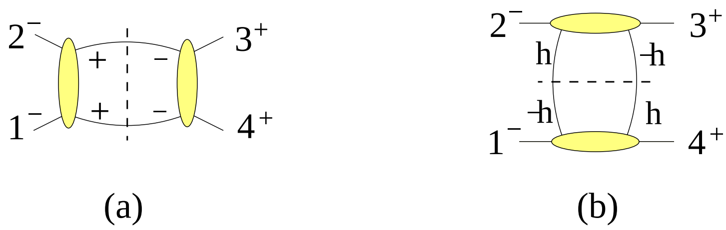


Fig. 4. Singlet and nonsinglet cuts of a one-loop four-gluon amplitude.

A reinterpretation of the equation (2.46) allows one to make use of the recent sophisticated techniques for evaluating Feynman integrals: identities, modern reduction techniques, differential equations, reduction to master integrals, *etc.* Indeed, besides representing the discontinuity of the amplitude, the right-hand-side of that equation also represents the part of the amplitude which contains the cut propagators. In fact, the right hand side of that equation contains a combination of parts of the amplitude containing two, three up to $(L+1)$ propagators.

It is not hard to see that separately each of these pieces are given by products of on-shell lower-loop amplitudes. This observation, originally due to Bern, Dixon, Dunbar and Kosower [76] and improved at one-loop level by Britto, Cachazo and Feng [84], allows to "cut" more than $(L+1)$ propagators for an L-loop amplitude, generalizing the unitarity relation (2.46). Similarly to regular cuts, generalized cuts can be either of singlet and nonsinglet types. These properties open the possibility of going beyond reconstructing the amplitudes from dispersion integrals: instead, one identifies the pieces of an amplitude with some prescribed set of propagators. Analyzing sufficiently many combinations of propagators one is guaranteed to be able to reconstruct the complete amplitude. Indeed, the fact that Feynman rules express scattering amplitudes as a sum of terms containing propagators and vertices implies that, after integral reduction, each term in the result contains part of the propagators present in the initial Feynman diagrams. By analyzing all possible generalized cuts one probes all possible combinations of propagators and thus all possible terms originating from the Feynman diagrams underlying the scattering process.

The argument above assumes that the (generalized) cuts are constructed in the regularized theory, *i.e.* in d-dimensions (perhaps with $d = 4 - 2\varepsilon$). In practice, however, it is much simpler to start by analyzing four-dimensional cuts, as one can saturate them with four-dimensional helicity states and also make use of the simplifying consequences of the supersymmetric Ward identities, such as (2.20). Four-dimensional cuts however potentially miss terms arising from the (-2ε)-dimensional components of the momenta in the momentum-dependent vertices. Such terms must be separately accounted for (either by considering d-dimensional cuts or by other means). In supersymmetric theories one can argue [77], based on the improved power-counting of the theory, that at one-loop level such terms do not exist through $\mathcal{O}(\varepsilon^0)$ (in the sense that through $\mathcal{O}(\varepsilon^0)$ one-loop amplitudes follow from four-dimensional cut calculations).

Let us illustrate this discussion with a simple example — that of the four gluon scattering amplitude in $\mathcal{N} = 4$ SYM. We will organize the calculation in terms of regular, two-particle cuts reinterpreted in the spirit of generalized unitarity-based method. There are two cuts — in the s and in the t-channels. Depending upon the external helicity configuration either one or both cuts are of non-singlet type, with the complete $\mathcal{N} = 4$ supermultiplet crossing it. As discussed previously, the helicity information in any MHV amplitude (such as this one) is carried by an overall factor of the tree-level amplitude (2.22). The remaining function may be thus computed by choosing the most convenient helicity configuration. Choosing $(1^-2^-3^+4^+)$ and evaluating the four-dimensional s-channel cut (Fig. 4(a)) one finds without difficulty that

$$A(l_2, 1^-, 2^-, l_1)A(-l_1, 3^+, 4^+, -l_2) = i s_{12} s_{23} A(1^-2^-3^+4^+) \times \frac{1}{(l_2+k_1)^2(l_2-k_4)^2} \,. \quad (2.48)$$

Here $s_{i\ldots j} = (k_i + k_{i+1} + \cdots + k_j)^2$ and we have used the fact that the cut condition allows one to write $2k_1 \cdot l_2 = (k_1 + l_2)^2$. In the propagator-like structures one recognizes the cut of a scalar box integral in ϕ^3 theory (that is, the integrand of a box integral in ϕ^3 theory in which two propagators have been removed and the on-shell condition for the corresponding momenta is imposed). At this stage one can argue based on the ultraviolet behavior of $\mathcal{N} = 4$ SYM that the full answer is given by the box integral whose s-channel cut we have just computed. Indeed, any other scalar integral diverges in a smaller number of dimensions than $\mathcal{N} = 4$ SYM and thus cannot appear in the final result. The conclusion of this argument can be confirmed by the evaluation of the (nonsinglet) t-channel cut (Fig. 4(b)). The simplest way to see this is to make use again of the equation (2.22) and note that up to the tree-level factor, the t-channel cut in the configurations $(1^-2^-3^+4^+)$ and $(1^+2^-3^-4^+)$ are the same. The latter is again a singlet cut, being given by a relabeling of equation (2.48). To summarize, we find [76] that

$$\mathcal{M}_4^{(1)} = \frac{i}{\pi^{d/2}} s_{12} s_{23} \int d^d l \, \frac{1}{l^2 (l - k_1)^2 (l - k_{12})^2 (k + k_4)^2} \equiv -\frac{1}{2} st \, I_4(s,t) \,, \tag{2.49}$$

thus reproducing the well-known result of [78].

The fact that a scalar box integral appears in the result of this calculation is not surprising. On general grounds one can show that in any four-dimensional massless theory, any one-loop scattering amplitude may be expressed as a linear combination of scalar box, triangle and bubble integrals (*i.e.* integrals with four, three and two propagators, respectively, and no loop-momentum factors appearing in the numerator — see Fig. 5(a), (b) and (c), respectively) with rational coefficients and a rational function which has no cuts in any channel. It was shown in [76] that in a supersymmetric theory this rational contributions are absent and that in such theories one-loop amplitudes are constructible using four-dimensional cuts.

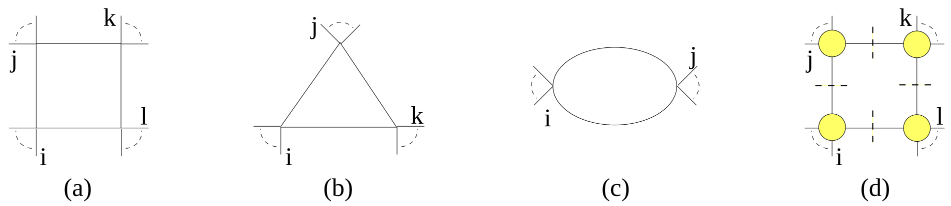


Fig. 5. Box (a), triangle (b) and bubble (c) scalar integrals. The clusters at each corner are constructed from color-adjacent external legs. If more than one external leg is present at a corner, then that corner is "massive" as the total momentum is no longer light-like. A quadruple cut (d) of an amplitude.

For one-loop amplitudes in $\mathcal{N}=4$ SYM one can do much better than the above by noticing [76] that the one-loop amplitudes with external states belonging to the same $\mathcal{N}=1$ vector multiplet may be written as a sum of box integrals. Besides a massless box integral which occurs only for four-gluon scattering, these integrals fall in five different classes: one-mass, easy two-mass, hard two-mass, three- and four-mass box integrals, depending on whether massive or massless momenta are injected at the corner of the box. The first two classes are shown in Fig. 6. The box integrals are defined and given in Ref. [79, 80] (with the four-mass boxes from Ref. [81–83]).

Since each box integral has a unique set of four propagators (*cf.* Fig. 5(a)), a quadruple cut (*i.e.* the result of eliminating four propagators and using the on-shell condition for their momenta) isolates a unique box integral and its coefficient [84]. The quadruple cut of the amplitude is, following the previous discussion, simply given by the product of four tree amplitudes evaluated on the solution of the on-shell conditions for the four propagators (*cf.* Fig. 5(d)). Thus:

$$c_{ijkl} = \frac{1}{2}\sum_{h_{q_i}} A(q_1, i \ldots j-1, -q_2) A(q_2, j \ldots k-1, -q_3) \times A(q_3, k \ldots l-1, -q_4) A(q_4, l \ldots i-1, -q_1)\Big|_{q_1^2=q_2^2=q_3^2=q_4^2=0}, \quad (2.50)$$

where the labels i,j,k,l are cyclic indices and label the first external leg at each corner of the box, counting clockwise. The sum runs over all possible helicity assignments on the internal lines. The factor of $1/2$ above is due to the four on-shell conditions having two solutions with equal values of the quadruple-cut box integrals. The sum over these solutions is implicit in the sum in equation (2.50). A further, implicit assumption is made in writing this expression. Any amplitude contains at least one box integral with one three-point corner. In Minkowski signature, *i.e.* with real momenta, the corresponding tree-level three-point amplitude vanishes identically. A non-vanishing result requires interpreting the loop momentum as complex, which is what we do.

We will later need the expression for the one-loop MHV amplitude. As we discussed, the four-point amplitude is given by (2.49). For an arbitrary number of external legs (larger than four), the result initially obtained in [76] (which can be reproduced using quadruple cuts and complex momenta) reads:

$$\mathcal{M}^{(1)}_{n=2m+1} = -\frac{1}{2}\sum_{r=2}^{m-1}\sum_{i=1}^{n}\left(t_{i-1}^{[r+1]}t_i^{[r+1]} - t_i^{[r]}t_{i+r+1}^{[n-r-2]}\right) I^{\rm 2m\,e}_{4;r;i} - \frac{1}{2}\sum_{i=1}^{n} t_{i-3}^{[2]}t_{i-2}^{[2]}\, I^{\rm 1m}_{4;i}$$

$$\mathcal{M}^{(1)}_{n=2m} = -\frac{1}{2}\sum_{r=2}^{m-2}\sum_{i=1}^{n}(t_{i-1}^{[r+1]}t_i^{[r+1]} - t_i^{[r]}t_{i+r+1}^{[n-r-2]})\, I^{2\mathrm{m\,e}}_{4;r;i} - \frac{1}{2}\sum_{i=1}^{n} t_{i-3}^{[2]}t_{i-2}^{[2]}\, I^{1\mathrm{m}}_{4;i}$$
$$-\frac{1}{2}\sum_{r=2}^{m-2}\sum_{i=1}^{n}(t_{i-1}^{[m]}t_i^{[m]} - t_i^{[m-1]}t_{i+m}^{[n-m-1]})\, I^{2\mathrm{m\,e}}_{4;m-1;i} \qquad (2.51)$$

where $I^{1\mathrm{m}}_{4;i}$ and $I^{2\mathrm{m\,e}}_{4;r;i}$ are the one-mass (Fig. 6(a)) and easy two-mass (Fig. 6(b)) integrals and $t_i^{[r]}$ are multi-particle invariants[9] $t_i^{[r]} = (k_i + \cdots + k_{i+r-1})^2$.

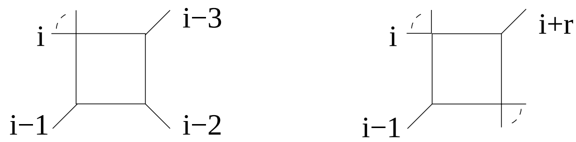


Fig. 6. The one-mass (a) and easy two-mass (b) integrals.

2.4. Calculations at higher loops

Higher loop calculations in $\mathcal{N} = 4$ SYM enjoy similar simplifications, though to a lesser extent. An analog of the 1-loop integral basis is not available, in the sense that the members of all proposed bases are in fact functionally dependent integrals[10]; moreover, not all integrals have sufficiently many propagators such that the cut condition on all of them does not completely freeze the integrals. It was pointed out [87] that under certain circumstances, after all propagators have been set on-shell, an additional propagator-like structure appears which can be used to set an additional on-shell condition. The lack of independence of the integral basis does not allow however a straightforward identification of the resulting product of tree amplitudes with the coefficient of the integral which is isolated by these cuts.

Generalized cuts can nevertheless be used to great effect to isolate parts of the full amplitude containing some prescribed set of propagators. One needs to ensure that integrals are not double-counted and that all cuts are consistent with each other. The previous arguments continue to hold and imply that the complete amplitude can be reconstructed from its d-dimensional generalized cuts. A detailed, general algorithm for assembling the amplitude was described in [88]. In a nutshell, starting from one (generalized) cut, one corrects it iteratively such that all the other cuts are correctly reproduced.

[9] This is a more compact notation for $s_{i\ldots(i+r-1)}$.

[10] Notable examples are the two-loop four-point integral basis with massless external legs [85] and the two-loop four-point integral basis with one massive external leg [86].

While fundamentally all cuts have equal importance, some of them exhibit more structure, which makes them useful starting points for the reconstruction of the amplitude. Such are the iterated two-particle cuts, defined as a sequence of two-particle cuts which at each stage reduces the number of loops by one unit[11]. Their importance stems from the fact that two-particle cuts with MHV amplitudes on both sides are naturally proportional to another MHV tree amplitude:

$$A^{\rm tree}(l_2^+ 1^+,\ldots,m_1^-,\ldots,m_j^-,\ldots,c_2^+,l_1^+)A^{\rm tree}(-l_1^-,(c_2+1)^+,\ldots,n^+,-l_2^-)$$
$$\propto A^{\rm tree}(1^+,\ldots,m_1^-,\ldots,m_j^-,\ldots,n^+)\,. \qquad (2.52)$$

The proportionality coefficient can be partial-fractioned into a sum of terms recognizable as cuts of box integrals with polynomial coefficients in external invariants. Repeatedly sewing an MHV tree amplitude onto such a construct yields another MHV tree amplitude as natural common factor.

For a four-particle amplitude the iteration of two-particle cuts can be explicitly solved and yields the so-called rung rule [103]. It states that the L-loop integrals which follow from iterated two-particle cuts can be obtained from the $(L-1)$-loop amplitudes by adding a rung in all possible (planar) ways and in the process multiplying the numerator by i times the invariant constructed from the momenta of the lines connected by the rung. This rule is illustrated in Fig. 7.

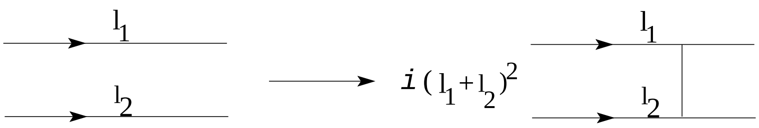


Fig. 7. The rung rule.

For higher multiplicity amplitudes the rung rule is less effective and it is necessary to explicitly evaluate the relevant iterated cuts. The strategy discussed in this section can be used to compute quite high loop amplitudes in $\mathcal{N}=4$ SYM. In the next section some explicit results obtained in this way will be discussed. It is important to keep in mind that, in contrast to one-loop calculations, four-dimensional cut calculations are not necessarily sufficient. Indeed, $\mathcal{O}(\varepsilon)$ terms at one-loop order may combine with singular terms from other loops to yield pole terms and/or finite terms at higher loops. Besides the obvious one-loop $\mathcal{O}(\varepsilon)$ arising from integrals whose integrand manifestly exhibit d-dimensional Lorentz-invariance, such terms may

[11] It is fairly clear that *a priori* there exist integrals which do not exhibit any two-particle cuts. Such contributions to the amplitude are not captured in this way. An example is provided by the four-loop four-gluon planar amplitude [60].

also arise from integrals containing explicitly the (-2ε) components of the loop momenta. Usually called "μ-integrals", at higher loops they contain explicitly the (-2ε) components of any number of the loop momenta[12]. One may decide whether such terms, not constructible from four-dimensional cuts, are present in the amplitude by comparing the infrared divergences emerging from a four-dimensional cut calculation with the expected structure implied by the soft and collinear factorization theorem.

An apparently alternative method for determining the four-dimensional cut-constructible part of scattering amplitudes was suggested in [101]. It is based on the observation that an amplitude possess singularities for specific momentum configurations, determined by their construction in terms of Feynman diagrams. These singularities must be correctly reproduced by any presentation of the amplitude in terms of simpler integrals. Moreover, singularities exhibited by these simpler integrals but not present in the sum of Feynman diagrams are spurious and should cancel out. The identification of the leading singularities of amplitudes proceeds by cutting the largest possible number of propagators and matching the result against a judicious choice of a(n overcomplete) basis of integral functions. At L-loops, integrals with $4L$ propagators are completely localized. Integrals with fewer propagators are however not. Additional propagator-like structures appear sometimes due to Jacobians coming from solving the cut conditions which are manifest. "Cutting" these additional "propagators" leads to a complete localization of the integrals and expresses the result in terms of product of tree-level amplitudes. This proposal has been tested for all the amplitudes constructed by independent means and appears to correctly reproduce their four-dimensional cut-constructible parts. The odd part of the two-loop six-point amplitude was constructed only through this method [115].

2.5. Some explicit higher loop results at low multiplicity

Using generalized unitarity, a number of higher loop amplitudes have been explicitly constructed and their properties analyzed. Due to the increase in the number of kinematic invariants with the number of external

[12] Such appear already at one-loop level if one is interested in expressions valid to all orders in ε. An example is provided by a parity-odd $\mathcal{O}(\varepsilon)$ term in the one-loop five-point amplitude [89]:

$$\mathcal{M}_5^{(1)\mu} \propto \int \frac{d^4p d^{-2\varepsilon}\mu}{(2\pi)^d} \times \frac{\varepsilon(k_1,k_2,k_3,k_4)\,\mu^2}{(p^2-\mu^2)((p-k_1)^2-\mu^2)((p-k_{12})^2-\mu^2)((p+k_{45})^2-\mu^2)((p+k_5)^2-\mu^2)}\,.$$

Similar integrals occur in all higher-multiplicity one-loop amplitudes. Two-loop analogs of such integrals will appear in Section 2.7.

particles, the complexity of the analysis increases as the number of external legs is increased. Here we discuss some of the available results, in increasing order of their complexity. First we will discuss the four-point amplitudes at two- and three-loops. As we have seen previously, splitting amplitudes provide a link between the lower and higher-point amplitudes; we will review them next and then proceed to the five-point amplitude.

The integrand of the four-point scattering amplitude at two-loops was found in [103] and evaluated in [104] using the results of [105]. It can be evaluated by a considering a double two-particle cut as in Fig. 8(a). As previously mentioned, they are correctly captured by the rung rule. It is instructive to follow the details of the calculation in this relatively simple case and in the process also have an explicit example of the application of the rung rule; they may in fact be constructed by iteratively using the equation (2.48). For the purpose of the calculation it is necessary to pick some helicity assignment; we will choose $(1^-2^-3^+4^+)$. Thus, we need to evaluate

$$A_4^{\rm tree}(l_2, k_1^-, k_2^-, l_1) A_4^{\rm tree}(-l_1, -l_4, -l_3, -l_2) A_4^{\rm tree}(l_4, k_3^+, k_4^+, l_3)\,, \quad (2.53)$$

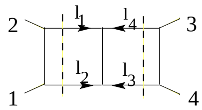
where the helicities of the cut lines are fixed by the requirement that the tree-level amplitudes are nonvanishing. The product of the first two tree-level amplitudes may be easily reorganized following equation (2.48) to be

$$\begin{aligned} &A_4^{\rm tree}(l_2, k_1^-, k_2^-, l_1) A_4^{\rm tree}(-l_1, -l_4, -l_3, -l_2) \\ &= i s_{12} (k_2 - l_4)^2 A(-l_3, 1^-, 2^-, -l_4) \frac{1}{(l_2 - k_1)^2 (l_2 + l_3)^2}\,. \end{aligned} \quad (2.54)$$

Further application of equation (2.48) leads to a final expression for the product in equation (2.53):

Fig. 8. The double two-particle and the three-particle cuts determining the four-gluon scattering amplitude in $\mathcal{N} = 4$ SYM at two-loop level.

$$
\begin{aligned}
&A_4^{\rm tree}(l_2,k_1^-,k_2^-,l_1)A_4^{\rm tree}(-l_1,-l_4,-l_3,-l_2)A_4^{\rm tree}(l_4,k_3^+,k_4^+,l_3)\\
&= is_{12}(k_2-l_4)^2\frac{1}{(l_2-k_1)^2(l_2+l_3)^2}A_4^{\rm tree}(-l_3,1^-,2^-,-l_4)A_4^{\rm tree}(l_4,k_3^+,k_4^+,l_3)\\
&= A_4^{\rm tree}(k_1^-,k_2^-,k_3^+,k_4^+)\\
&\quad\times\left[is_{12}(k_2-l_4)^2\frac{1}{(l_2-k_1)^2(l_2+l_3)^2}\right]\left[is_{12}s_{23}\frac{1}{(l_3-k_1)^2(l_3+k_4)^2}\right]\\
&= -s_{12}^2s_{23}A_4^{\rm tree}(k_1^-,k_2^-,k_3^+,k_4^+)
\end{aligned}
$$

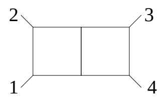


, (2.55)

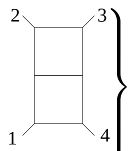
where we have used again the cut conditions to organize the result in terms of propagators. One notes without difficulty that momentum conservation implies the cancellation of the numerator factor $(k_2-l_4)^2$ against the denominator factor $(l_3-k_1)^2$ in the last equality. This cancellation is crucial for having a Feynman integral interpretation for the generalized cut in equation (2.53). The conclusion of this calculation is that the two-loop four-gluon amplitude contains the double-box integral whose cut appears in the equation above. This calculation also illustrates the application of the rung rule (*cf.* Fig. 7).

The other double two-particle cuts are obtained by simple relabeling of the previous calculation. Thus, one finds that they imply that the two-loop four-gluon amplitude in $\mathcal{N}=4$ SYM is (for any choice of helicity assignment) given by [103]

$$\mathcal{M}_4^{(2)}(k_1,k_2,k_3,k_4)=-\frac{1}{4}s_{12}s_{23}\left\{s_{12}\right.$$



$$+s_{23}$$



$$\left.\right\}. \quad (2.56)$$

The ultraviolet behavior of $\mathcal{N}=4$ SYM suggests[13] that this is indeed the complete amplitude, a fact confirmed by the evaluation of the three-particle cut.

Similar (though somewhat lengthier) manipulations or repeated application of the rung rule leads to the three-loop four-gluon amplitude [49, 103].

[13] Superspace arguments [102] imply that at two-loops, $\mathcal{N}=4$ SYM is logarithmically-divergent in seven dimensions. This is however only suggestive of (2.56) being the full answer, as there may exist more divergent contributions whose leading ultraviolet behavior cancels out.

The notable fact is that, unlike the two-loop amplitude, the three-loop integrand retains some dependence of the loop momentum in its numerator.

A link between lower and higher point amplitudes at any number of loops is provided by the splitting amplitudes introduced in equation (2.15). A unitarity-based all-order proof of those equations as well as a means of directly evaluating the splitting amplitudes was discussed in [31] for arbitrary gauge theories. Similar to scattering amplitudes, they are determined by tree-level information up to the appropriate treatment of the intermediate momentum p in equation (2.15) which must be kept massive throughout the calculation. The one-loop splitting amplitudes can be obtained without difficulty either by considering collinear limits of higher loop amplitudes [32] or by direct evaluation [106]. The two-loop splitting amplitude in $\mathcal{N}=4$ SYM theory have been computed and their properties have been analyzed in [88, 104].

2.5.1. A possible integral basis at higher loops; Conformal integrals

$\mathcal{N}=4$ SYM is a conformal field theory at the quantum level; conformal invariance may be observed in correlation functions of operators of definite (anomalous) dimension. In the context of the AdS/CFT correspondence this symmetry is related to the existence of an exact SO(2,4) isometry of the anti-de-Sitter space. At the level of on-shell scattering amplitudes however (super)conformal invariance is obscured beyond tree level; after removing the effects of the (infrared) regulator which explicitly breaks it, the momentum space realization of the generators of the conformal group still exhibits anomalies analogous to the holomorphic anomaly of collinear operators [109].

It was observed in [110] by explicitly inspecting the known results for the one-, two- and three-loop four-gluon amplitudes that the integrals appearing in the rescaled amplitude $\mathcal{M}_4$ exhibit, if regularized by keeping the external legs off-shell, (in a sense we will describe below) an SO(2,4) symmetry apparently unrelated to the four-dimensional conformal group. To expose this symmetry one solves the momentum conservation constraint at each vertex by writing each momentum as a difference of two variables

$$p_i = x_i - x_{i+1}\,. \tag{2.57}$$

We use the notation p_i here to denote generically both external momenta as well as loop momenta. These variables define the position x_i of the vertices of the dual graph. This way the momentum conservation constraint is replaced by an invariance under uniform shifts of the dual coordinates x_i. Moreover, their Lorentz transformation properties are identical to those of the momenta. Since the dual variables are unconstrained one may also define an inversion operator

$$I = \sum_i I_i \,, \qquad I : x_i^\mu \mapsto \frac{x_i^\mu}{x_i^2} \,. \tag{2.58}$$

An off-shell regularization of infrared divergences allows the construction of planar loop integrals which are invariant under such a transformation. Indeed, properties of dual graphs imply that in any planar integral, any inverse propagators can be written as the square of a difference of two x_i-s. Thus, propagators transform homogeneously (with weight $(+1)$ in each of the two x_i-s) under the transformation (2.58). The weight of each x_i in the transformation of all propagators defining the integral equals twice the number of propagators containing this variable. The four-dimensional loop integration measure transforms homogeneously (with weight (-4)). It therefore follows that a numerator factor transforming homogeneously with the appropriate weight would render the integral invariant under simultaneous inversion of all dual variables x_i. Simple graphical rules capturing the transformation under inversion of an integral are illustrated in Fig. 9. Let us illustrate the details by discussing a simple example — the one-loop box integral shown in Fig. 9(a). Up to numerator factors, the relevant integral is

$$I_a = \int d^4x_5 \, \frac{1}{x_{51}^2 x_{52}^2 x_{53}^2 x_{54}^2} \,, \tag{2.59}$$

each of the propagators present is denoted by a solid line in Fig. 9(a). As mentioned, under inversion this integral transforms as

$$I : I_a \mapsto \int \frac{d^4x_5}{(x_5^2)^4} \, \frac{(x_5^2 x_1^2)(x_5^2 x_2^2)(x_5^2 x_3^2)(x_5^2 x_4^2)}{x_{51}^2 x_{52}^2 x_{53}^2 x_{54}^2} = x_1^2 x_2^2 x_3^2 x_4^2 \, I_a \,, \tag{2.60}$$

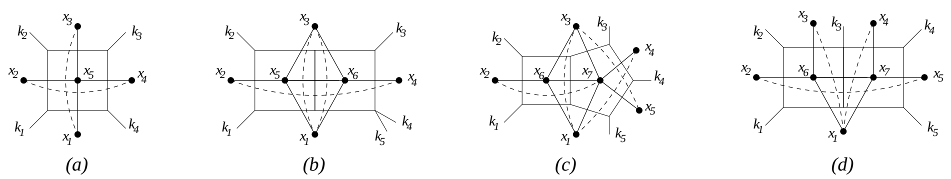


Fig. 9. Examples of pseudo-conformal integrals. Points x_i label the dual graph, a solid line connecting two points x_i and x_j corresponds to a factor of $1/x_{ij}^2$ while a dashed line corresponds to a factor of x_{ij}^2. The integral is pseudo-conformal if the difference between the number of solid lines and dashed lines at a vertex equals 4 if the vertex is inside the loops of the original graph and zero for all other points x_i. The graphs (b), (c) and (d) show that the integrals appearing in the even part of the five-point two-loop amplitude [107, 108] are pseudo-conformal.

i.e. it transforms homogeneously with weight (+2) for each of the coordinates x_i unrelated to the loop momentum. If the external momenta are massless, $k_i^2 = 0$, then the only way to compensate for this transformation is by adding a factor of $s_{12}s_{23} = x_{13}^2 x_{24}^2$ since

$$I : x_{13}^2 x_{24}^2 \mapsto \frac{x_{13}^2 x_{24}^2}{x_1^2 x_2^2 x_3^2 x_4^2} \tag{2.61}$$

and thus $s_{12}s_{23}I_a$ is invariant. If two opposite external legs are massive — say $k_1^2 \neq 0$ and $k_3^2 \neq 0$ — a further numerator factor is possible since $k_1^2 k_3^2 = x_{12}^2 x_{34}^2$ no longer vanishes and transforms as

$$I : x_{12}^2 x_{34}^2 \mapsto \frac{x_{12}^2 x_{34}^2}{x_1^2 x_2^2 x_3^2 x_4^2} \,. \tag{2.62}$$

Further possibilities occur if more of the external legs are massive. A similar discussion may be carried out at higher loops [110].

One can also define a dilatation generator, under which all integrals transform homogeneously and carry the same weight as under rescaling of momentum variables. Together with translations of the dual variables and their inversion this generate an SO(2,4) algebra called *dual conformal symmetry.*

It turns out that all integrals which appear in the four-gluon amplitude through three-loops are invariant under dual conformal transformations if they are regularized with an off-shell regulator. The amplitudes are however constructed assuming dimensional regularization; due to the change in the dimension of the integration measure this regularization breaks the inversion invariance. Dimensionally-regularized integrals which, if regularized with an off-shell regulator are invariant under dual conformal transformations are called *pseudo-conformal integrals* [169]. It is interesting to note that by this definition μ-integrals are also pseudo-conformal. Indeed, with an off-shell regulator the integrand is treated as four-dimensional and thus vanishes identically for these integrals.

The appearance of pseudo-conformal integrals is not limited to four-point amplitudes in $\mathcal{N} = 4$ SYM; they also generate the scalar factor of n-point one-loop MHV amplitudes, the even part of the two-loop five-point amplitude (*cf.* Fig. 9) and the even part of the two-loop six-point amplitude [111] which we shall review shortly.

It is not clear what is the underlying reason for the appearance of dual conformal invariance at weak coupling. It is, moreover, not clear whether its appearance persists to all loop orders (perhaps up to integrals whose integrands vanish identically in four dimensions [111]). It is nevertheless a useful guide in organizing higher loop calculations. If it indeed survives to all orders in perturbation theory it provides a general (though nevertheless overcomplete at higher loops) basis of integrals organizing parts of higher loop amplitudes in $\mathcal{N}=4$ SYM.

2.5.2. Soft/Collinear factorization

As it is clear from the discussion in the previous section, a general feature of massless gauge theories in four dimensions is the existence of infrared singularities[14]. Unlike ultraviolet divergences they cannot be renormalized away, but rather should cancel once gluon scattering amplitudes are combined to compute infrared-safe quantities. Their structure has been thoroughly studied and understood (see *e.g.* [36–48]). Here we briefly review some of the results specializing them, following [49], to the case of $\mathcal{N}=4$ SYM in the planar limit.

In a gauge theory, infrared singularities of scattering amplitudes come from two sources: the small energy region of some virtual particle

$$\int \frac{d\omega}{\omega^{1+\varepsilon}} \propto \frac{1}{\varepsilon} \tag{2.63}$$

and the region in which some virtual particle is collinear with some external one

$$\int \frac{dk_{\rm T}}{k_{\rm T}^{1+\varepsilon}} \propto \frac{1}{\varepsilon}\,. \tag{2.64}$$

Since they can occur simultaneously, at L-loops the infrared singularities lead to an $1/\varepsilon^{2L}$ pole.

The structure of soft and collinear singularities in a massless gauge theory in four dimensions has been extensively studied. The realization that soft and virtual collinear effects can be factorized in a universal way, together with the fact [50] that the soft radiation can be further factorized from the (harder) collinear one led to a three-factor structure for gauge theory scattering amplitudes [51–53]:

$$\begin{aligned}\mathcal{M}_n \;=\; & \left[\prod_{i=1}^{n} J_i\left(\frac{Q}{\mu},\alpha_{\rm s}(\mu),\varepsilon\right)\right] \\ & \times S\left(k,\frac{Q}{\mu},\alpha_{\rm s}(\mu),\varepsilon\right)\times h_n\left(k,\frac{Q}{\mu},\alpha_{\rm s}(\mu),\varepsilon\right)\,.\end{aligned} \tag{2.65}$$

Here the product runs over all the external lines, Q is the factorization scale, separating soft and collinear momenta, μ is the renormalization scale and $\alpha_{\rm s}(\mu)=(g(\mu)^2)/(4\pi)$ is the running coupling at scale μ.

[14] Ultraviolet divergences may of course be present as well; as previously mentioned, our focus is $\mathcal{N}=4$ SYM theory, which is free of such divergences.

Both $h_n(k, Q/\mu, \alpha_s(\mu), \varepsilon)$ and the rescaled amplitude $\mathcal{M}_n$ are vectors in the space of color configurations available for the scattering process. The soft function $S(k, Q/\mu, \alpha_s(\mu), \varepsilon)$ is a matrix acting on this space and it is defined up to a multiple of the identity matrix. It captures the soft gluon radiation and it is responsible for the purely infrared poles. For this reason it can be computed in the eikonal approximation in which the hard partonic lines are replaced by Wilson lines. The "jet" functions $J_i(Q/\mu, \alpha_s(\mu), \varepsilon)$ do not alter the color flow and contain the complete information on collinear dynamics of virtual particles. Finally, $h_n(k, Q/\mu, \alpha_s(\mu), \varepsilon)$ contains the effects of highly virtual fields and is finite as $\varepsilon \to 0$. The jet and soft functions can be independently defined in terms of specific matrix elements.

The factorization scale Q is arbitrary (within some physical limits); it is simply used to construct the equation (2.65). While it enters in each of the three factors on the right hand side, the (rescaled) amplitude $\mathcal{M}_n$ is independent of it. This independence, akin to the independence on the renormalization scale μ, leads to an evolution equation for the soft function.

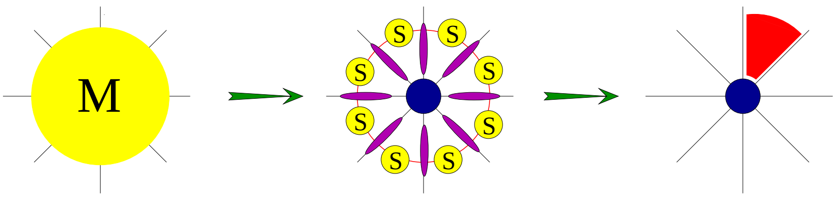


Fig. 10. Soft/Collinear factorization and its planar limit.

In the planar limit the soft/collinear factorization formulae simplify significantly. Since in this limit there is a single color structure, all color-space vectors reduce to a single component. The fact that the soft function is defined only up to an overall function implies that, in the planar limit, it can be completely absorbed in the jet functions J_i. The planar limit implies that all interactions included in the thus redefined jet functions are confined to adjacent gluons, *cf.* Fig. 10. In this limit it is then instructive to consider a two-gluon process — simply the decay of a color-singlet state into two gluons. Direct application of the factorization equation identifies then the square of the jet function with the amplitude of this process which is, by definition, the Sudakov form factor $\mathcal{M}^{[gg\to 1]}(\lambda(s_{i,i+1}/\mu), s_{i,i+1}, \varepsilon)$ if the two gluons have momenta k_i and k_{i+1}. It therefore follows that, in the planar limit, a generic n-point scattering amplitude factorizes as

$$\mathcal{M}_n = \left[\prod_{i=1}^{n} \mathcal{M}^{[gg\to 1]}\left(\frac{s_{i,i+1}}{\mu}, \lambda(\mu), \varepsilon\right)\right]^{1/2} h_n \,, \tag{2.66}$$

where $\lambda(\mu)=g(\mu)^2N$ is the 't Hooft coupling. As before, here $\mathcal{M}_n$ denotes a generic resumed amplitude, rescaled by the corresponding tree-level amplitude.

Similarly to the soft and jet functions, the factorization (2.66) implies an evolution equation and a renormalization group equation for the factors $\mathcal{M}^{[gg\to 1]}\big(Q^2/\mu^2,\lambda(\mu),\varepsilon\big)$. The same equations follow independently from the gauge invariance and the properties of the form factor. They read

$$\frac{d}{d\ln Q^2}\mathcal{M}^{[gg\to 1]}\left(\frac{Q^2}{\mu^2},\lambda,\varepsilon\right) = \frac{1}{2}\left[K(\varepsilon,\lambda)+G\left(\frac{Q^2}{\mu^2},\lambda,\varepsilon\right)\right] \times\mathcal{M}^{[gg\to 1]}\left(\frac{Q^2}{\mu^2},\lambda,\varepsilon\right), \tag{2.67}$$

where the function K contains only poles and no scale dependence. The functions K and G themselves obey renormalization group equations [37–39], [54, 55]

$$\begin{aligned}\left(\frac{d}{d\ln\mu}+\beta(\lambda)\frac{d}{dg}\right)(K+G) &= 0\,,\\ \left(\frac{d}{d\ln\mu}+\beta(\lambda)\frac{d}{dg}\right)K(\varepsilon,\lambda) &= -\gamma_K(\lambda)\,.\end{aligned} \tag{2.68}$$

In $\mathcal{N}=4$ SYM they may be solved exactly and explicitly in terms of the expansion coefficients of the cusp anomalous dimension

$$f(\lambda)\equiv\gamma_K(\lambda)=\sum_l a^l\gamma_K^{(l)} \tag{2.69}$$

and another set of coefficients defining the expansion of G:

$$G\left(\frac{Q^2}{\mu^2},\lambda,\varepsilon\right)=\sum_l \mathcal{G}_0^{(l)}a^l\left(\frac{Q^2}{\mu^2}\right)^{l\varepsilon}\,, \tag{2.70}$$

where $a=\lambda/(8\pi^2)\,(4\pi e^{-\gamma})^{\varepsilon}$ the coupling constant customarily used in higher loop calculations. An important ingredient in solving these equations is that in the dimensionally-regularized $\mathcal{N}=4$ SYM theory the beta function is

$$\beta(\lambda)=-2\varepsilon\lambda\,, \tag{2.71}$$

i.e. in the presence of the dimensional regulator the theory is infrared-free.

The solution for K and G may then be used to reconstruct the Sudakov form factor (2.67) which, in turn, leads to the following expression for the factorized amplitude [49]:

$$\begin{aligned}\mathcal{M}_n &= \exp\left[-\frac{1}{8}\sum_{l=1}^{\infty} a^l \left(\frac{\gamma_K^{(l)}}{(l\varepsilon)^2}+\frac{2\mathcal{G}_0^{(l)}}{l\varepsilon}\right)\sum_{i=1}^{n}\left(\frac{\mu^2}{-s_{i,i+1}}\right)^{l\varepsilon}\right] \times\ h_n \\ &= \exp\left[\sum_{l=1}^{\infty} a^l \left(\frac{1}{4}\gamma_K^{(l)}+\frac{l}{2}\mathcal{G}_0^{(l)}\right)\hat{I}_n^{(1)}(l\varepsilon)\right] \times\ h_n\,. \qquad (2.72)\end{aligned}$$

The definition of $\hat{I}_n^{(1)}(\varepsilon)$ may be easily seen to be

$$\hat{I}_n^{(1)} = -\frac{1}{\varepsilon^2}\sum_{i=1}^{n}\left(\frac{\mu^2}{-s_{i,i+1}}\right)^{\varepsilon}\,. \qquad (2.73)$$

This function captures the divergences of the planar one-loop n-point amplitudes in $\mathcal{N}=4$ SYM.

The first few coefficients in the weak coupling expansion of the cusp anomalous dimension and G function (2.70) have been evaluated directly [49, 56–62] with the result

$$f(\lambda) = \frac{\lambda}{2\pi^2}\left(1-\frac{\lambda}{48}+\frac{11\,\lambda^2}{11520}-\left(\frac{73}{1290240}+\frac{\zeta_3^2}{512\pi^6}\right)\lambda^3+\cdots\right), \quad (2.74)$$

$$\begin{aligned}G(\lambda) &= -\zeta_3\left(\frac{\lambda}{8\pi^2}\right)^2+\left(6\zeta_5+5\zeta_2\zeta_3\right)\left(\frac{\lambda}{8\pi^2}\right)^3 \\ &\quad -2(77.56\pm 0.02)\left(\frac{\lambda}{8\pi^2}\right)^4+\cdots\,. \qquad (2.75)\end{aligned}$$

The three-loop cusp anomaly was initially extracted [57], from the impressive calculations [63, 64] of the QCD splitting functions, using the maximal transcendentality conjecture; its value was confirmed by a direct amplitude calculation in [49]. Similarly, the value of the three-loop correction to the G-function can be independently confirmed by using maximal transcendentality conjecture and the QCD results for the quark and gluon form factors [65, 66].

Using the integrability of the gauge theory dilatation operator, Beisert, Eden and Staudacher (BES) [3] constructed an integral equation whose solution is the universal scaling function (conjecturally equal to the cusp anomalous dimension) to all orders in perturbation theory. This equation was solved in a weak coupling expansion [3] and also in a strong coupling expansion [67–69]. Using the AdS/CFT correspondence the first few coefficients in the strong coupling expansion were evaluated in [70–72]. The leading term in the strong coupling expansion of G was computed in [73]:

$$f(\lambda) \;=\; \frac{\sqrt{\lambda}}{\pi}\left(1-\frac{3\ln 2}{\sqrt{\lambda}}-\frac{\mathrm{K}}{\lambda}+\cdots\right), \qquad \lambda\to\infty\,, \tag{2.76}$$

$$G(\lambda) \;=\; (1-\ln 2)\frac{\sqrt{\lambda}}{8\pi}+\cdots\,, \qquad \lambda\to\infty\,; \tag{2.77}$$

here $\mathrm{K}=\sum_{n\geq 0}(-1)^n/(2n+1)^2 \simeq 0.9159656\ldots$ is the Catalan constant.

The properties of the collinear anomalous dimension G were discussed in detail in [75] where this function was identified with the sum of the first subleading term in the large spin expansion of the anomalous dimension of twist-2 operators and the coefficient of the subleading pole in the expectation value of the cusp Wilson line with edges of finite length.

2.6. The BDS conjecture and potential departures from it

In Section 2.5 we discussed, following [49, 103], higher loop corrections to the four-gluon amplitude. The direct evaluation of the integrals in the two-loop four-gluon amplitude [104] reveals a surprising structure: up to terms of order ε,

$$\mathcal{M}_4^{(2)}(\varepsilon) \;=\; \tfrac{1}{2}\left(\mathcal{M}_4^{(1)}(\varepsilon)\right)^2 + f^{(2)}(\varepsilon)\mathcal{M}_4^{(1)}(2\varepsilon)+C^{(2)}+\mathcal{O}(\varepsilon)\,. \tag{2.78}$$

Equally surprisingly, the same expression holds for the two-loop splitting amplitude [104]. Such an iterative behavior is to be expected for the infrared-singular part of the amplitudes; indeed, it is only a consequence of the soft/collinear factorization theorem discussed previously (*cf.* Eq. (2.72)). The surprising fact is that this structure extends to the finite part of the amplitude, in particular that $C^{(2)}$ is a constant.

The fact that splitting amplitudes provide a link between higher and lower-point amplitudes at fixed loop order suggests a generalization of the iteration relation above to arbitrary number of external legs. Indeed, an ansatz, due to Anastasiou, Bern, Dixon and Kosower [104], which correctly captures the behavior of MHV amplitudes in all collinear limits as well as their infrared singularities is

$$\mathcal{M}_n^{(2)}(\varepsilon) \;=\; \tfrac{1}{2}\left(\mathcal{M}_n^{(1)}(\varepsilon)\right)^2 + f^{(2)}(\varepsilon)\mathcal{M}_n^{(1)}(2\varepsilon)+C^{(2)}+\mathcal{O}(\varepsilon)\,. \tag{2.79}$$

Similarly to the explicit calculation of the four-point amplitude, the main feature of this ansatz is that $C^{(2)}$ and $f^{(2)}(\varepsilon)$ are independent of the external momenta and also of the number of external legs. The five-gluon amplitude at two-loops obeys this ansatz; the same cannot be said however about the six-gluon amplitude, as we shall discuss in Section 2.7.

A similarly surprising result followed [49] from the evaluation of the three-gluon amplitude; throughout the finite part, it obeys an iterative structure similar to that of the two-loop amplitude.

$$\mathcal{M}_4^{(3)}(\varepsilon)=-\frac{1}{3}\left(\mathcal{M}_4^{(1)}(\varepsilon)\right)^3+\mathcal{M}_4^{(2)}(\varepsilon)\mathcal{M}_4^{(1)}(\varepsilon)+f^{(1)}(\varepsilon)\mathcal{M}_4^{(1)}(3\varepsilon)+C^{(3)}+\mathcal{O}(\varepsilon)\,.$$

This equation as well as (2.78) are consistent with the resumed amplitude taking an exponential form with the exponent given in terms of the one-loop four-gluon amplitude. Assuming that the same is true for the splitting amplitude, Bern, Dixon and Smirnov [49] suggested that, to all loop orders, the rescaled n-point MHV amplitude is given by

$$\mathcal{M}_n = \exp\left[\sum_{l=1}^{\infty} a^l f^{(l)}(\varepsilon)\mathcal{M}_n^{(1)}(l\varepsilon) + C^{(l)} + \mathcal{O}(\varepsilon)\right]\,, \tag{2.80}$$

where the coefficients

$$f^{(l)}(\varepsilon) = f_0^{(l)} + \varepsilon f_1^{(l)} + \varepsilon^2 f_2^{(l)} \tag{2.81}$$

are independent of the number of external legs. The ε-independent part, $f_0^{(l)}$, are the Taylor coefficients of the cusp anomalous dimension or universal scaling function (2.69)

$$f(\lambda) = 4\sum_{l=0}^{\infty} a^l f_0^{(l)}\,. \tag{2.82}$$

The appearance of the cusp anomalous dimension is, of course, dictated by the infrared structure of the amplitude. Similarly, $f_1^{(l)}$ and $f_2^{(l)}$ define the functions

$$g(\lambda) = 2\sum_{l=2}^{\infty}\frac{a^l}{l}f_1^{(l)} \equiv 2\int\frac{d\lambda}{\lambda}\,G(\lambda)\,, \qquad k(\lambda) = -\frac{1}{2}\sum_{l=2}^{\infty}\frac{a^l}{l^2}f_2^{(l)}\,, \tag{2.83}$$

the former being twice the first logarithmic integral of G entering in the Sudakov form factor (2.70).

In the construction of (2.80) it was assumed that the splitting amplitude obeys an all-order exponentiation similar to the infrared-singular part of the amplitude:

$$r_{\rm S} = \exp\left[\sum_{l=1}^{\infty} a^l f^{(l)}(\varepsilon) r_{\rm S}^{(1)}(l\varepsilon)\right]\,. \tag{2.84}$$

This relation may be justified using the dual conformal invariance introduced in Section 2.5.1. Indeed, as shown in [112] and reviewed elsewhere in this volume, the four- and five-point amplitudes are uniquely fixed by requiring that this symmetry. Then, taking the collinear limit of the five-point amplitude immediately yields (2.84).

The infrared poles are apparent in the equation (2.80) and, using equation (2.72), may be readily isolated together with the associated dependence on the two-particle invariants:

$$\mathrm{Div}_n = -\sum_{i=1}^{n}\left[\frac{1}{8\varepsilon^2}f^{(-2)}\left(\frac{\lambda\mu_{\mathrm{IR}}^{2\varepsilon}}{(-s_{i,i+1})^{\varepsilon}}\right)+\frac{1}{4\varepsilon}g^{(-1)}\left(\frac{\lambda\mu_{\mathrm{IR}}^{2\varepsilon}}{(-s_{i,i+1})^{\varepsilon}}\right)\right], \quad (2.85)$$

where the invariants $s_{i,i+1}$ are assumed to be negative. The functions $f^{(-2)}$ and $g^{(-1)}$ are, respectively, the second and first logarithmic integrals of the functions $f(\lambda)$ and $G(\lambda)$. Extracting this divergent part defines the finite remainder $F_n^{(1)}(0)$.

$$\ln\mathcal{M}_n = \mathrm{Div}_n + \frac{f(\lambda)}{4}F_n^{(1)}(0) + nk(\lambda) + C(\lambda)\,, \quad (2.86)$$

with $C(\lambda)=\sum_{l=1}^{\infty}C^{(l)}a^l$. In the simplest case of the four-gluon amplitude the finite remainder $F_n^{(1)}(0)$ takes the form

$$F_4^{(1)}(0) = \frac{1}{2}\left(\ln\frac{s_{12}}{s_{23}}\right)^2 + 4\zeta_2\,. \quad (2.87)$$

For more than four external legs the finite remainders $F_n^{(1)}(0)$ have a more complicated form:

$$F_n^{(1)}(0) = \frac{1}{2}\sum_{i=1}^{n} g_{n,i}\,, \quad (2.88)$$

where

$$g_{n,i} \;=\; -\sum_{r=2}^{\lfloor n/2\rfloor-1}\ln\left(\frac{-t_i^{[r]}}{-t_i^{[r+1]}}\right)\ln\left(\frac{-t_{i+1}^{[r]}}{-t_i^{[r+1]}}\right) + D_{n,i} + L_{n,i} + \frac{3}{2}\zeta_2\,, \quad (2.89)$$

in which $\lfloor x\rfloor$ is the greatest integer less than or equal to x and, as in (2.51), $t_i^{[r]}=(k_i+\cdots+k_{i+r-1})^2$ are momentum invariants. (All indices are understood to be $\mod n$.) The form of $D_{n,i}$ and $L_{n,i}$ depends upon whether n

is odd or even. For the even case ($n = 2m$) these quantities are given by

$$D_{2m,i} = -\sum_{r=2}^{m-2} \mathrm{Li}_2\left(1 - \frac{t_i^{[r]} t_{i-1}^{[r+2]}}{t_i^{[r+1]} t_{i-1}^{[r+1]}}\right) - \frac{1}{2}\mathrm{Li}_2\left(1 - \frac{t_i^{[m-1]} t_{i-1}^{[m+1]}}{t_i^{[m]} t_{i-1}^{[m]}}\right),$$
$$L_{2m,i} = \frac{1}{4}\ln^2\left(\frac{-t_i^{[m]}}{-t_{i+1}^{[m]}}\right). \tag{2.90}$$

In the odd case ($n = 2m+1$), we have

$$D_{2m+1,i} = -\sum_{r=2}^{m-1} \mathrm{Li}_2\left(1 - \frac{t_i^{[r]} t_{i-1}^{[r+2]}}{t_i^{[r+1]} t_{i-1}^{[r+1]}}\right),$$
$$L_{2m+1,i} = -\frac{1}{2}\ln\left(\frac{-t_i^{[m-1]}}{-t_i^{[m+1]}}\right)\ln\left(\frac{-t_{i+1}^{[m]}}{-t_{i-1}^{[m+1]}}\right). \tag{2.91}$$

These expressions for $D_{n,i}$ and $L_{n,i}$ are found [76] by inserting the explicit values of the box integrals into equation (2.51).

By construction, the BDS conjecture captures the correct infrared singularities as well as the correct behavior under collinear limits. Thus, departures from it should contain no infrared singularities and moreover should have vanishing collinear limits in all channels.

Additional constraints may be found if one assumes that dual conformal invariance is a property of MHV amplitudes to all orders in perturbation theory [112]. While it is a plausible assumption especially in light of the strong coupling prescription for the calculation of scattering amplitudes [73] which we will discuss shortly, this assumption needs to be verified on a case by case basis. Nevertheless, if this assumption holds, it leads to the conclusion that departures from the BDS ansatz must exhibit dual conformal invariance as a consequence of their finiteness. Thus, similarly to two-dimensional conformal field theories, such corrections must be functions of invariants under the inversion transformations (2.58). Dual conformal invariants can be constructed for any kinematics with at least six momenta. In this simplest case they are[15]

$$u_1 = \frac{s_{12}s_{45}}{s_{123}s_{345}}, \qquad u_2 = \frac{s_{23}s_{56}}{s_{234}s_{123}}, \qquad u_3 = \frac{s_{34}s_{61}}{s_{345}s_{234}}. \tag{2.92}$$

The number of such cross-ratios — *i.e.* $x_{ij}x_{kl}/x_{ik}x_{jl}$ with the difference between any two labels of at least two units — grows with the number of

[15] Parity-odd dual conformal invariants can also be constructed. Explicit calculations [115] show that, at least at two-loop order, all parity-odd terms in the six-point amplitude exponentiate following the BDS ansatz.

external points. Clearly, dual conformal invariance would imply a reduction in the number of independent arguments of (the finite part of) MHV amplitudes.

To probe the structure of amplitudes it is useful to define the *remainder function* R_A:

$$R_{An}(a) = \ln\left(1 + \sum_l a^l \mathcal{M}_n^{(l)}\right) - \left(\sum_l a^l f_l(\varepsilon)\mathcal{M}_n^{(1)}(l\varepsilon) + C(a)\right) . \quad (2.93)$$

This is a finite, dual conformally invariant function of the coupling constant which encodes the departure of the n-point MHV rescaled amplitude from the BDS ansatz. The $\mathcal{O}(a^2)$ part may be readily extracted and reads

$$R_{An}^{(2)} \equiv \lim_{\varepsilon\to 0}\left[M_n^{(2)}(\varepsilon) - \left(\frac{1}{2}\big(M_n^{(1)}(\varepsilon)\big)^2 + f^{(2)}(\varepsilon)\, M_n^{(1)}(2\varepsilon) + C^{(2)}\right)\right] . \quad (2.94)$$

Note that the terms in parenthesis are just the ABDK ansatz (2.79) for the 2-loop MHV amplitude with arbitrary multiplicity.

2.7. The six-point MHV amplitude at two-loops and the BDS ansatz

As previously mentioned, the ABDK/BDS ansatz was constructed based on explicit calculations of four gluon amplitudes at two- and three-loop orders as well as of the collinear splitting amplitudes at two-loop order and was subsequently tested through the calculation of the five-point amplitude at two-loops. Assuming that dual conformal invariance holds to all loop orders, the fact that no conformal cross-ratios can be constructed for four- and five-point kinematics suggests that these amplitudes are determined to all orders by their infrared singularities. In later sections we will discuss to what extent this interpretation is accurate; the ABDK/BDS ansatz will obey an anomalous Ward identity for dual conformal transformations which has a unique solution for four- and five-point kinematics. Thus, in these cases, the ABDK/BDS ansatz necessarily reproduces the scattering amplitudes.

For higher-point kinematics scattering amplitudes may contain in principle additional information, beyond that related to its infrared divergences, which is captured by finite functions of conformal ratios and vanishes in all collinear limits.

Similarly to the five-point MHV amplitude at two-loops, the two-loop six-point MHV amplitude contains an even and an odd part. The even part was evaluated in [111] and information on the odd part was found in [115]. Projecting the ABDK ansatz (2.79) onto parity-even and parity-odd components, it follows quickly that similar iteration relations should hold separately for the even and odd parts of rescaled amplitudes. It turns out that, while the odd part of the amplitude obeys the iteration relation [115], the even part does not, signaling departures from the ABDK/BDS ansatz.

Following the discussion in previous sections, to reconstruct the amplitude one should analyze all of its generalized cuts. The known ultraviolet behavior of the theory however implies certain simplifications. As mentioned before, this is quite analogous to the observation that one-loop amplitudes can be written solely in terms of box integrals. The analogous statement for two-loop amplitudes of arbitrary multiplicity is that they are completely determined by (d-dimensional) iterated two-particle cuts. The relevant topologies are listed in Fig. 11.

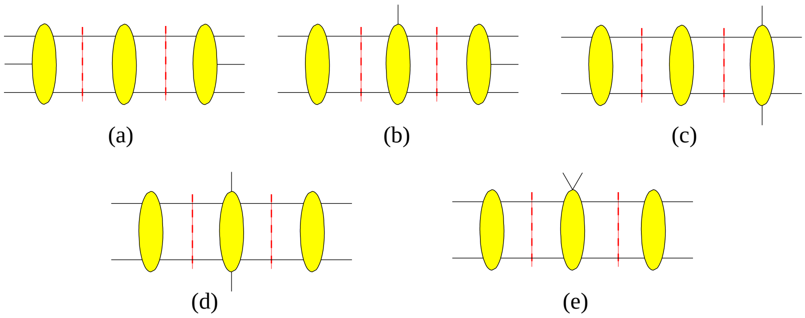


Fig. 11. Cuts capturing the complete structure of the six gluon amplitude at two loops.

Here, as well as for more general amplitudes, it is useful to organize the result as the sum of the part constructible from four-dimensional cuts $M_6^{(2),D=4}(\varepsilon)$ and the part accessible only through some (partial) d-dimensional cuts $M_6^{(2),\mu}(\varepsilon)$

$$M_6^{(2),D=4-2\varepsilon}(\varepsilon) = M_6^{(2),D=4}(\varepsilon) + M_6^{(2),\mu}(\varepsilon)\,. \tag{2.95}$$

The latter terms are built from μ-integrals and their four-dimensional generalized cuts vanish identically (in the sense that all cut propagators are considered four-dimensional).

The four-dimensional cut-constructible parity-even part of the amplitude is given entirely in terms of pseudo-conformal integrals [111];

$$\begin{aligned} M_6^{(2),D=4}(\varepsilon) = \frac{1}{16}\sum_{12\text{ perms.}}\Bigg[&\frac{1}{4}c_1 I^{(1)}(\varepsilon) + c_2 I^{(2)}(\varepsilon) + \frac{1}{2}c_3 I^{(3)}(\varepsilon) + \frac{1}{2}c_4 I^{(4)}(\varepsilon) \\ &+ c_5 I^{(5)}(\varepsilon) + c_6 I^{(6)}(\varepsilon) + \frac{1}{4}c_7 I^{(7)}(\varepsilon) + \frac{1}{2}c_8 I^{(8)}(\varepsilon) + c_9 I^{(9)}(\varepsilon) \\ &+ c_{10} I^{(10)}(\varepsilon) + c_{11} I^{(11)}(\varepsilon) + \frac{1}{2}c_{12} I^{(12)}(\varepsilon) + \frac{1}{2}c_{13} I^{(13)}(\varepsilon)\Bigg]\,. \end{aligned} \tag{2.96}$$

The integrals I_i are listed in Fig. 12 and the corresponding coefficients for the $(1,2,3,4,5,6)$ permutation are:

$$\begin{aligned}
c_1 &= s_{61}s_{34}s_{123}s_{345} + s_{12}s_{45}s_{234}s_{345} + s_{345}^2(s_{23}s_{56} - s_{123}s_{234})\,,\\
c_2 &= 2s_{12}s_{23}^2\,,\\
c_3 &= s_{234}(s_{123}s_{234} - s_{23}s_{56})\,,\\
c_4 &= s_{12}s_{234}^2\,,\\
c_5 &= s_{34}(s_{123}s_{234} - 2s_{23}s_{56})\,,\\
c_6 &= -s_{12}s_{23}s_{234}\,,\\
c_7 &= 2s_{123}s_{234}s_{345} - 4s_{61}s_{34}s_{123} - s_{12}s_{45}s_{234} - s_{23}s_{56}s_{345}\,,\\
c_8 &= 2s_{61}(s_{234}s_{345} - s_{61}s_{34})\,,\\
c_9 &= s_{23}s_{34}s_{234}\,,\\
c_{10} &= s_{23}(2s_{61}s_{34} - s_{234}s_{345})\,,\\
c_{11} &= s_{12}s_{23}s_{234}\,,\\
c_{12} &= s_{345}(s_{234}s_{345} - s_{61}s_{34})\,,\\
c_{13} &= -s_{345}^2s_{56}\,.
\end{aligned} \tag{2.97}$$

It is not hard to check that all terms appearing in (2.96) are indeed pseudo-conformal integrals. Their relative coefficients are $0, \pm1, \pm2$ and ±4, which represents a chance of patterns from the four- and five-point amplitudes where they were only 0 and ±1. It is currently unclear what is the origin of this change.

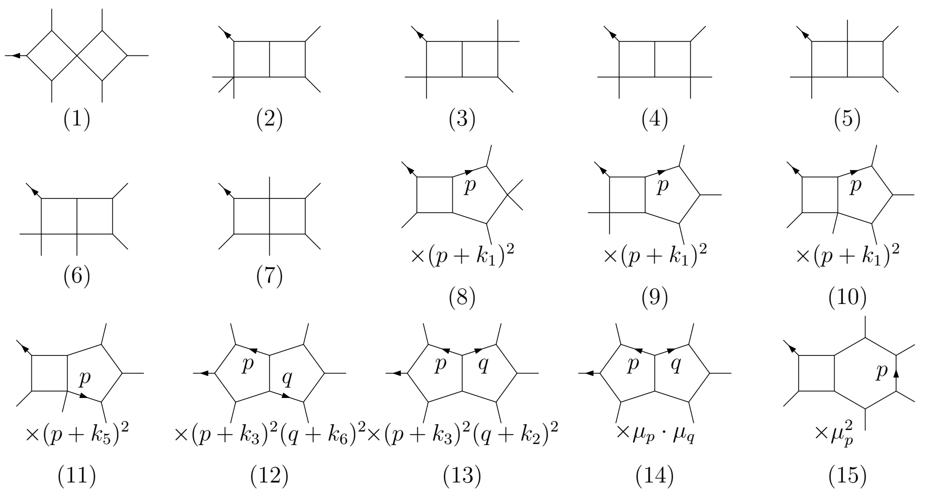


Fig. 12. Integral topologies appearing in the even part of the six gluon amplitude at two loops. All momenta are considered to be outgoing and the arrow on an external line denotes the line carrying momentum k_1. As before, μ_p and μ_q denote the (-2ε)-dimensional part of the loop momenta.

The remaining parity-even part of the amplitude, which may be determined by performing generalized cuts with at least one two-particle d-dimensional cut is [111]

$$M_6^{(2),\mu}(\varepsilon) = \frac{1}{16}\sum_{12\ \text{perms.}}\left[\frac{1}{4}c_{14}I^{(14)}(\varepsilon)+\frac{1}{2}c_{15}I^{(15)}(\varepsilon)\right], \qquad (2.98)$$

the coefficients for the identity permutation $(1,2,3,4,5,6)$ are

$$\begin{aligned} c_{14} &= -2s_{345}(s_{123}s_{234}s_{345}-s_{61}s_{34}s_{123}-s_{12}s_{45}s_{234}-s_{23}s_{56}s_{345}),\\ c_{15} &= 2s_{61}(s_{123}s_{234}s_{345}-s_{61}s_{34}s_{123}-s_{12}s_{45}s_{234}-s_{23}s_{56}s_{345}). \end{aligned} \qquad (2.99)$$

Their dual conformal properties are somewhat nontransparent; following the definition given in Section 2.5.1 they may be interpreted as pseudo-conformal as their integrand vanishes identically in four dimensions. Explicit calculation [111] shows that they do not contribute to the remainder function (2.94); thus, their presence may be ascribed to the infrared structure of the amplitude, interpretation strengthened by the fact that they either integrate to $\mathcal{O}(\varepsilon)$ (I_{14}) or they exhibit infrared poles (I_{15}).

The analytic evaluation of the integrals appearing in the six-point amplitude remains a difficult open problem, with potential applications beyond $\mathcal{N}=4$ SYM. To test for the structure and the conformal properties of the remainder function, [111] evaluated the amplitude at a variety of kinematic points $K^{(0)}$ through $K^{(5)}$. For the state of the art in the evaluation of Feynman integrals we refer the reader to [116–119]. Two of the kinematic points, $K^{(0)}$ and $K^{(1)}$, were chosen to have the same cross-ratios while the momentum invariants are different. The results [111] of the evaluation of the remainder function R_{A6} are shown in Table I.

TABLE I

The numerical remainder compared with the ABDK ansatz (2.79) for various kinematic points. The second column gives the conformal cross-ratios introduced in (2.92).

Kinematic point	(u_1,u_2,u_3)	R_{A6}
$K^{(0)}$	$(1/4,1/4,1/4)$	1.0937 ± 0.0057
$K^{(1)}$	$(1/4,1/4,1/4)$	1.076 ± 0.022
$K^{(2)}$	$(0.547253, 0.203822, 0.881270)$	-1.659 ± 0.014
$K^{(3)}$	$(28/17,16/5,112/85)$	-3.6508 ± 0.0032
$K^{(4)}$	$(1/9,1/9,1/9)$	5.21 ± 0.10
$K^{(5)}$	$(4/81,4/81,4/81)$	11.09 ± 0.50

The main conclusion is that the remainder function is nonzero to a high level of confidence thus suggesting that the ABDK/BDS ansatz captures only part of the amplitude. It is also important to note that the remainder functions at kinematic points $K^{(0)}$ and $K^{(1)}$ are equal within errors. This strongly suggests that R_{A6} is indeed a function of only conformal cross-ratios, *i.e.* is invariant under dual conformal transformations.

This calculation implies therefore that the BDS ansatz should be modified at six points and beyond. For this purpose it is instructive to identify the origin of the remainder function within the arguments that led to this ansatz. In short, the full structure of collinear limits for n-point amplitudes with $n \geq 6$ is somewhat more involved. One may consider limits in which more than two particles are simultaneously collinear:

$$k_i = z_i k \quad \text{for } i = 1 \ldots m \qquad \sum_{i=1}^{m} z_i = 1\,, \qquad z_i \leq 1 \qquad k^2 \to 0\,. \quad (2.100)$$

For the six-point amplitude only a triple-collinear limit (*i.e.* $m = 3$ above) exists. While vanishing in all double-collinear limits, the remainder function for the six-point amplitude has a nontrivial triple-collinear limit, which in fact allows its complete reconstruction. We refer the reader to [111] for more detailed discussions in this direction.

3. Scattering amplitudes at strong coupling

The AdS/CFT correspondence [5–7] provides the only direct access to the strong coupling regime of the $\mathcal{N} = 4$ SYM; it relates four-dimensional $\mathcal{N} = 4$ SYM theory and type IIB string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ space through the identification of string states and gauge-invariant operators. The two gauge theory parameters — the 't Hooft coupling λ and the rank of the gauge group N — are expressed in terms of the radius of curvature of the space and the string coupling by the well-known relations

$$\sqrt{\lambda} \equiv \sqrt{g_{\mathrm{YM}}^2 N} = \frac{R^2}{\alpha'}\,, \qquad \frac{1}{N} \sim g_s\,. \quad (3.1)$$

Thus, in the limit of a large number of colors the splitting and joining of strings is suppressed and in the limit of large 't Hooft coupling, the string theory lives on a weakly curved space. In this regime the string theory is completely described by a weakly-coupled worldsheet sigma-model.

By appending an open string sector to closed string theory in $\mathrm{AdS}_5 \times \mathrm{S}^5$ gluon scattering amplitudes could in principle be directly computed, on the string side of the AdS/CFT correspondence, in terms of integrated correlation functions of vertex operators. Due to the presence of color factors it is, however, unclear how much of the resulting structure can be captured entirely in terms of closed string data. We will argue, following [73, 74], that the color-stripped partial amplitudes do have such a description, to leading order in the strong coupling expansion.

As discussed in the previous section, one needs to introduce an infrared regulator in order to define perturbative scattering amplitudes properly. Dimensional regularization and variants thereof remain the preferred gauge theory regularization scheme.

A choice of regularization is also needed to define scattering amplitudes on the string theory side of the AdS/CFT correspondence. We will discuss two such choices: first, to set up the calculation, we will use as a regulator a D-brane cutting off the infrared part of AdS_5. After formulating and understanding the prescription for computing scattering amplitudes at strong coupling we will modify the regulator to one akin to the gauge theory dimensional regulator; this will allow a direct comparison with the strong coupling limit of (2.80).

3.1. The general construction

As a first IR regulator we consider a D-brane localized in the radial direction. We start with the AdS_5 metric written in Poincaré coordinates

$$ds^2 = R^2 \frac{dy_{3+1}^2 + dz^2}{z^2} \,. \tag{3.2}$$

The boundary is located at $z = 0$ while the horizon is located at $z = \infty$. Then we place a D-brane at some fixed large value of $z = z_{\rm IR}$ and extending along the x_{3+1} coordinates. The asymptotic states are open strings that end on the D-brane. We then consider the scattering of these open strings.

The proper momentum of the strings is $k_{\rm pr} = k z_{\rm IR}/R$, where k is the momentum conjugate to x_{3+1}, plays the role of gauge theory momentum and will be kept fixed as we take away the IR cut-off, $z_{\rm IR} \to \infty$. Therefore, due to the warping of the metric, the proper momentum is very large, so we are considering the scattering of strings at fixed angle with very large momentum. Amplitudes in such regime were studied in flat space by Gross, Mende and Manes, [120, 121]. The key feature of their computation is that the amplitudes are dominated by a saddle point of the classical action. In our case we need to consider classical strings on AdS.

We need then to consider a world-sheet with the topology of a disk with vertex operator insertions on its boundary, which correspond to the external states (see Fig. 13). Each color ordered amplitude corresponds to a disk amplitude with a fixed ordering of the open string vertex operators. The boundary conditions for the world-sheet are the following: the worldsheet boundary is located at $z = z_{\rm IR}$ since the open strings are attached to the regulator D-brane and, in the vicinity of each vertex operator, the momentum of the external state fixes the form of the worldsheet field configuration.

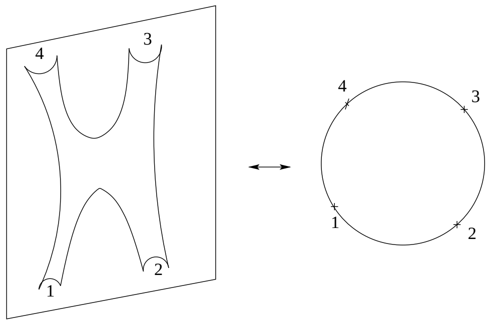


Fig. 13. World-sheet corresponding to the scattering of four open strings.

To state more simply the boundary conditions for the world-sheet, it is convenient to describe the solution in terms of T-dual coordinates x^μ, defined as

$$ds^2 = w^2(z) dy_\mu dy^\mu \to \partial_\alpha x^\mu = i w^2(z) \varepsilon_{\alpha\beta} \partial_\beta y^\mu \,. \tag{3.3}$$

Note that we do not T-dualize along the radial direction z. The boundary conditions for the original coordinates x^μ, which are that they carry momentum k^μ, translates into the condition that x^μ has "winding"

$$\Delta x^\mu = 2\pi k^\mu \,. \tag{3.4}$$

After defining $r = R^2/z$ we end up again with the AdS_5 metric

$$ds^2 = R^2 \frac{dx_\mu dx^\mu + dr^2}{r^2} \,. \tag{3.5}$$

Now the boundary of the world-sheet is located at $r = R^2/z_{\rm IR}$ and is a particular line constructed as follows (see Fig. 14)

- For each particle of momentum k^μ, draw a segment joining two points separated by $\Delta x^\mu = 2\pi k^\mu$.
- Concatenate the segments according to the insertions on the disk (corresponding to a particular color ordering).
- As gluons are massless, the segments will be light-like. Due to momentum conservation, the diagram is closed.

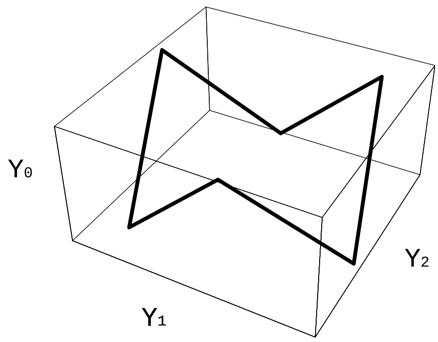


Fig. 14. Polygon of light-like segments corresponding to the momenta of the external particles.

The world-sheet, when expressed in T-dual coordinates, will then end on such a sequence of light-like segments (see for example Fig. 14 for the sequence corresponding to the scattering of six gluons) located at $r = R^2/z_{\rm IR}$

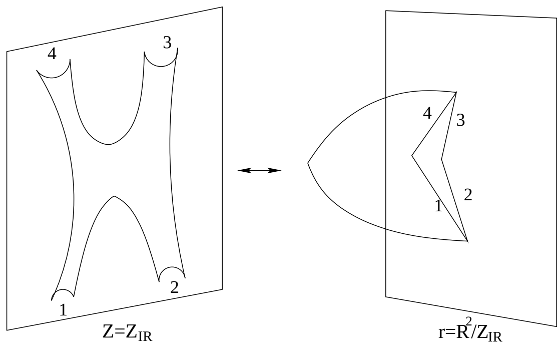


Fig. 15. Comparison of the world sheet in original and T-dual coordinates.

As we take away the IR cut-off, $z_{\rm IR} \to \infty$, the boundary of the world-sheet moves towards the boundary of the T-dual metric, at $r = 0$. At leading order in the strong coupling expansion, the computation that we are doing is formally the same as the one we would do if we were computing the expectation value of a Wilson loop given by a sequence of light-like segments.

Our prescription is then that the leading exponential behavior of the n-point scattering amplitude is given by the area A of the minimal surface that ends on a sequence of light-like segments on the boundary

$$\mathcal{A}_n \sim e^{-\frac{\sqrt{\lambda}}{2\pi} A(k_1,\ldots,k_n)}\,. \tag{3.6}$$

The area A contains the kinematical information through its boundary conditions. We stress that our computation is blind to the polarization of the gluons, which contribute to prefactors in 3.6 and are subleading in $1/\sqrt{\lambda}$[16].

In the following, we will show in detail how our prescription works for the scattering of four gluons and compare our results with field theory expectations.

3.2. Four gluon scattering

The simplest scattering process involves four particles and is characterized by the usual Mandelstam invariants

$$s = -(k_1 + k_2)^2\,, \qquad t = -(k_2 + k_3)^2\,. \tag{3.7}$$

The discussion in the previous section suggests that, in the strongly coupled $\mathcal{N} = 4$ SYM theory, the amplitude for this process is governed by the minimal surface ending on the light-like polygon shown in Fig. 16.

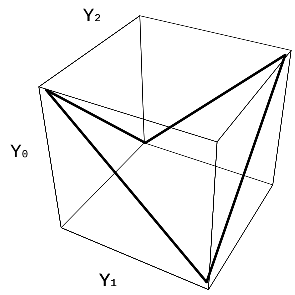


Fig. 16. Polygon corresponding to the scattering of four gluons.

As implied by the equation (3.4), the difference between the coordinates of each of the corners of the polygon (cusps) are, up to a factor of (2π), the momenta of the corresponding gluon. In drawing Fig. 16, it was assumed that the third component of the momentum (described by the boundary coordinate y_3) vanishes. This choice imposes no kinematic restrictions, as it does not imply any relations between the two Mandelstam invariants.

The area of a surface embedded in a higher dimensional space is simply given by the integral of the induced metric

$$A = \int d\sigma d\tau \sqrt{-\det \partial_\alpha x^\mu \partial_\beta x^\nu g_{\mu\nu}(x)}\,. \tag{3.8}$$

[16] For instance, if we consider amplitudes of the form $A(++++\ldots)$, then such prefactors should vanish.

Finding minimal surfaces amounts simply to treating this area as an action (the Nambu–Goto action) and solving the classical equations of motion, subject to the desired boundary conditions. Then, the area is obtained by evaluating (3.8) on the resulting configuration.

3.2.1. The single cusp solution

As a warm up exercise let us discuss the solution near the cusp where two of the light-like lines meet. This problem was originally considered in [125] and it will prove useful for generating the solution relevant for the four-gluon scattering. The surface can be embedded into an AdS_3 subspace of AdS_5

$$ds^2 = \frac{-dx_0^2 + dx_1^2 + dr^2}{r^2} \,. \tag{3.9}$$

We are interested in finding the surface ending on a light-like Wilson line which is along $x^1 = \pm x^0$, $x^0 > 0$[17] (see Fig. 17). This configuration has both boost and scale symmetry, which are made manifest by the following ansatz:

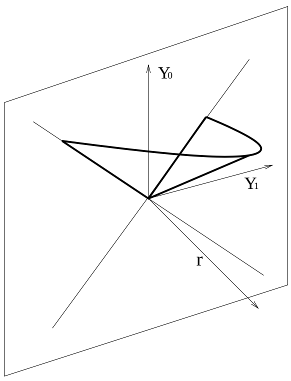


Fig. 17. Single cusp solution.

$$x_0 = e^\tau \cosh\sigma \,, \qquad x_1 = e^\tau \sinh\sigma, \qquad r = e^\tau w(\tau) \,. \tag{3.10}$$

boosts in the $(0,1)$ plane and scale transformations are then simply given by shifts of σ and τ, respectively.

Equations for the remaining function $w(\tau)$ may be found by evaluating the equations of motion on the ansatz (3.10). Alternatively, one may simply evaluate the Nambu–Goto action on the ansatz (3.10) and derive an equation

[17] One can also consider Wilson loops along $x^0 = \pm x^1$, $x^1 > 0$. The basic difference with the ones considered here is that their worldsheet is Lorentzian and z is imaginary.

for $w(\tau)$ by varying the result with respect to it. Choosing the second path, the Nambu–Goto action becomes

$$S_{\rm NG} = \frac{R^2}{2\pi\alpha'}\int d\sigma d\tau \frac{\sqrt{1-(w(\tau)+w'(\tau))^2}}{w(\tau)^2}\,. \tag{3.11}$$

One can then explicitly check that $w(\tau)=\sqrt{2}$ solves the equations of motion and has the correct boundary conditions. Hence the surface is given by

$$r = \sqrt{2}\sqrt{x_0^2 - x_1^2}\,. \tag{3.12}$$

Notice that the surface lies entirely outside the light-cone of the origin, hence it is Euclidean.

3.2.2. Four cusps solution

The four cusps solution is closely related to the single cusp solution discussed above. The relevant solution of the Nambu Goto action can be embedded in a AdS_4 subspace of AdS_5, parametrized by $(r, x_0, x_1, x_2, x_3=0)$. Furthermore, we fix reparametrization invariance by choosing $(\sigma_1, \sigma_0) = (x_1, x_2)$. With these choices, the Nambu–Goto action describes the dynamics of two fields, r and x_0, living in the space parametrized by x_1 and x_2

$$S = \frac{R^2}{2\pi\alpha'}\int dx_1 dx_2 \frac{\sqrt{1+(\partial_i r)^2-(\partial_i x_0)^2-(\partial_1 r\partial_2 x_0-\partial_2 r\partial_1 x_0)^2}}{r^2}\,. \tag{3.13}$$

The classical equations of motion should then be supplemented by the appropriate boundary conditions. We consider first the case with $s=t$, where the projection of the Wilson lines is a square. By scale invariance, we can choose the edges of the square to be at $x_1, x_2 = \pm 1$. The boundary conditions can be easily seen to be

$$r(\pm 1, x_2) = r(x_1, \pm 1) = 0\,, \quad x_0(\pm 1, x_2) = \pm x_2\,, \quad y_0(x_1, \pm 1) = \pm x_1\,. \tag{3.14}$$

The form of the solution near each of the cusps can be obtained by rotations and boosts from the single cusp solution (3.12). The following field configuration satisfies the boundary conditions and has the correct properties near each of the cusps

$$x_0(x_1, x_2) = x_1 x_2\,, \qquad r(x_1, x_2) = \sqrt{(1-x_1^2)(1-x_2^2)}\,. \tag{3.15}$$

Remarkably it turns out to be a solution of the equations of motion! In principle, we could plug this solution into the action and compute the amplitude as strong coupling for this particular configuration. However, in order to capture the kinematic dependence of the area[18] we need to consider more general solutions with $s \neq t$. In this case the projection of the surface to the (x_1, x_2) plane will not be an square but a rhombus, with s and t given by the square of the distance between opposite vertices, as shown in figure 18.

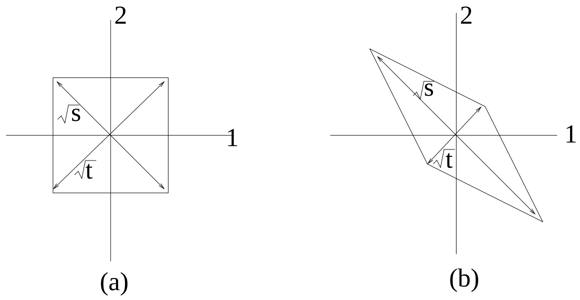


Fig. 18. Projection to the plane (x_1, x_2) of the surface for the cases $s = t$ and $s \neq t$.

The symmetry generators of anti-de-Sitter space act nonlinearly on the Poincaré patch coordinates (3.5). They are however useful for generating new and interesting worldsheet configurations from known ones, since they can change the Mandelstam variables s and t; it would therefore be useful to linearize their action. This is realized by passing to the so-called embedding coordinates, in which AdS_5 is viewed as a hypersurface embedded in $\boldsymbol{R}^{2,4}$

$$-X_{-1}^2 - X_0^2 + X_1^2 + X_2^2 + X_3^2 + X_4^2 = -1\,. \tag{3.16}$$

Clearly, this constraint equation is manifestly invariant under the SO(2,4) symmetry group of AdS_5, which is also the Lorentz group of the embedding space. The Poincaré coordinates $(\boldsymbol{x}, r)$ are but a particular solution of the constraint equation (3.16):

$$\begin{aligned} X^\mu &= \frac{x^\mu}{r}\,, \qquad \mu = 0, ..., 3 \\ X_{-1} + X_4 &= \frac{1}{r}\,, \qquad X_{-1} - X_4 = \frac{r^2 + x_\mu x^\mu}{r}\,. \end{aligned} \tag{3.17}$$

In terms of embedding coordinates, the minimal surface describing the scattering of four gluons with the $s = t$ kinematics is

$$X_0 X_{-1} = X_1 X_2\,, \qquad X_3 = X_4 = 0\,. \tag{3.18}$$

In this form it is then easy to use the SO(2,4) to generate from (3.15) new minimal surfaces, corresponding to $s \neq t$ gauge theory kinematics.

[18] On dimensional grounds the area, if finite, should be a function of the form $f(s/t)$.

Through the AdS/CFT correspondence, the SO(2,4) of the dual AdS space should have a gauge theory counterpart. Due to the T-duality transformation relating the boundary coordinates of this space with gauge theory momenta this symmetry is necessarily different from the usual position space conformal invariance of $\mathcal{N}=4$ SYM theory. As discussed in Section 2.5.1, the action of a "dual conformal group" can be identified at the level of perturbative scattering amplitudes. While *a priori* these two symmetries are unrelated (as *e.g.* the former exists at strong coupling while the latter at weak coupling), it is nevertheless tempting to interpret the latter as the weak coupling version of the former.

Solutions with $s \neq t$ can be obtained by starting from (3.15) and performing a boost in the $(0,4)$ plane. In this way we change the distance between opposite vertices of the square.

$$\begin{aligned} X_0 X_{-1} &= X_1 X_2 \,, \\ X_4 &= 0 \to X_4 - v X_0 = 0 \,, \\ \sqrt{1-v^2} X_0 X_{-1} &= X_1 X_2 \,. \end{aligned} \tag{3.19}$$

After the boost, we end up with a two-parameter solution, one related to the size of the initial square and another related to the boost parameter. The solution can be conveniently written as

$$\begin{aligned} r &= \frac{a}{\cosh u_1 \cosh u_2 + b \sinh u_1 \sinh u_2} \,, \\ x_0 &= \frac{a\sqrt{1+b^2} \sinh u_1 \sinh u_2}{\cosh u_1 \cosh u_2 + b \sinh u_1 \sinh u_2} \\ x_1 &= \frac{a \sinh u_1 \cosh u_2}{\cosh u_1 \cosh u_2 + b \sinh u_1 \sinh u_2} \,, \\ x_2 &= \frac{a \cosh u_1 \sinh u_2}{\cosh u_1 \cosh u_2 + b \sinh u_1 \sinh u_2} \,, \end{aligned} \tag{3.20}$$

where we have written the surface as a solution to the equations of motion in conformal gauge

$$iS = -\frac{R^2}{2\pi\alpha'} \int \mathcal{L} = -\frac{R^2}{2\pi} \int du_1 du_2 \frac{1}{2} \frac{(\partial r \partial r + \partial x_\mu \partial x^\mu)}{r^2} \tag{3.21}$$

a and b encode the kinematic information of the scattering as follows

$$-s(2\pi)^2 = \frac{8a^2}{(1-b)^2} \,, \qquad -t(2\pi)^2 = \frac{8a^2}{(1+b)^2} \,, \qquad \frac{s}{t} = \frac{(1+b)^2}{(1-b)^2} \,. \tag{3.22}$$

To obtain the four point scattering amplitude at strong coupling it should suffice, following the discussion in Section 3.1, to evaluate the classical action on the solution (3.20). However, in doing so, one finds a divergent answer. That is of course the case, since we have ignored the infrared regulator. In order to obtain a finite answer we need to reintroduce a regulator.

3.2.3. Dimensional regularization at strong coupling

Gauge theory amplitudes are regularized by considering the theory in $D = 4 - 2\varepsilon$ dimensions. More precisely (see discussion in Section 2), one starts with $\mathcal{N} = 1$ in ten dimensions and then dimensionally reduces to $4 - 2\varepsilon$ dimensions. For integer 2ε this is precisely the low energy theory living on a Dp-brane, where $p = 3 - 2\varepsilon$. We regularize the amplitudes at strong coupling by considering the gravity dual of these theories.

$$\begin{aligned} ds^2 &= f^{-1/2} dx^2_{4-2\varepsilon} + f^{1/2}\left[dr^2 + r^2 d\Omega^2_{5+2\varepsilon}\right] , \\ f &= (4\pi^2 e^{\gamma})^{\varepsilon}\Gamma(2+\varepsilon)\mu^{2\varepsilon}\frac{\lambda}{r^{4+2\varepsilon}} . \end{aligned} \tag{3.23}$$

Following the steps described above, one is led to the following action

$$S = \frac{\sqrt{c_\varepsilon \lambda}\mu^{\varepsilon}}{2\pi}\int \frac{\mathcal{L}_{\varepsilon=0}}{r^{\varepsilon}} , \tag{3.24}$$

where $\mathcal{L}_{\varepsilon=0}$ is the Lagrangian density in the absence of the regulator. The presence of the factor r^{ε} will have two important effects. On one hand, previously divergent integrals will now converge. On the other hand, the equations of motion will now depend on ε and it turns out to be very difficult to find the solution for general ε. However, we are interested in computing the amplitude only up to finite terms as we take $\varepsilon \to 0$. In that case, it turns out to be sufficient to plug the original solution into the ε-deformed action[19]. The evaluation of the integrals leads to [73]

$$S \approx \sqrt{\lambda}\frac{\mu^{\varepsilon}}{a^{\varepsilon}}\, {}_2F_1\left(\frac{1}{2}, -\frac{\varepsilon}{2}, \frac{1-\varepsilon}{2}; b^2\right) . \tag{3.25}$$

Finally, expanding in powers of ε yields the final answer

$$\mathcal{A} = e^{iS} = \exp\left[iS_{\rm div} + \frac{\sqrt{\lambda}}{8\pi}\left(\ln\frac{s}{t}\right)^2 + \tilde{C}\right] , \tag{3.26}$$

$$S_{\rm div} = 2S_{\rm div,s} + 2S_{\rm div,t} , \tag{3.27}$$

$$iS_{\rm div,s} = -\frac{1}{\varepsilon^2}\frac{1}{2\pi}\sqrt{\frac{\lambda\mu^{2\varepsilon}}{(-s)^{\varepsilon}}} - \frac{1}{\varepsilon}\frac{1}{4\pi}(1-\ln 2)\sqrt{\frac{\lambda\mu^{2\varepsilon}}{(-s)^{\varepsilon}}} . \tag{3.28}$$

[19] To be more precise, the ε-corrected solution can be constructed close to the cusps. The contribution of the correction terms to the minimal area turns out to be independent of the kinematics.

This should be compared with the field theory expectations, equations (2.85) and (2.86), specialized to the case $n=4$:

$$\mathcal{A} \sim (\mathcal{A}_{\rm div,s})^2 (\mathcal{A}_{\rm div,t})^2 \exp\left\{ \frac{f(\lambda)}{8} \left(\ln\frac{s}{t}\right)^2 + \text{const.}\right\} \tag{3.29}$$

$$\mathcal{A}_{\rm div,s} = \exp\left\{ -\frac{1}{8\varepsilon^2} f^{(-2)}\left(\frac{\lambda\mu^{2\varepsilon}}{s^\varepsilon}\right) - \frac{1}{4\varepsilon} g^{(-1)}\left(\frac{\lambda\mu^{2\varepsilon}}{s^\varepsilon}\right)\right\}. \tag{3.30}$$

It is important to notice that the general structure is in perfect agreement with the general structure of infrared divergences in $\mathcal{N}=4$ SYM theory. It is not hard to see that the leading divergence has the correct coefficient, given by the strong coupling limit of the cusp anomalous dimension [125][20]

$$f(\lambda) = \frac{\sqrt{\lambda}}{\pi}. \tag{3.31}$$

Moreover, from (3.26) one could extract the strong coupling behavior of the function $g(\lambda)$ introduced in equation (2.83):

$$g(\lambda) = \frac{\sqrt{\lambda}}{2\pi}(1-\ln 2). \tag{3.32}$$

Notice that due to the scheme dependence of $g(\lambda)$, it should be computed using the same regularization as in perturbative computations, that of course, is not the case for $f(\lambda)$. Finally, the kinematic dependence of the finite term (3.29) reproduces the strong coupling limit of the BDS prediction (2.86)–(2.87) for the four gluon scattering amplitude.

3.2.4. Radial cut-off

A more common regularization scheme for computing minimal areas in AdS is to introduce a cut-off in the radial direction. The correct procedure would be to impose the boundary conditions at some small $r=r_{\rm c}$. It turns out, however, that in order to compute the finite piece as $r_{\rm c}\to 0$ it suffices to use the original solution and cut the integral giving the area at $r={r_{\rm c}}$[21].

[20] The appearance of the cusp anomalous dimension in the equations (3.29) may appear surprising at first sight. Indeed, by analogy with weak coupling arguments based on finiteness of physical quantities constructed from gluon scattering amplitudes, the natural quantity entering (3.29) should be the large spin limit of the anomalous dimension of twist-2 operators. It was however shown in [151] that worldsheet calculations of the cusp anomaly and of the large spin limit of the anomalous dimension of twist-2 operators are related by an analytic continuation and and target space symmetry transformations. Thus, similarly to the weak coupling result of [158–160], the cusp anomaly equals the large spin limit of the anomalous dimension of twist-2 operators to all orders in the $1/\sqrt{\lambda}$ expansion.

[21] This finite part arises in a similar way when using the conjectured string theory version of gauge theory dimensional regularization.

In order to compute the regularized area for the scattering of four gluons it is convenient to work in conformal gauge. In this case, the problem reduces to the calculation of the area enclosed by the curve

$$\frac{a}{\cosh u_1 \cosh u_2 + b \sinh u_1 \sinh u_2} = r_{\rm c}\,. \tag{3.33}$$

One way to compute the area is by expanding the integrand in power series of $r_{\rm c}/a$ and integrating term by term. Equivalently, one can use Green's theorem and express the area as a one dimensional integral over the boundary of the worldsheet. The result is

$$\begin{aligned} iS &= -\frac{\sqrt{\lambda}}{2\pi} A\,, \\ A &= \frac{1}{4}\left(\ln\left(\frac{r_{\rm c}^2}{-8\pi^2 s}\right)\right)^2 + \frac{1}{4}\left(\ln\left(\frac{r_{\rm c}^2}{-8\pi^2 t}\right)\right)^2 - \frac{1}{4}\ln^2\left(\frac{s}{t}\right) + {\rm const.} \end{aligned} \tag{3.34}$$

Several comments are in order. First, notice that the structure of infrared divergences is of the form $\ln^2(r_{\rm c}^2/s)$ [22], and the coefficient in front of double logs and the finite piece is proportional to the cusp anomalous dimension at leading strong coupling, as in the case of dimensional regularization. Second, single logs have been absorbed into the double logs. Finally, the finite term reproduces, up to an additive constant, the results of dimensional regularization. Hence, the computation of amplitudes at strong coupling does not need to be done by using dimensional regularization, unless we are interested in computing the function $g(\lambda)$ and the constant $C(\lambda)$ entering in equations (2.85) and (2.86), respectively, and computed by using dimensional regularization.

3.2.5. Structure of infrared poles at strong coupling

Even if the relevant solutions for minimal surfaces for the cases $n > 4$ are presently unknown, the IR structure of amplitudes at strong coupling for the general case of $n-$point amplitudes can easily be understood.

Given the cusp formed by a pair of neighboring gluons with momenta k_i and k_{i+1} we associate the kinematic invariant $s_{i,i+1} = (k_i + k_{i+1})^2$. We expect the following structure for the infrared-divergent part of the action

$$iS_{\rm div} = -\frac{\sqrt{\lambda}}{2\pi} \sum_i I\left(\frac{r_{\rm c}^2}{s_{i,i+1}}\right)\,, \tag{3.35}$$

[22] Notice that a very similar structure appears when using the off-shell regularization suggested in Ref. [128].

where $I(r_{\rm c}^2/s_{i,i+1})$ can be computed following [129], either by using dimensional regularization or a radial cut-off. For later applications the later scheme will be more useful to us, in this case

$$4I = \int_{\xi}^{1}\int_{\frac{\xi}{X^-}}^{1} \frac{1}{X^-X^+} = \frac{1}{2}\ln^2\xi\,, \qquad \xi = \frac{r_{\rm c}^2}{-8\pi^2 s_{i,i+1}}\,. \tag{3.36}$$

Hence, when using a radial cut-off as regulator, we expect the following structure for scattering amplitudes at strong coupling

$$iS_n = -\frac{\sqrt{\lambda}}{16\pi}\sum_{i=1}^{n}\ln^2\left(\frac{r_{\rm c}^2}{-8\pi^2 s_{i,i+1}}\right) + {\rm Fin}(k_i)\,. \tag{3.37}$$

It is easy to check that the general form of the amplitude for the case $n=4$ is consistent with this general expression.

For the discussion of the next subsection, it will be important to consider a radial cut-off that depends on the point at the boundary we are approaching, *i.e.* $r_{\rm c}(x)$. In that case, the structure of the amplitude turns out to be as follows

$$iS_n = -\frac{\sqrt{\lambda}}{16\pi}\sum_{i=1}^{n}\ln^2\left(\frac{r_{\rm c}^2(x_i)}{-8\pi^2 s_{i,i+1}}\right) + {\rm Fin}(k_i) + \sum_{i=1}^{n}E^i_{\rm edge}(r_{\rm c})\,. \tag{3.38}$$

The last sum in this expression corresponds to finite extra contributions coming from the edges

$$E^i_{\rm edge} = \frac{\sqrt{\lambda}}{2\pi}\int_0^1 \frac{ds}{s}\ln\left(\frac{r_{\rm c}(s)r_{\rm c}(1-s)}{r_{\rm c}(0)r_{\rm c}(1)}\right)\,, \tag{3.39}$$

where s running from zero to one parametrizes the i-th edge, namely $x^\mu(s) = x_i^\mu + s(x_{i+1}^\mu - x_i^\mu)$ and $r_{\rm c}(s)$ is a shorthand notation for $r_{\rm c}(x(s))$.

3.3. A conformal Ward Identity

An important ingredient in the construction of the minimal surface governing the four-gluon scattering amplitude was the existence of a dual SO(2,4) symmetry[23]. This symmetry allowed the construction of new solutions and fixed the finite piece of the scattering amplitude. Naively, this

[23] In principle this symmetry is unrelated to the original conformal symmetry. It has been suggested [113] that, at the level of the worldsheet sigma-model, the symmetries of the dual AdS space are in fact part of the hidden (non-local) symmetries of the original AdS space sigma-model [114].

conformal symmetry would imply that the amplitude is independent of s and t, since they can be sent to arbitrary values by a dual conformal symmetry. The whole dependence on s and t arises due to the necessity of introducing an infrared regulator. However, we will see that, after keeping track of the dependence on the infrared regulator, the amplitude is still determined by the dual conformal symmetry.

Symmetries manifest themselves through the existence of (potentially anomalous) Ward identities, the simplest is the quantum version of the conservation of the Noether current. More complicated Ward identities describe the action of symmetry generators on gauge-invariant quantities and potentially constrain their quantum expressions. In this direction it is possible to construct Ward identities for the dual SO(2,4) symmetry and study the constraints they impose on scattering amplitudes. For this purpose it is convenient to regularize the amplitude calculation with a radial cut-off.

Given the momenta k_i of the external gluons, the boundary of the worldsheet contains cusps located at x_i, with $2\pi k_i = x_i - x_{i+1}$. Now imagine that we regularize the area by choosing a cut-off $r_{\rm c}$. Moreover, we would like this cut-off to depend on the point at the boundary we are approaching, *i.e.* $r_{\rm c} \to r_{\rm c}(x)$. From the discussion above we expect the regulated area to have the general form

$$A_n^{\rm reg} = f(\lambda) \sum_{i=1}^{n} \ln^2 \left(\frac{r_{\rm c}^2(x_i)}{-2x_{i-1,i+1}^2} \right) + {\rm Fin}(x_i)\,, \tag{3.40}$$

where we have ignored extra terms coming from the edges of the contour as they can be seen not to affect the following argument. SO(2,4) transformations will then act on the points x_i and $r_{\rm c}(x_i)$. By requiring the area to be invariant under the action of special conformal transformations generated by $\boldsymbol{K}^\mu$

$$\boldsymbol{K}^\mu A_n^{\rm reg} = \left(\sum_{i=1}^{n} 2x_i^\mu (x_i \cdot \partial_{x_i} + r(x_i)\partial_{r(x_i)}) - x_i^2 \partial_{x_i^\mu} \right) A_n^{\rm reg} = 0 \tag{3.41}$$

one may derive an equation for the finite part of the amplitude[24]. At weak coupling this equation was constructed in [112] from the analysis of the dual conformal properties of Wilson loops in dimensional regularization. Its strong coupling counterpart was constructed in [130] using the strong coupling version of dimensional regularization discussed in Section 3.2.3.

[24] As the introduction of a infrared cut-off breaks (dual) conformal invariance, one obtains terms that depend explicitly on $r_{\rm c}$ on the right hand side of (3.41). As we take $r_{\rm c} \to 0$, conformal invariant is recovered and such terms vanish.

It turns out, that for the case of $n = 4$ and $n = 5$, this equation fixes uniquely the form of the finite piece, to be the one in the BDS conjecture. It is however not completely clear whether the dual conformal symmetry is an exact property of the all-order planar amplitudes. Explicit calculations show that it indeed exists to the first few orders in the weak coupling expansion and to leading order in the strong coupling expansion. Assuming that it dual conformal transformations are indeed a symmetry of on-shell scattering amplitudes leads to the conclusion that the BDS conjecture for four and five gluons is correct.

3.4. The BDS ansatz, scattering amplitudes and Wilson loops

As already mentioned in great detail, the relation between scattering amplitudes and Wilson loops described above emerged from strong coupling considerations. The conclusion however may be formulated without reference to the value of the coupling constant; this suggests that a weak coupling relation between Wilson loops and scattering amplitudes is possible. This was the standpoint taken in [112, 153] where it was shown that one-loop MHV amplitudes rescaled by their tree-level counterparts are indeed equal to the expectation value of the corresponding null cusped Wilson loops. This observation allows for a direct test of the BDS ansatz at strong coupling.

3.4.1. Rectangular configuration with a large number of gluons

As mentioned previously, the construction of the relevant minimal surfaces for some number of sides larger than four is difficult. The problem simplifies substantially in the limit of a large number of sides arranged in a specific configuration. In particular, following [74], we consider a zig-zag configuration with a large number of edges that approximates the rectangular Wilson loop, as shown in Fig. 19.

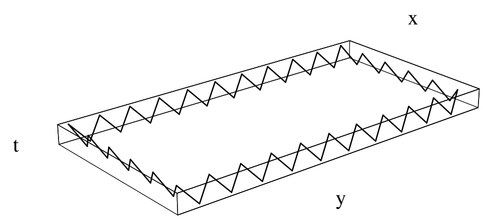


Fig. 19. Zig-zag configuration approaching the space-like rectangular Wilson loop.

We, moreover, consider the limit of very large T and L and for $T \gg L$; in this limit one may ignore the contribution of the shorter sides and thus the Wilson loop becomes identical to the one yielding the potential between a quark and an anti-quark [123,124]. The the results of the weak and strong coupling calculations are

$$\ln\langle W^{\rm weak}_{\rm rect}\rangle = \frac{\lambda}{8\pi}\frac{T}{L}\,, \qquad \ln\langle W^{\rm strong}_{\rm rect}\rangle = \frac{\sqrt{\lambda}4\pi^2}{\Gamma(1/4)^4}\frac{T}{L}\,. \tag{3.42}$$

The observation of [112, 153] on the relation between one-loop amplitudes and the expectation value of null Wilson loops allows us, following [73], to reconstruct the prediction of the BDS ansatz for the particular configuration of gluons corresponding to the Wilson loop in Fig. 19. Indeed, the BDS ansatz, explicit calculations and the dual conformal Ward identity imply that the expression of logarithm of the resumed and rescaled MHV amplitudes takes the form

$$\ln M_n = {\rm Div}_n + \frac{f(\lambda)}{4}a_1(k_1,...,k_n) + R_A(\boldsymbol{u}) + h(\lambda) + nk(\lambda) \tag{3.43}$$

with a_1 the one loop amplitude, $R_A(\boldsymbol{u})$ is the amplitude "remainder function" depending only on the conformally invariant cross-ratios u_i and $h(\lambda)$ and $k(\lambda)$ are functions that are independent on the kinematics and the number of gluons. The BDS ansatz predicts that $R_A(\boldsymbol{u}) = 0$. Explicit computations [112, 153] show that $a_1 = w_1$ where w_1 is the one-loop expectation value of the corresponding Wilson loop. It therefore follows that

$$\ln M^{\rm BDS}_{\rm fig\,19} = \frac{\sqrt{\lambda}}{4}\frac{T}{L} + \dots\,, \tag{3.44}$$

where we used the known string coupling limit of the universal scaling function $f(\lambda)$ and the ellipsis stand for terms whose dependence on L and T is *not* of the form T/L.

This expression clearly differs from $\ln\langle W^{\rm strong}_{\rm rect}\rangle$ suggesting that the BDS ansatz needs to be revised for a large number of gluons. A two-loop calculation of the expectation value of the rectangular Wilson loop shows that, to this order and for a large number of gluons, either the BDS ansatz or the relation between scattering amplitudes and Wilson loops should be modified. The two-loop calculation of the six gluon amplitude discussed in detail in Section 2.7 supports the conclusion above that the BDS ansatz is incomplete. Finding an analytic expression and an interpretation for the appropriate modification remains an important open problem.

4. Conclusions

In these lectures we described modern techniques for the calculation of scattering amplitudes at weak and strong coupling. Tools such as the MHV vertex rules, on-shell recursion relations and generalized unitarity can lead to substantial simplifications of analytic calculations in weakly coupled in $\mathcal{N} = 4$ SYM theory as well as in theories with reduced or without supersymmetry. We also discussed how the AdS/CFT correspondence implies that, to leading order in the strong coupling expansion, scattering amplitudes of the $\mathcal{N} = 4$ SYM theory are related to special light-like Wilson loops with cusps. This relation, confirmed by a direct two-loop calculation of the six-point amplitude, cast doubts on a conjectured all-order resummation of MHV amplitudes.

It is clear that additional structure, waiting to be uncovered, is present in $\mathcal{N} = 4$ SYM theory and that it may be sufficiently powerful to completely determine, at least in some sectors, the kinematic dependence of the scattering matrix of the theory. There are many other amplitudes of interest in the $\mathcal{N} = 4$ SYM theory which are yet uncalculated; the techniques described here offer means for attacking such calculations while nonetheless leaving room for improvement.

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