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A SPIN CHAIN FROM STRING THEORY*

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We study the semiclassical spectrum of bosonic string theory on $\mathrm{AdS}_3 \times S^1$ in the limit of large AdS angular momentum. At leading semiclassical order, this is a subsector of the IIB superstring on $\mathrm{AdS}_5 \times S^5$. The theory includes strings with $K \geq 2$ spikes which approach the boundary in this limit. We show that, for all K, the spectrum of these strings exactly matches that of the corresponding operators in the dual gauge theory up to a single universal prefactor which can be identified with the cusp anomalous dimension. We propose a precise map between the dynamics of the spikes and the classical $\mathrm{SL}(2,\mathbb{R})$ spin chain which arises in the large-spin limit of $\mathcal{N}=4$ Super Yang–Mills theory.

PACS numbers:

1. Introduction

Logarithmic scaling of anomalous dimensions with Lorentz spin (S) is a characteristic feature of composite operators in four-dimensional gauge theory [1, 2] (for a recent discussion see [3]). Although initially observed in perturbative gauge theory, the AdS/CFT correspondence has provided strong evidence that $\log S$ scaling persists at strong coupling [4]. The best understood example is the anomalous dimension of twist-two operators in the planar limit of $\mathcal{N}=4$ SUSY Yang–Mills, which has the form,

$$\gamma \;=\; 2\Gamma(\lambda)\,\log(S) + O(S^0)\,.$$

Here Γ is a function of the 't Hooft coupling $\lambda = g^2_{\mathrm{YM}}N$, known as the cusp anomalous dimension, with the following behaviour at weak and strong coupling, respectively[1],

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[1] The emergent integrability of the $\mathcal{N}=4$ theory has subsequently lead to a conjecture [17] for $\Gamma(\lambda)$ which should hold for all values of λ.

$$\Gamma(\lambda) = \frac{\lambda}{4\pi^2} + O(\lambda^2)\,, \qquad \text{for } \lambda \ll 1\,,$$
$$= \frac{\sqrt{\lambda}}{2\pi} + O(\lambda^0)\,, \qquad \text{for } \lambda \gg 1\,.$$

The strong coupling result was first obtained in [4] by evaluating the classical energy of the corresponding state in the dual string theory on $\mathrm{AdS}_5 \times S^5$ in the limit $S \to \infty$. One of the aims of this paper is to extend this analysis to string states dual to operators of arbitrary twist.

In one-loop gauge theory, the large-S spectrum of operators of fixed twist has been studied in detail by Belitsky, Gorsky, Korchemsky and collaborators (see [5–7] and references therein). We will focus on operators of the form

$$\hat{O} \sim \mathrm{Tr}_N \left[\mathcal{D}_+^{s_1} Z \mathcal{D}_+^{s_2} Z \ldots \mathcal{D}_+^{s_J} Z \right] \tag{1}$$

having total Lorentz spin $S = \sum_{l=1}^{J} s_l$ and twist J. Here $\mathcal{D}_+$ is a covariant derivative with conformal spin plus one and Z is one of the three complex adjoint scalar fields of the $\mathcal{N} = 4$ theory. For $S \gg 1$, the resulting one-loop spectrum of anomalous dimensions lies in a band

$$\gamma_{\min} = \frac{\lambda}{4\pi^2} 2 \log(S) \leq \gamma_{1-\mathrm{loop}} \leq \gamma_{\max} = \frac{\lambda}{4\pi^2} J \log(S)\,. \tag{2}$$

More precisely, for each positive integer K with $2 \leq K \leq J$, the large-S theory contains families of states labelled by $K-1$ non-negative integers $l_k \sim S$ with $\sum_{k=1}^{K-1} l_k = S$. The spectrum of these states is,

$$\gamma_{1-\mathrm{loop}}[l_1, l_2, \ldots, l_{K-1}] = \frac{\lambda}{4\pi^2} \Bigg(K \log S + H_K \left[\frac{l_1}{S}, \frac{l_2}{S}, \ldots, \frac{l_{K-1}}{S} \right] + C_{1-\mathrm{loop}} + O(1/\log^2 S) \Bigg)\,, \tag{3}$$

where $C_{1-\mathrm{loop}}$ is a constant of order S^0 which may depend on K and J but not on the integers l_k. The function H_K is the Hamiltonian of a certain *classical* spin chain with K sites. This chain can be thought of as a classical $S \to \infty$ limit of the quantum spin chain which governs the complete one-loop spectrum of anomalous dimensions (*i.e.* for all values of S and J), in the planar $\mathcal{N} = 4$ theory [8–10]. The dynamical variables are classical "spins" $\mathcal{L}_k^{\pm}$, $\mathcal{L}_k^0$, for $k = 1, 2, \ldots, K$ whose Poisson brackets provide a representation of the $\mathrm{sl}(2, \mathbb{R})$ Lie algebra at each site,

$$\{\mathcal{L}_k^+, \mathcal{L}_{k'}^-\} = 2i\delta_{kk'}\mathcal{L}_k^0\,, \qquad \{\mathcal{L}_k^0, \mathcal{L}_{k'}^{\pm}\} = \pm i\delta_{kk'}\mathcal{L}_k^{\pm}\,. \tag{4}$$

The corresponding quadratic Casimir at each site is appropriate for a representation of zero spin,

$$\mathcal{L}_k^+ \mathcal{L}_k^- + \left(\mathcal{L}_k^0\right)^2 = 0\,, \tag{5}$$

for $k = 1, \ldots, K$. As we review in the next section, the classical chain is integrable [11] with a continuous spectrum governed by a spectral curve of genus $K-2$. The discrete spectrum (3) is obtained by applying appropriate Bohr–Sommerfeld semiclassical quantization conditions.

In this paper we will study the corresponding large-S limit of the dual string theory. Importantly, we are interested in the $S \to \infty$ limit with J fixed. Thus the string states we seek are dual to the states of a spin chain of *fixed* length. For earlier work relevant to this limit see [3, 7, 12–18] and especially [19]. Our main result is a calculation the semiclassical string spectrum at large S. We find precise agreement with the gauge theory spectrum (3) up to a single overall function of the coupling which can be identified with the cusp anomalous dimension $\Gamma(\lambda)$. We will also propose a mechanism whereby the gauge theory spin chain emerges as a decoupled subsector of semiclassical string theory in the large-S limit. The main results are briefly described in the remainder of this introductory section.

For large 't Hooft coupling, operators of the form (1) are dual to semiclassical strings moving on an $\mathrm{AdS}_3 \times S^1$ submanifold of $\mathrm{AdS}_5 \times S^5$. Here spin S corresponds to angular momentum on AdS_3 and twist J to angular momentum on S^1. Generic solutions of the equation of motion can be constructed (somewhat implicitly) by the method of finite gap integration [22] (see also [20, 21, 23, 24]). The leading-order semiclassical spectrum is then obtained by applying the Bohr–Sommerfeld conditions derived in [23, 24]. Solutions are classified by the number of gaps, K, in the spectrum of the auxiliary linear problem. These solutions are analogous to classical solutions with K oscillator modes turned on in flat space string theory[2].

There are some superficial similarities between the spectrum of K-gap strings and that of the classical spin chain described above. In particular both are governed by a hyperelliptic spectral curve and solutions correspond to linear motion on the Jacobian in both cases. However, there are also important differences which reflect the fact that the string has an infinite number of degrees of freedom while the chain has a only one degree of freedom on each of its K sites. As we review in Section 3, the infinite tower of string modes leads to essential singularities in the spectral data of the string which are absent in the corresponding description of the spin

[2] In fact this is not just an analogy. In a limit where the equations of motion linearize around a pointlike string there is a one-to-one correspondence between the mode expansion of the linear problem and the gaps of the non-linear one.

chain. A related issue is that the string theory curve is only determined implicitly by the existence of a certain normalised abelian differential dp. The normalisation conditions for dp are transcendental and generally cannot be solved to give an unconstrained parametrisation of the curve[3].

In the following we will consider the large-S limit of the finite gap construction at fixed K and J. We will find that large AdS angular momentum leads to drastic simplifications which allow the solution of the period conditions in closed form. The main result is that the resulting semiclassical string spectrum coincides precisely with the spectrum (3) of the spin chain up to the overall coupling dependence. In particular we find,

$$\gamma_{\rm string}[l_1, l_2, \ldots, l_{K-1}] = \frac{\sqrt{\lambda}}{2\pi}\left(K \log S + H_K\left[\frac{l_1}{S}, \frac{l_2}{S}, \ldots, \frac{l_{K-1}}{S}\right] + C_{\rm string} + O(1/\log^2 S)\right), \tag{6}$$

where H_K is the spin chain Hamiltonian, $C_{\rm string}$ is an undetermined constant and $l_i \sim S$ are positive integers such that $\sum_{i=1}^{K-1} l_i = S$. Worldsheet σ-model loop corrections to this semiclassical formula are suppressed by powers of $1/\sqrt{\lambda}$. The occurrence of the first term on the RHS of (6) has been verified in a previous studies of finite gap solutions [7, 15], the new feature of our result is the detailed agreement with the spin chain which emerges in the second term which is $O(S^0)$. The results are consistent with the conjecture,

$$\gamma[l_1, l_2, \ldots, l_{K-1}] = \Gamma(\lambda)\left(K \log S + H_K\left[\frac{l_1}{S}, \frac{l_2}{S}, \ldots, \frac{l_{K-1}}{S}\right] + C(\lambda)\right) + O(1/\log^2 S), \tag{7}$$

for the exact large-S spectrum where $\Gamma(\lambda)$ is the cusp anomalous dimension introduced above. A similar conjecture was made for the exact spectrum of a closely related set of operators in large-N QCD in [5][4]. The exactness of this relation to all loops was also established in [5], for the leading term of order $\log S$, using a relation between anomalous dimensions and Wilson lines.

The $S \to \infty$ limit considered here is quite different from the thermodynamic $J \to \infty$ limit of the chain where the full spectrum is determined by the Asymptotic Bethe Ansatz Equations (ABAE) [17, 25, 26]. *A priori* there is no reason why the ABAE should apply to a spin chain of fixed length.

[3] I would like to thank Harry Braden for emphasising this point.
[4] See Eq. (3.45) in this reference.

On the other hand, it was argued in [16], that the *lowest* operator dimension for fixed, large S is independent of J and can therefore be evaluated in the $J \to \infty$ limit using the ABAE[5]. The large-S semiclassical spectrum (6) obtained in this paper is also independent of J which suggests that the universality proposed in [16] should apply to *all* operators with (one) large spin, not just the operator of lowest dimension.

The exact agreement between the semiclassical spectrum of a discrete spin chain and a continuous string initially seems somewhat mysterious[6]. In the final part of the paper, we will propose a precise account of how the classical spins naturally emerge from the string at large S. The key phenomenon is already visible in the rotating folded string studied in [4]. Logarithmic scaling of the form $\Delta - S \simeq \sqrt{\lambda}/2\pi\, 2\log S$ naturally arises arises when the two folds of the string approach the boundary of AdS_3. It is natural to expect that the finite gap solutions studied in this paper correspond to configurations with K spikes which approach the boundary as $S \to \infty$ giving the scaling $\Delta - S \simeq \sqrt{\lambda}/2\pi\, K \log S$. Rigidly rotating solutions of this type were constructed in [19].

In static conformal gauge, the string σ-model coincides with the $\mathrm{SL}(2,\mathbb{R})$ Principal Chiral Model. The dynamical variable is the Noether current $j_\pm(\sigma,\tau) = g^{-1}\partial_\pm g$ with $g(\sigma,\tau) \in SL(2,\mathbb{R})$ corresponding to right multiplication in the group. In the limit $S \to \infty$ we will argue that the corresponding charge density becomes δ-function localised at the positions of the K spikes. This localisation leads to a natural proposal for spin degrees of freedom,

$$L_k^A \; = \; \lim_{S\to\infty} \left[\frac{\sqrt{\lambda}}{8\pi} \int_{\mu_k}^{\mu_{k+1}} d\sigma \, j_\tau^A(\sigma,\tau) \right] \tag{8}$$

for $k = 1, 2, \ldots K$ where the index $A = 0, 1, 2$ runs over the generators of $\mathrm{SL}(2,\mathbb{R})$. Here, the K-th spike is located at $\sigma = \sigma_k$ and μ_k is are arbitrary points on the string with $\mu_k < \sigma_k < \mu_{k+1}$. We propose that the variables defined in (8) are related to the spins introduced above as, $\mathcal{L}_k^0 = L_k^0$ and $i\mathcal{L}_k^\pm = L_k^1 \pm iL_k^2$. In particular one may then verify the Poisson brackets (4) and the quadratic Casimir relation (5). With this identification it can be shown that the monodromy matrix of the string reduces to that of the classical spin chain as $S \to \infty$. A related correspondence between the dynamics of spikes and the spin chain was suggested in [19]. The emergence of a spin chain from a limit of string theory was also discussed in [28].

The remainder of the paper is organised as follows. In Section 2 we provide a brief review of the semiclassical spin chain which arises in one-

[5] Indeed this strategy was used in [17] to obtain the conjectured exact form of $\Gamma(\lambda)$.

[6] Discreteness here refers to the fact that the spin chain lives on a spatial lattice. A related mystery for the case of the magnon dispersion relation is resolved in [27].

loop gauge theory. In Section 3 we introduce the finite gap construction of string solutions and the corresponding spectral curve Σ. In Section 4 we study the $S \to \infty$ limit for generic solutions and show that it corresponds to a particular degeneration of the spectral curve. In Section 5, we solve the period conditions for the differential dp in the degenerate limit and find the semiclassical spectrum of the model. In Section 6 we give an interpretation of our results in terms of spikey strings and propose a precise map between spikes and spins. Finally the results are discussed in Section 7.

2. The gauge theory spin chain

We consider the one-loop anomalous dimensions of operators in the non-compact rank one subsector of planar $\mathcal{N} = 4$ SUSY Yang–Mills (also known as the sl(2) sector). Generic single-trace operators in this sector are labelled by their Lorentz spin S and $\mathrm{U}(1)_{\mathrm{R}}$ charge J and have the form (1). The classical dimension of each operator is $\Delta_0 = J + S$ and its twist (classical dimension minus spin) is therefore equal to J.

The one-loop anomalous dimensions of operators in the sl(2) sector are determined by the eigenvalues of the Hamiltonian of the Heisenberg $\mathrm{XXX}_{-\frac{1}{2}}$ spin chain with J sites. Each site of this chain carries a representation of $\mathrm{SL}(2, \mathbb{R})$ with quadratic Casimir equal to minus one half. Our discussion of the chain in this section follows that of [5, 7] (see in particular Section 2.2 of [7]). Here we will focus on the large-spin limit of the chain: $S \to \infty$ with J fixed. This is a effectively a semiclassical limit where $1/S$ plays the role of Planck's constant $\hbar$ [5, 7]. In this limit the quantum spins are replaced by the classical variables $\mathcal{L}_k^{\pm}$, $\mathcal{L}_k^0$, for $k = 1, 2, \ldots, J$, introduced above. The commutators of spin operators are replaced by the Poisson brackets (4) of these classical spins. As the quadratic Casimir equal to $-1/2$ is negligable in the $S \to \infty$ limit, the classical spins at each site obey the relation (5) up to $1/S$ corrections. We will restrict our attention to states obeying the highest weight condition,

$$\sum_{k=1}^{J} \mathcal{L}_k^{\pm} \; = \; 0 \, . \tag{9}$$

Integrability of the classical spin chain starts from the existence of a Lax matrix,

$$\mathbb{L}_k(u) \; = \; \begin{pmatrix} u + i\mathcal{L}_k^0 & i\mathcal{L}_k^+ \\ i\mathcal{L}_k^- & u - i\mathcal{L}_k^0 \end{pmatrix} ,$$

where $u \in \mathbb{C}$ is a spectral parameter. A tower of conserved quantities are obtained by constructing the monodromy,

$$\begin{aligned} t_J(u) &= \mathrm{Tr}_2\left[\mathbb{L}_1(u)\mathbb{L}_2(u)\ldots\mathbb{L}_J(u)\right] \\ &= 2u^J + q_2 u^{J-2} + \ldots + q_{J-1}u + q_J\,. \end{aligned} \tag{10}$$

At large-S we find $q_2 = -S^2$ up to corrections of order of $1/S$. One may check starting from the Poisson brackets (4) that the conserved charges, q_j, $j = 2, 3, \ldots J$ are in involution: $\{q_j, q_k\} = 0\ \forall\ j,\ k$. Taking into account the highest-weight constraint (9), this is a sufficient number of conserved quantities for complete integrability of the chain.

The one-loop spectrum of operator dimensions at large-S is determined from the semiclassical spectrum of the spin chain. It has different branches, labelled by an integer $K \leq J$, corresponding to the highest non-zero conserved charge [7],

$$q_K \neq 0\,, \qquad q_j = 0\,, \qquad \text{for all } j > K\,.$$

The one-loop anomalous dimensions are given as,

$$\gamma_{\rm one-loop} = \Delta - J - S = \frac{\lambda}{8\pi^2}\log\left(q_K\right) + C_{1-\rm loop} + O(1/\log^2 S)\,, \tag{11}$$

where $C_{1-\rm loop}$ is an undetermined constant which is independent of the moduli q_j. We call the branch with $K = J$ the highest sector. For each $K < J$ there is also a sector of states isomorphic to the highest sector of a shorter chain with only K sites. In the limit of large-S, the conserved charge q_j scales as S^j for $j = 2, \ldots, K$. Hence (11) exhibits the expected logarithmic scaling with S. In the following it will be useful to introduce rescaled charges $\hat{q}_j$, such that $q_j = S^j\,\hat{q}_j$. In particular $\hat{q}_2 = -1$ up to corrections of the order of $1/S$.

At the classical level, the conserved charges $\hat{q}_j$ vary continuously. The discrete spectrum described in the Introduction arises from imposing appropriate Bohr–Sommerfeld quantisation conditions. To describe these we introduce the spectral curve of the spin chain,

$$\begin{aligned} \Gamma_K\ : \qquad y^2 &= \prod_{l=1}^{K-2}(x - x_l) \\ &= x^{2K}\left[1 - \frac{1}{4}\hat{\mathbb{P}}_K\left(\frac{1}{x}\right)^2\right] \end{aligned}$$

with

$$\hat{\mathbb{P}}_K\left(\frac{1}{x}\right) = 2 - \frac{1}{x^2} + \frac{\hat{q}_3}{x^3} + \ldots + \frac{\hat{q}_K}{x^K}\,,$$

which is a hyperelliptic Riemann surface of genus $K-2$. The rescaled spectral parameter $x = u/S$ is held fixed as $S \to \infty$ and the rescaled conserved charges $\hat{q}_j$, $j = 3, \ldots, K$ correspond to moduli of the curve. Notice that the curve is highly non-generic in that the positions of the $2K-2$ branch points x_l are determined in terms of $K-2$ parameters $\hat{q}_j$. The curve Γ_K corresponds to a double cover of the x plane as shown in Fig. 1. We also define $K-1$ one-cycles α_j, $j = 1, \ldots, J-1$ as shown in the figure[7].

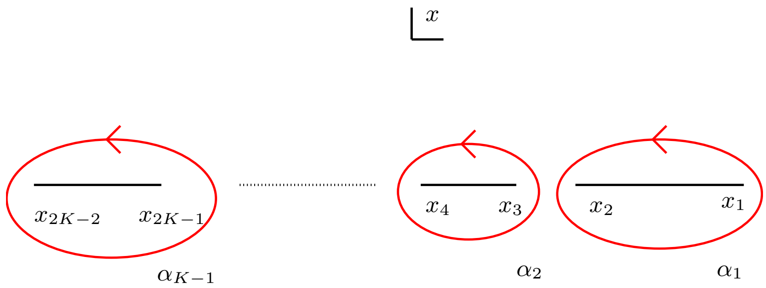

Fig. 1. The cut x-plane corresponding to the curve Γ_K.

The Bohr–Sommerfeld conditions are expressed in terms of a certain meromorphic differential on Γ_K,

$$d\hat{p} \;=\; -i\,\frac{dx}{x^2}\frac{\hat{\mathbb{P}}'_K\left(\frac{1}{x}\right)}{\sqrt{\hat{\mathbb{P}}_K\left(\frac{1}{x}\right)^2 - 4}} \tag{12}$$

and they read,

$$-\frac{1}{2\pi i}\oint_{\alpha_j} x\, d\hat{p} \;=\; \frac{l_j}{S}\,, \qquad l_j \in \mathbb{Z}^+\,, \tag{13}$$

for $j = 1, 2, \ldots, K-1$. The integers l_j which label the states in the spectrum obey the conditions,

$$\sum_{j=1}^{K-1} l_j = S\,, \qquad \sum_{j=1}^{K-1} j l_j = 0 \bmod K\,. \tag{14}$$

The first equality is related to the fact that the integers l_j count numbers of Bethe roots associated with each cut. Each root carries one unit of spin and thus the total number of roots is equal to S. The second condition is

[7] To state the main results of [5,7], we will not need a to introduce a full basis of cycles on Γ_K.

imposed by the cyclicity of the trace and corresponds to a vanishing of the total momentum of the chain.

In order for a leading order semiclassical approach for any quantum mechanical problem to be valid it is necessary that the quantum numbers are large. Thus we must take $l_j \sim O(S)$ as $S \to \infty$ for each j. Then both sides of Eq. (13) scale like S^0. The $K-2$ independent equations (13) determine the charges,

$$\hat{q}_j \;=\; \hat{q}_j\left[\frac{l_1}{S}, \frac{l_2}{S}, \ldots, \frac{l_{K-1}}{S}\right] ,$$

$j = 3, \ldots, K$. Finally the spectrum of one-loop anomalous dimensions is given as,

$$\gamma[l_1, \ldots, l_{K-1}] \;=\; \frac{\lambda}{4\pi^2}\Bigg(K \log S + H_K\left[\frac{l_1}{S}, \frac{l_2}{S}, \ldots, \frac{l_{K-1}}{S}\right] + C_{1-\mathrm{loop}} + O(1/\log^2 S)\Bigg) , \qquad (15)$$

where $H_K = \log \hat{q}_K$.

3. Semiclassical string theory

At large 't Hooft coupling $\sqrt{\lambda} \gg 1$, gauge theory operators of the sl(2) sector are dual to semiclassical strings moving on $\mathrm{AdS}_3 \times S^1$. The U(1) R-charge J corresponds to momentum in the S^1 direction and the conformal spin S corresponds to angular momentum in AdS_3. We introduce string worldsheet coordinates $\sigma \sim \sigma + 2\pi$ and τ and the corresponding lightcone coordinates $\sigma_{\pm} = (\tau \pm \sigma)/2$ and we define lightcone derivatives $\partial_{\pm} = \partial_\tau \pm \partial_\sigma$. The space-time coordinates correspond to fields on the string worldsheet: we introduce $\phi(\sigma, \tau) \in S^1$ and parametrize AdS_3 with a group-valued field $g(\sigma, \tau) \in \mathrm{SL}(2, \mathbb{R}) \simeq \mathrm{AdS}_3$. The $\mathrm{SL}(2,\mathbb{R})_{\mathrm{R}} \times \mathrm{SL}(2,\mathbb{R})_{\mathrm{L}}$ isometries of AdS_3 correspond to left and right group multiplication. The Noether current corresponding to right multiplication is $j_{\pm} = g^{-1}\partial_{\pm} g$. Following [22], we work in static, conformal gauge with a flat worldsheet metric and set,

$$\phi(\sigma, \tau) \;=\; \frac{J}{\sqrt{\lambda}}\tau .$$

In this gauge, the string action becomes that of the $\mathrm{SL}(2, \mathbb{R})$ Principal Chiral model,

$$S_\sigma \;=\; \frac{\sqrt{\lambda}}{4\pi}\int_0^{2\pi} d\sigma\, \frac{1}{2}\mathrm{Tr}_2\left[j_+ j_-\right] .$$

String motion is also subject to the Virasoro constraint,

$$\frac{1}{2}\mathrm{Tr}_2\left[j_\pm^2\right] \;=\; \frac{J^2}{\lambda}\,.$$

Classical integrability of string theory on $\mathrm{AdS}_3 \times S^1$ follows from the construction of the monodromy matrix [29],

$$\Omega\left[x;\tau\right] \;=\; \mathcal{P}\exp\left[\frac{1}{2}\int_0^{2\pi} d\sigma\left(\frac{j_+}{x-1}+\frac{j_-}{x+1}\right)\right] \in SL(2,\mathbb{R})\,,$$

whose eigenvalues $w_\pm = \exp(\pm i\,p(x))$ are τ-independent for all values of the spectral parameter x. It is convenient to consider the analytic continuation of the monodromy matrix $\Omega[x;\tau]$ and of the quasi-momentum $p(x)$ to complex values of x. In this case Ω will take values in $\mathrm{SL}(2,\mathbb{C})$ and appropriate reality conditions must be imposed to recover the physical case.

The eigenvalues $w_\pm(x)$ are two branches of an analytic function defined on the spectral curve,

$$\Sigma_\Omega : \qquad \det\left(w\mathbb{I} - \Omega[x;\tau]\right) = w^2 - 2\cos p(x)\,w + 1 \;=\; 0\,, \qquad w, x \in \mathbb{C}\,.$$

This curve corresponds to a double cover of the complex x-plane with branch points at the simple zeros of the discriminant $D = 4\sin^2 p(x)$. In addition the monodromy matrix defined above is singular at the points above $x = \pm 1$. Using the Virasoro constraint, one may show that, $p(x)$ has a simple poles at these points,

$$p(x) \;\sim\; \frac{\pi J}{\sqrt{\lambda}}\frac{1}{(x\pm 1)^2} \;+\; O\left((x\pm 1)^0\right) \tag{16}$$

as $x \to \mp 1$. Hence the discriminant D has essential singularities at $x = \pm 1$ and D must therefore have an infinite number of zeros which accumulate at these points. Formally we may represent the discriminant as a product over its zeros and write the spectral curve as

$$\Sigma_\Omega : \qquad y_\Omega^2 = 4\sin^2 p(x) \;=\; \prod_{j=1}^{\infty}\left(x - x_i\right)\,.$$

For generic solutions the points $x = x_i$ are distinct and the curve Σ_Ω has infinite genus.

In order to make progress it is necessary to focus on solutions for which the discriminant has only a finite number $2K$ of simple zeros and the spectral curve Σ_Ω has finite genus. The infinite number of additional zeros of the discriminant D must then have multiplicity two or higher. These are known as finite gap solutions[8]. In this case, dp is a meromorphic differential on the hyperelliptic curve,

$$\Sigma : \qquad y^2 = \prod_{i=1}^{2K} (x - x_i)$$

of genus $g = K - 1$ which is obtained by removing the double points of $\hat{\Sigma}$ (see [23]). For ease of presentation we will consider only even values of K, the generalisation to odd values is straightforward.

In the following we will focus on curves where all the branch points lie on the real axis and outside the interval $[-1, +1]$. This corresponds to string solutions where only classical oscillator modes which carry positive spin are activated. In the dual gauge theory these solutions are believed to correspond to operators of the form (1) where only the covariant derivative $\mathcal{D}_+$, which carries positive spin, appears [22,30]. We label the branch points of the curve according to,

$$\Sigma : \qquad y^2 \; = \; (x - b_+)\,(x - b_-) \prod_{i=1}^{K-1} \left(x - a_+^{(i)}\right)\left(x - a_-^{(i)}\right) , \tag{17}$$

with the ordering,

$$\begin{aligned} a_-^{(K-1)} &\leq a_-^{(K-2)} \; \ldots \; \leq a_-^{(1)} \leq b_- \leq -1 \, , \\ a_+^{(K-1)} &\geq a_+^{(K-2)} \; \ldots \; \geq a_+^{(1)} \geq b_+ \geq +1 \, . \end{aligned}$$

The branch points are joined in pairs by cuts $C_I^{\pm}$, $I = 1, 2, \ldots, K/2$ as shown in Fig. 2. We also define a standard basis of one-cycles, $\mathcal{A}_I^{\pm}$, $\mathcal{B}_I^{\pm}$. Here $\mathcal{A}_I^{\pm}$ encircles the cut $C_I^{\pm}$ on the upper sheet in an anti-clockwise direction and $\mathcal{B}_I^{\pm}$ runs from the point at infinity on the upper sheet to the point at infinity

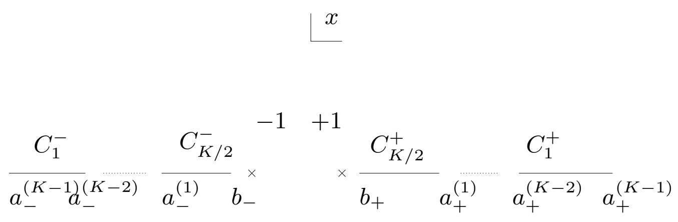

Fig. 2. The cut x-plane corresponding to the curve Σ.

[8] Strictly speaking these are not generic solutions of the string equations of motion. However, as K can be arbitrarily large, it is reasonable to expect that generic solutions could be obtained by an appropriate $K \to \infty$ limit.

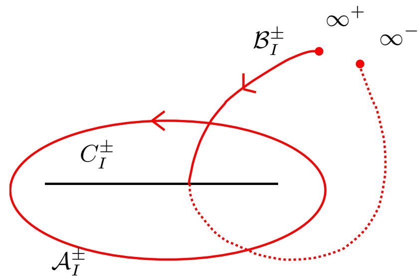

Fig. 3. The cycles on Σ. The index I runs from 1 to $K/2$.

on the lower sheet passing through the cut $C_I^\pm$, as shown in Fig. 3. For any $x_0 \in \mathbb{C}$, we will sometimes use the notation $x_0^\pm$ to denote the two points on Σ where $x = x_0$.

The quasi-momentum $p(x)$ gives rise to a meromorphic abelian differential dp on Σ. From (16) we see that the differential has second-order poles at the points above $x = +1$ and $x = -1$ on Σ. On the top sheet we have,

$$dp \longrightarrow -\frac{\pi J}{\sqrt{\lambda}} \frac{dx}{(x \pm 1)^2} + O\left((x \pm 1)^0\right) \tag{18}$$

as $x \to \mp 1$. There are also two second-order poles at the points $x = \mp 1$ on the lower sheet related by the involution $dp \to -dp$. The value of the Noether charges Δ and S is encoded in the asymptotic behaviour of dp near the points $x = 0$ and $x = \infty$ on the top sheet,

$$dp \longrightarrow -\frac{2\pi}{\sqrt{\lambda}} (\Delta + S) \frac{dx}{x^2} \qquad \text{as } x \to \infty\,, \tag{19}$$

$$dp \longrightarrow -\frac{2\pi}{\sqrt{\lambda}} (\Delta - S)\, dx \qquad \text{as } x \to 0\,. \tag{20}$$

For a valid semiclassical description, the conserved charges J, S and Δ should all be $O(\sqrt{\lambda})$ with $\sqrt{\lambda} \gg 1$.

In addition to the above relations, dp must obey $2K$ normalisation conditions,

$$\oint_{\mathcal{A}_I^\pm} dp = 0\,, \qquad \oint_{\mathcal{B}_I^\pm} dp = 2\pi n_I^\pm\,, \tag{21}$$

with $I = 1, 2, \ldots, K/2$. The integers $n_I^\pm$ correspond to the mode numbers of the string. In the following we will assign the mode numbers so as to pick out the K lowest modes of the string which carry positive angular momentum, including both left and right movers. This is accomplished by setting $n_I^\pm = \pm I$ for $I = 1, \ldots, K/2$.

To find the spectrum of classical string solutions we must first construct the meromorphic differential dp with the specified pole behaviour (18). The most general possible such differential has the form,

$$\begin{aligned} dp &= dp_1 + dp_2 = -\frac{dx}{y}\left[f(x) + g(x)\right] , \\ f(x) &= \sum_{\ell=0}^{K-2} C_\ell \, x^\ell , \\ g(x) &= \frac{\pi J}{\sqrt{\lambda}} \left[\frac{y_+}{(x-1)^2} + \frac{y_-}{(x+1)^2} + \frac{y'_+}{(x-1)} - \frac{y'_-}{(x+1)} \right] , \end{aligned} \tag{22}$$

with $y_\pm = y(\pm 1)$ and

$$y'_\pm = \left. \frac{dy}{dx} \right|_{x=\pm 1} .$$

Here the second term dp_2 is a particular differential with the required poles and the first term dp_1 is a general holomorphic differential on Σ. The resulting curve Σ and differential dp depend on $3K-1$ undetermined parameters $\{b_\pm, a_\pm^{(i)}, C_\ell\}$ with $i = 1, \ldots, K-1$, $\ell = 0, 1, \ldots, K-2$. We then obtain $2K$ constraints on these parameters from the normalisation equations (21), leaving us with a $K-1$ dimensional moduli space of solutions [22]. As mentioned above, a significant difficulty with this approach is that the normalisation conditions are transcendental and cannot be solved in closed form.

A convenient parametrisation for the moduli space is given in terms of the K *filling fractions*,

$$\mathcal{S}_I^\pm = \frac{1}{2\pi i} \frac{\sqrt{\lambda}}{4\pi} \oint_{\mathcal{A}_I^\pm} \left(x + \frac{1}{x} \right) dp ,$$

with $I = 1, \ldots, K/2$, subject to the level matching constraint,

$$\sum_{I=1}^{K/2} n_I^+ \mathcal{S}_I^+ + n_I^- \mathcal{S}_I^- = 0 .$$

Here the total AdS angular momentum is given as

$$S = \sum_{I=1}^{K/2} \mathcal{S}_I^+ + \mathcal{S}_I^-$$

and is regarded as one of the moduli of the solution. The significance of the filling fractions is that they constitute a set of normalised action variables for the string[9]. They are canonically conjugate to angles $\varphi_I \in [0, 2\pi]$ living on the Jacobian torus $\mathcal{J}(\Sigma)$. Evolution of the string solution in both worldsheet coordinates, σ and τ, corresponds to linear motion of these angles [23].

The constraints described above uniquely determine (Σ, dp) for given values of $\mathcal{S}_I^{\pm}$, and one may then extract the string energy from the asymptotics (19,20) which imply,

$$\begin{aligned}\Delta + S &= -\frac{\sqrt{\lambda}}{2\pi} C_{K-2}\,, \\ \Delta - S &= -\frac{\sqrt{\lambda}}{2\pi}\frac{C_0}{y(0)} + \frac{J}{2y(0)}\left(y_+ + y_- - y'_+ - y'_-\right)\,.\end{aligned}$$

In this way, one obtains a set of transcendental equations which determine the string energy as a function of the filling fractions,

$$\Delta \;=\; \Delta\left[\mathcal{S}_1^+, \mathcal{S}_1^-, \ldots, \mathcal{S}_{K/2}^+, \mathcal{S}_{K/2}^-\right]\,.$$

Finally the leading order semiclassical spectrum of the string is obtained by imposing the Bohr–Sommerfeld conditions which impose the integrality of the filling fractions [23, 24]: $\mathcal{S}_I^{\pm} \in \mathbb{Z}$, $I = 1, 2, \ldots, K/2$. For uniform validity of the semi-classical approach we should focus on states where $\mathcal{S}_I^{\pm} \sim \sqrt{\lambda}$ for each I. Higher-loop corrections in the string σ-model are then suppressed by powers of $1/\sqrt{\lambda}$.

4. The large-S limit

In this section we will take an $S \to \infty$ limit with fixed J for the generic K-gap solution. The 't Hooft coupling $\lambda \gg 1$ is also held fixed in the limit. In the genus one ($K = 2$) case this limit has been studied in [7]. At the level of the curve (17), the "outer" branch points $a_{\pm}^{(1)}$ of the $K = 2$ curve scale linearly with S approaching infinity in the large-S limit, while the inner branch points $b_{\pm}$ approach the singular points of dp at $x = \pm 1$. For $K > 2$ we will take a similar limit where the $2K - 2$ branch points $a_{\pm}^{(i)}$ will all scale linearly with the spin. To implement this we set,

$$a_{\pm}^{(i)} \;=\; \rho\tilde{a}_{\pm}^{(i)}$$

[9] The symplectic structure of the string was analysed in detail for the case of strings on $S^3 \times \mathbb{R}$ in [23, 24]. The resulting string σ-model was an SU(2) principal chiral model (PCM). In the context of the finite gap construction one works with a complexified Lax connection and results for the SU(2) and SL(2, $\mathbb{R}$) PCMs differ only at the level of reality conditions which do not affect the conclusion that the filling fractions are the canonical action variables of the string.

for $i = 1, 2, \ldots, K-1$ and take the limit $\rho \to \infty$ with $\tilde{a}^{(i)}_{\pm}$ held fixed. The remaining branch points, $b_{\pm}$, are treated as $O(\rho^0)$. Eventually we will see that $S \sim \rho$ and also that we are forced to take $b_{\pm} \to 1$ as $\rho \to \infty$ as in the genus one case of [7]. A related limit of the K-gap solution was studied in [15]. For convenience we will also set $b_+ = -b_- = b \geq 1$ although the same results are obtained without this condition.

Our main concern is to analyse the limiting behaviour of the equations (17), (21), (22) which define the pair (Σ, dp). The limit has a convenient description in terms of a *degeneration* of the spectral curve Σ. The relevant degeneration is one where the closed cycle $\hat{\mathcal{B}} = \mathcal{B}^+_{K/2} - \mathcal{B}^-_{K/2}$ on Σ pinches at two points as shown in Fig. 4. The result is that the curve Σ, which has genus $K-1$, factorises into two components,

$$\Sigma \longrightarrow \tilde{\Sigma}_1 \cup \tilde{\Sigma}_2 \,, \tag{23}$$

where $\tilde{\Sigma}_1$ is a curve of genus $K-2$ and $\tilde{\Sigma}_2$ is a curve of genus zero. There are two additional marked points on each component where the two curves touch. The degeneration of the curve is determined by the condition that the differential dp has a good limit as $\rho \to \infty$. The main point is that, as $\rho \to \infty$ the normalisation conditions for the differential dp on Σ reduce to conditions on two meromorphic differentials $d\tilde{p}_1$ and $d\tilde{p}_2$ defined on the curves $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$, respectively.

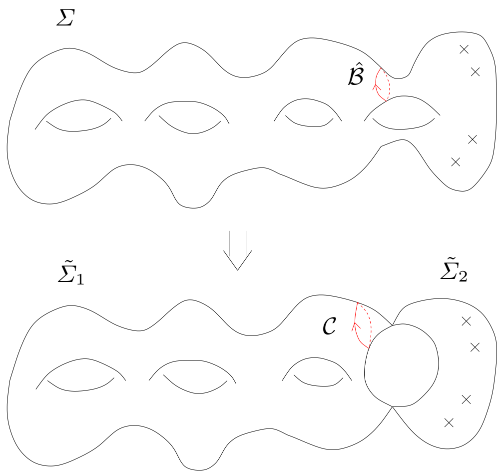

Fig. 4. The degeneration $\Sigma \to \tilde{\Sigma}_1 \cup \tilde{\Sigma}_2$. The four singular points $\pm 1^{\pm}$ are marked with crosses on the curve.

Starting from the original spectral curve,

$$\Sigma: \qquad y^2 \;=\; (x - b_+)\,(x - b_-) \prod_{i=1}^{K-1} \left(x - a^{(i)}_+\right)\left(x - a^{(i)}_-\right)$$

the curve $\tilde{\Sigma}_1$ is obtained by "blowing-up" the region near $x = \infty$. Thus we set

$$x = \rho \tilde{x}\,, \qquad y = \rho^K \tilde{x} \tilde{y}_1 \tag{24}$$

holding $\tilde{x}$ and $\tilde{y}_1$ fixed as $\rho \to \infty$. Thus we obtain the curve,

$$\tilde{\Sigma}_1 : \qquad \tilde{y}_1^2 = \prod_{i=1}^{K-1} \left(\tilde{x} - \tilde{a}_+^{(i)}\right)\left(\tilde{x} - \tilde{a}_-^{(i)}\right).$$

This is a generic hyper-elliptic curve of genus $K-2$. It can be represented as a double cover of the complex $\tilde{x}$-plane with $K-2$ cuts, $\tilde{C}_I^{\pm}$, with $I = 1, 2, \ldots, K/2-1$, and $\tilde{C}_0$ arranged as shown in Fig. 5. We introduce a corresponding set of one-cycles, $\tilde{\mathcal{A}}_I^{\pm}$, $\tilde{\mathcal{A}}_0$ which encircle the cuts $\tilde{C}_I^{\pm}$ and $\tilde{C}_0$, respectively, as shown in Fig. 6. The conjugate cycles $\tilde{\mathcal{B}}_I^{\pm}$, $\tilde{\mathcal{B}}_0$ run from the point at infinity on the top sheet to the point at infinity on the lower sheet, passing through the corresponding cut as also shown in this figure. The curve also has punctures at the two points $0^{\pm}$ above $\tilde{x} = 0$, which correspond to the shrinking cycle[10].

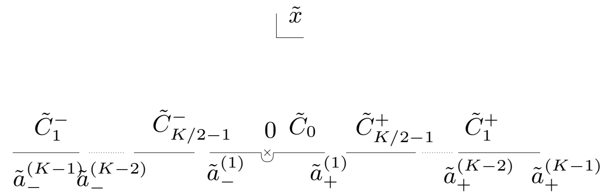

Fig. 5. The cut $\tilde{x}$-plane corresponding to the curve $\tilde{\Sigma}_1$.

We now consider the $\rho \to \infty$ limit of the differential $dp = dp_1 + dp_2$. The first term in (22), denoted dp_1, involves $K-1$ arbitrary constants C_ℓ, $\ell = 0, \ldots, K-2$. Its limiting behaviour is,

$$dp_1 \;\to\; -\frac{d\tilde{x}}{\tilde{y}_1} \sum_{\ell=0}^{K-2} \rho^{\ell+1-K} C_\ell \tilde{x}^{\ell-1}\,.$$

We will choose to scale the undetermined coefficient C_ℓ so as to retain $K-1$ free parameters in the resulting differential on $\tilde{\Sigma}_1$. Thus we set $C_\ell = \tilde{C}_\ell \rho^{K-\ell-1}$ and hold $\tilde{C}_\ell$ fixed. In this case dp_1 has a finite limit as $\rho \to \infty$ while $dp_2 \to 0$. The net result is,

$$dp = dp_1 + dp_2 \;\to\; d\tilde{p}_1 = -\frac{d\tilde{x}}{\tilde{y}_1} \sum_{\ell=0}^{K-2} \tilde{C}_\ell \tilde{x}^{\ell-1}\,. \tag{25}$$

[10] More precisely the resulting meromorphic differential $d\tilde{p}_1$ on $\tilde{\Sigma}_1$ discussed below has poles at these points.

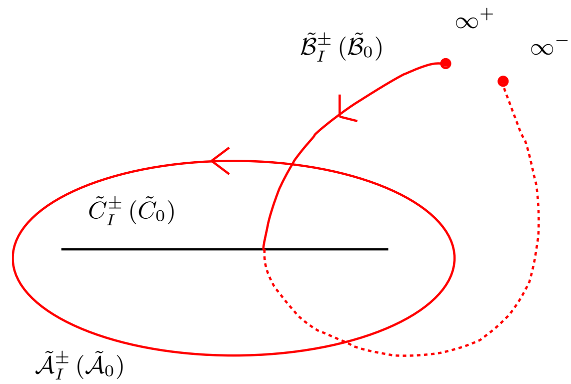

Fig. 6. The cycles on $\tilde{\Sigma}_1$. The index I runs from 1 to $K/2-1$.

This is a meormorphic differential on $\tilde{\Sigma}_1$. It has simple poles at the punctures $0^{\pm}$ above $\tilde{x}=0$ with residues $\pm\tilde{C}_0/\tilde{Q}$ and no other singularities on $\tilde{\Sigma}_1$.

It is easy to check that all but one of the periods of dp on Σ go over to corresponding periods of $d\tilde{p}_1$ on $\tilde{\Sigma}_1$. In particular we find,

$$\lim_{\rho\to\infty}\left[\oint_{\mathcal{A}_I^{\pm}} dp\right] = \oint_{\tilde{\mathcal{A}}_I^{\pm}} d\tilde{p}_1\,, \qquad \lim_{\rho\to\infty}\left[\oint_{\mathcal{B}_I^{\pm}} dp\right] = \oint_{\tilde{\mathcal{B}}_I^{\pm}} d\tilde{p}_1\,,$$

for $I=1,2,\ldots,K/2-1$ and also,

$$\lim_{\rho\to\infty}\left[\oint_{\bar{\mathcal{A}}} dp\right] = \oint_{\tilde{\mathcal{A}}_0} d\tilde{p}_1\,, \qquad \lim_{\rho\to\infty}\left[\oint_{\mathcal{B}_{K/2}^{\pm}} dp\right] = \pm\oint_{\tilde{\mathcal{B}}_0} d\tilde{p}_1\,,$$

where $\bar{\mathcal{A}}=\tilde{\mathcal{A}}_{K/2}^{+}+\tilde{\mathcal{A}}_{K/2}^{-}$. The above results are straightforwardly obtained by making the change of variables (24) in each period integral and keeping only the leading contribution as $\rho\to\infty$.

As shown in Fig. 4, the vanishing cycle $\hat{\mathcal{B}}=\mathcal{B}_{K/2}^{+}-\mathcal{B}_{K/2}^{-}$ becomes a closed contour $\mathcal{C}$ surrounding the marked point on the top sheet above $\tilde{x}=0$, so we also have,

$$\lim_{\rho\to\infty}\left[\oint_{\bar{\mathcal{A}}} dp\right] = \oint_{\mathcal{C}} d\tilde{p}_1 = 2\pi K\,. \tag{26}$$

Comparing with (25), we see that this integral is equal to the residue of $d\tilde{p}_1$ at the point 0^+. Then Eq. (26) is solved by setting $\tilde{C}_0=\tilde{Q}K$.

To summarise the above discussion the defining conditions for the differential dp on Σ have reduced to a set of conditions for the meromorphic differential $d\tilde{p}_1$ on $\tilde{\Sigma}_1$,

- $d\tilde{p}_1$ has *simple poles* at the points $O^{\pm}$ above $\tilde{x}=0$ with residues $\pm K/i$, and no other singularities on $\tilde{\Sigma}$. Thus,

$$d\tilde{p}_1 \longrightarrow \pm\frac{K}{i}\frac{d\tilde{x}}{\tilde{x}}\,, \qquad \text{as } \tilde{x}\to 0\,,$$

- $d\tilde{p}_1$ obeys the normalisation conditions,

$$\oint_{\tilde{\mathcal{A}}_I^{\pm}} d\tilde{p}_1 = 0\,, \qquad \oint_{\tilde{\mathcal{B}}_I^{\pm}} d\tilde{p}_1 = \pm 2\pi I\,, \tag{27}$$

for $I=1,2,\ldots,K/2-1$ and,

$$\oint_{\tilde{\mathcal{A}}^0} d\tilde{p}_1 = 0\,, \qquad \oint_{\tilde{\mathcal{B}}^0} d\tilde{p}_1 = \pi\, K\,. \tag{28}$$

The second component in (23), the curve $\tilde{\Sigma}_2$, arises from blowing up the region around the points $0^{\pm}$ above $x=0$. We scale the coordinates as,

$$y = \tilde{Q}\rho^{K-1}\tilde{y}_2 \qquad \text{with} \qquad \tilde{Q}^2 = \prod_{i=1}^{K-1}\tilde{a}_+^{(i)}\tilde{a}_-^{(i)} \tag{29}$$

and take the limit $\rho\to\infty$ with x and $\tilde{y}_2$ held fixed to get the curve,

$$\tilde{\Sigma}_2: \qquad \tilde{y}_2^2 \;=\; x^2 - b^2\,,$$

which has genus zero. This curve contains the original four singular points $\pm 1^{\pm}$ and also has two new punctures at the points $\infty^{\pm}$ corresponding to the vanishing cycles.

The differential dp_2 on Σ gives rise to the following meromorphic differential on $\tilde{\Sigma}_2$ with double poles at the points over $x=\pm 1$;

$$\begin{aligned} d\tilde{p}_2 \;=\; -\frac{i\pi J}{\sqrt{\lambda}}\Bigg[& \sqrt{b^2-1}\left(\frac{1}{(x-1)^2}+\frac{1}{(x+1)^2}\right) \\ & -\frac{1}{\sqrt{b^2-1}}\left(\frac{1}{(x-1)}-\frac{1}{(x+1)}\right)\Bigg]\frac{dx}{\tilde{y}_2}\,. \end{aligned} \tag{30}$$

The surface $\tilde{\Sigma}_2$ enters when considering the limit of the final period condition corresponding to the cycle $\hat{\mathcal{A}} = \mathcal{A}^+_{K/2}$. The analysis of this condition is presented in the Appendix. In particular we find the following limit for this equation,

$$\lim_{\rho\to\infty}\left[\oint_{\hat{\mathcal{A}}} dp\right] = \int_{\hat{\mathcal{A}}_1^{\rm reg}} d\tilde{p}_1 + \int_{\hat{\mathcal{A}}_2} d\tilde{p}_2 = 0\,, \tag{31}$$

where $\hat{\mathcal{A}}_1^{\rm reg}$ and $\hat{\mathcal{A}}_2$ are suitably regulated "chains" (*i.e.* open contours) on $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$ as shown in Fig. 7. More precisely, we define,

$$\int_{\hat{\mathcal{A}}_1^{\rm reg}} d\tilde{p}_1 = \int_{\epsilon^-}^{\epsilon^+} d\tilde{p}_1\,, \qquad \int_{\hat{\mathcal{A}}_2} d\tilde{p}_2 = -\int_{\infty^-}^{\infty^+} d\tilde{p}_2\,,$$

where $\epsilon = b/\rho$ and $\epsilon^{\pm}$ are the two points above $\tilde{x} = \epsilon$ on $\tilde{\Sigma}_1$ and $\infty^{\pm}$ are the two points above $x = \infty$ on $\tilde{\Sigma}_2$.

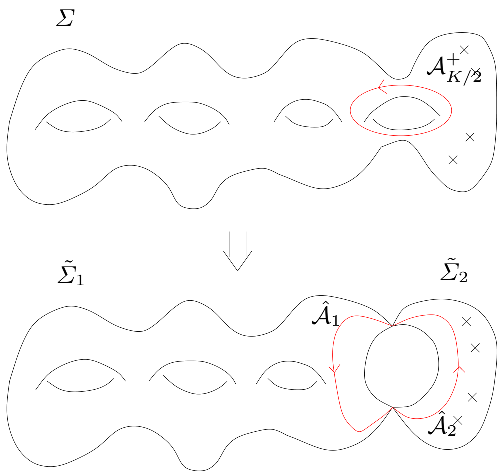

Fig. 7. The "extra" cycle $\mathcal{A}^+_{K/2}$ becomes the sum of the chains $\hat{\mathcal{A}}_1$ and $\hat{\mathcal{A}}_2$.

Finally in the $\rho \to \infty$ limit, the conserved charges have the behaviour,

$$\begin{aligned} \Delta + S &\simeq \frac{\sqrt{\lambda}}{2\pi}\rho \qquad \to \infty\,, \\ \Delta - S &\simeq \frac{\sqrt{\lambda}}{2\pi}\frac{K}{b} + \frac{J}{b}\left(\sqrt{b^2-1} + \frac{1}{\sqrt{b^2-1}}\right), \end{aligned} \tag{32}$$

up to corrections which vanish as $\rho \to \infty$.

5. The solution

To solve for the spectrum in the large-S limit we need to determine the pair $(\tilde{\Sigma}_1, d\tilde{p}_1)$ and then solve the matching condition (31). The first task is similar in nature to the original problem of finding dp, in that we must solve the normalisation conditions for the meromorphic differential $d\tilde{p}_1$ on a generic hyperelliptic curve $\tilde{\Sigma}_1$. There is however, an important difference: while the original differential dp had double poles at four points on Σ, the new differential $d\tilde{p}_1$ has only simple poles with integral residues,

$$d\tilde{p}_1 \;\longrightarrow\; \pm\frac{K}{i}\frac{d\tilde{x}}{\tilde{x}} \qquad \text{as } \tilde{x} \to 0\,, \tag{33}$$

and no other singularities. The resulting problem of reconstructing $(\tilde{\Sigma}_1, d\tilde{p}_1)$ is then a standard one which arises for example in the study of the F-terms of $\mathcal{N} = 2$ SUSY gauge theories [31][11]. We now describe its solution.

Integrating (33) we find that,

$$\tilde{p}_1(\tilde{x}) \;\longrightarrow\; \pm\frac{K}{i}\log\tilde{x}\,, \qquad \text{as } \tilde{x} \to 0\,,$$

and thus we have,

$$\exp\left(\pm i\tilde{p}_1(\tilde{x})\right) \;\longrightarrow\; (\tilde{x})^{\pm K}\,, \qquad \text{as } \tilde{x} \to 0\,. \tag{34}$$

Now consider the function,

$$f(\tilde{x}) \;=\; 2\cos\tilde{p}_1(\tilde{x}) = \exp\left(+i\tilde{p}_1(\tilde{x})\right) + \exp\left(-i\tilde{p}_1(\tilde{x})\right)\,.$$

As the periods of $d\tilde{p}_1$ are normalised in integer units f is analytic on the complex $\tilde{x}$ plane. According to Eq. (34) it has a pole of order K at $\tilde{x} = 0$ and no other singularities. Its behaviour at infinity is inherited from that of $p(x)$;

$$\tilde{p}_1(\tilde{x}) \to 0\,, \qquad \text{as } \tilde{x} \to \infty$$

and thus $f \to 2$ as $\tilde{x} \to \infty$. The most general analytic function obeying these conditions can be parametrised in terms of $K-1$ undetermined coefficients $\tilde{q}_j$, with $j = 2, \ldots, K$, as,

$$f(\tilde{x}) = \mathbb{P}\left(\frac{1}{\tilde{x}}\right) \;=\; 2 + \frac{\tilde{q}_2}{\tilde{x}^2} + \frac{\tilde{q}_3}{\tilde{x}^3} + \ldots + \frac{\tilde{q}_K}{\tilde{x}^K}\,. \tag{35}$$

Thus we have an explicit solution for $\tilde{p}_1(\tilde{x}) = \cos^{-1}(f/2)$ which yields a meromorphic differential,

$$d\tilde{p}_1 \;=\; -i\,\frac{d\tilde{x}}{\tilde{x}^2}\frac{\mathbb{P}'_K\left(\frac{1}{\tilde{x}}\right)}{\sqrt{\mathbb{P}_K\left(\frac{1}{\tilde{x}}\right)^2 - 4}}\,. \tag{36}$$

[11] See in particular subsection 3.3 of this reference.

One may easily check that this differential satisfies the normalisation conditions (27), (28) and has poles with the required residues at the points over $\tilde{x} = 0$. The differential $d\tilde{p}_1$ is meromorphic on the curve,

$$\begin{aligned} \tilde{\Sigma}_1 : \qquad \tilde{y}_1^2 &= \prod_{i=1}^{K-1} \left(\tilde{x} - \tilde{a}_+^{(i)}\right)\left(\tilde{x} - \tilde{a}_-^{(i)}\right) \\ &= \frac{\tilde{x}^{2K}}{4\tilde{q}_2}\left[\mathbb{P}_K\left(\frac{1}{\tilde{x}}\right)^2 - 4\right], \end{aligned}$$

and we can rewrite (36) as,

$$d\tilde{p}_1 = -\frac{d\tilde{x}}{\tilde{y}_1}\sum_{\ell=0}^{K-2} \tilde{C}_\ell\, \tilde{x}^{\ell-1}, \qquad \text{with} \qquad \tilde{C}_\ell = -\frac{(K-\ell)\tilde{q}_{K-\ell}}{2\sqrt{-\tilde{q}_2}}.$$

Thus we have expressed the $2K-2$ parameters corresponding to the branch-points $\tilde{a}_\pm^{(i)}$ of the curve $\tilde{\Sigma}_1$ and the $K-1$ parameters corresponding to the undetermined coefficients $\tilde{C}_\ell$ in the differential $d\tilde{p}_1$ in terms $K-1$ parameters $\tilde{q}_j$, $j = 2, \ldots, K$. At this point we observe that the curve $\tilde{\Sigma}_1$ and differential $d\tilde{p}_1$ are essentially identical to the curve Γ_K and differential $d\hat{p}$ of the $\mathrm{SL}(2,\mathbb{R})$ spin chain.

The matching condition,

$$\tilde{p}_1(\tilde{x} = \epsilon) \; = \; -\frac{1}{2}\int_{\infty^-}^{\infty^+} d\tilde{p}_2\,,$$

with $\epsilon = b/\rho$ can now be evaluated explicitly in terms of the closed formulae (36), (30) for the differentials $d\tilde{p}_1$ and $d\tilde{p}_2$. It yields,

$$\frac{1}{i}\log\left(\frac{\rho^K \tilde{q}_K}{b}\right) \; = \; \frac{2\pi i J}{\sqrt{\lambda}}\frac{1}{\sqrt{b^2-1}}\,,$$

or equivalently,

$$\sqrt{b^2-1} = \frac{2\pi J}{\sqrt{\lambda}} \times \frac{1}{K\log\left(\rho\tilde{q}_K^{1/K}\right)}. \tag{37}$$

Thus $b \to 1$ and the inner branch points approach the punctures at the points $x = \pm 1$ as the scaling parameter ρ goes to infinity.

Finally, we can evaluate the conserved charges in the limit $\rho \to \infty$,

$$\begin{aligned} \Delta + S &= \frac{\sqrt{\lambda}}{2\pi}\sqrt{-\tilde{q}_2}\rho \qquad \to \infty\,, \\ \Delta - S &= \frac{\sqrt{\lambda}}{2\pi}K + \frac{J}{\sqrt{b^2-1}}\,. \end{aligned} \tag{38}$$

These relations confirm that $S \to \infty$ as $\rho \to \infty$ as anticipated. Using (37) we obtain,

$$\rho \simeq \frac{4\pi}{\sqrt{\lambda}} \frac{S}{\sqrt{-\tilde{q}_2}} \,. \tag{39}$$

Eliminating ρ from (38) then gives,

$$\Delta - S = \frac{\sqrt{\lambda}}{2\pi} \left[K \log S + \log(\tilde{q}_K/\sqrt{-\tilde{q}_2}) \right] + O(1/\log S) \,. \tag{40}$$

In classical string theory the parameters $\tilde{q}_i$ are continuous variables. To complete the solution of the model we must also consider the semiclassical quantisation conditions [23, 24]. As mentioned above, semiclassical quantization of string theory on $\mathrm{AdS}_3 \times S^1$ is accomplished by quantizing the filling fractions in integer units,

$$\mathcal{S}_I^{\pm} = -\frac{1}{2\pi i} \frac{\sqrt{\lambda}}{4\pi} \oint_{\mathcal{A}_I^{\pm}} \left(x + \frac{1}{x} \right) dp = l_I^{\pm} \in \mathbb{Z}^+ \tag{41}$$

for $I = 1, 2, \ldots, K/2$. The integers $l_I^{\pm}$ obey,

$$\sum_{I=1}^{K/2} \left(l_I^+ + l_I^- \right) = S \,, \qquad \sum_{I=1}^{K/2} I \left(l_I^+ - l_I^- \right) = 0 \,. \tag{42}$$

We now consider the limiting form of these conditions in the scaling limit $\rho \to \infty$. This is easily implemented by setting $x = \rho \tilde{x}$ in the integrals appearing on the LHS of (41) holding $\tilde{x}$ fixed in the limit. In this case the periods of the differential $(x + 1/x)dp$ on Σ go over to periods of $\tilde{x} d\tilde{p}_1$ on $\tilde{\Sigma}_1$ as $\rho \to \infty$. In particular we find,

$$-\frac{1}{2\pi i} \frac{1}{\sqrt{-\tilde{q}_2}} \oint_{\tilde{\mathcal{A}}_I^{\pm}} \tilde{x} \, d\tilde{p}_1 = \frac{l_I^{\pm}}{S} \tag{43}$$

for $I = 1, 2, \ldots, K/2 - 1$ and,

$$-\frac{1}{2\pi i} \frac{1}{\sqrt{-\tilde{q}_2}} \oint_{\tilde{\mathcal{A}}_0} \tilde{x} \, d\tilde{p}_1 = \frac{\bar{l}}{S} \,, \tag{44}$$

with $\bar{l} = l_{K/2}^+ + l_{K/2}^-$.

The quantization conditions (43), (44) lead to a discrete spectrum labeled by the $K-1$ integers $l_I^{\pm}$ and $\bar{l}$,

$$\begin{aligned}&\gamma\left[l_1^+, l_1^-, \ldots, l_{K/2-1}^+, l_{K/2-1}^-, \bar{l}\right]\\ &\simeq \frac{\sqrt{\lambda}}{2\pi}\left[K\log S + \log\left(\frac{\tilde{q}_K}{(-\tilde{q}_2)^{\frac{K}{2}}}\right) + C + O\left(\frac{1}{\log S}\right)\right]. \end{aligned} \tag{45}$$

Finally, one may check that the spectrum defined by equations (43)–(45) is identical to the result (6) given in the Introduction with the identifications,

$$\hat{q}_j = \frac{\tilde{q}_j}{(-\tilde{q}_2)^{\frac{K}{2}}}$$

for $j = 2, \ldots, K$, $H_K = \log \hat{q}_K$ and,

$$\begin{aligned} l_j &= l_j^+\,, & j &= 1, \ldots, K/2-1\,,\\ &= \bar{l}\,, & j &= K/2\,,\\ &= l_{K-j}^-\,, & j &= K/2+1, \ldots, K-1\,. \end{aligned}$$

In particular the conditions (42) ensure that the integers l_j obey the corresponding relations (13). Thus we see that the spectra of one-loop gauge theory and string theory differ only in the overall λ dependent prefactor which takes the value $\sqrt{\lambda}/2\pi$ in semiclassical string theory and $\lambda/4\pi^2$ in one-loop gauge theory.

Finally, one feature of the gauge theory results which remains unclear on the string side is the bound $K \leq J$. In fact the large-S string spectrum derived above does not depend on J at all. Despite this, our semiclassical analysis formally requires $J \sim \sqrt{\lambda} \gg 1$. Thus the upper bound is, therefore, reached for solutions with K spikes only when $K \sim \sqrt{\lambda} \gg 1$. It is unclear whether higher-loop worldsheet corrections remain suppressed when K scales with λ in this way and it may therefore require a more sophisticated analysis than the one presented above to detect the presence of an upper bound on K in string theory.

6. Interpretation

In the minimal case $K = 2$, the finite gap solution (with symmetric cuts) considered above reduces to the folded GKP string. Logarithmic scaling in S with prefactor $2\sqrt{\lambda}/2\pi$ arises when the two folding points of the string approach the boundary of AdS_3. Following [19], it is natural to expect that

$\log S$ scaling with prefactor $K\sqrt{\lambda}/2\pi$ corresponds to strings with K spikes[12] which approach the boundary as $S \to \infty$. Solutions with $\mathbb{Z}_K$ symmetry, where the spikes lie at the vertices of a regular polygon were constructed in [19]. Based on the scaling limit of the finite gap construction considered above, we expect to find $(K-2)$-parameter families of spikey solutions in this limit. More precisely there should be $K-2$ parameters corresponding to the conserved action variables for solutions with fixed S and an additional $K-2$ corresponding to the initial values of the conjugate angle variables. Solutions are also labelled by an orientation angle ψ_0 canonically conjugate to S. The generic solution need not have the symmetric form considered in [19], but rather should have variable angular separations between the spikes as shown in Fig. 8. The details of this picture will be presented elsewhere and will not be needed for the following arguments.

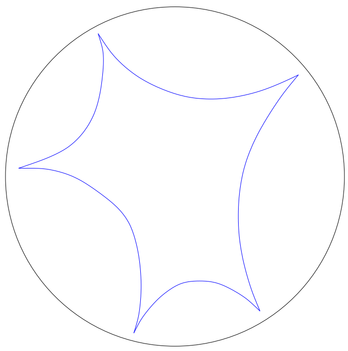

Fig. 8. A spinning string in AdS_3 with spikes approaching the boundary.

To analyse the $S \to \infty$ limit it will be convenient to use the representation of AdS_3 as the group manifold SU(1,1) with complex coordinates, Z_1, Z_2 obeying $|Z_1|^2 - |Z_2|^2 = 1$. The complex coordinates are related to the standard global coordinates (t, ρ, ψ) on AdS_3 by,

$$Z_1 = \cosh\rho\, \exp(it)\,, \qquad Z_2 = \sinh\rho\, \exp(i\psi)\,.$$

We introduce the group-valued worldsheet field,

$$g(\sigma,\tau) \;=\; \begin{pmatrix} Z_1 & Z_2 \\ \bar{Z}_2 & \bar{Z}_1 \end{pmatrix} \text{in } \mathrm{SU}(1,1)$$

and the conserved current corresponding to right multiplication in the group,

$$j_\pm(\sigma,\tau) = g^{-1}\,\partial_\pm g \;=\; \frac{1}{2}\eta_{AB} j^A s^B\,.$$

[12] In the $S \to \infty$ limit with fixed J, the motion on S^1 can be neglected so that the string effectively move in the two spatial dimensions of AdS_3 as do the solutions of [19]. More generally, motion in the extra S^1 will tend to smooth out the spikes.

Here s^A, with $A = 0, 1, 2$ are generators satisfying,

$$[s^A,\, s^B] \;=\; -2\varepsilon^{ABC} s_C$$

and,

$$\eta^{AB} \;=\; \frac{1}{2}\mathrm{Tr}_2\left[s^A s^B\right]\,,$$

where $\eta = \mathrm{diag}(-1, 1, 1)$ is the Killing form of the Lie algebra su(1,1) which is used to raise and lower indices with the usual summation convention. For a generic Lie algebra valued quantity,

$$X \;=\; \frac{1}{2}\eta_{AB} X^A s^B$$

we will sometimes use vector notation $\vec{X} = (X^0, X^1, X^2)$. In the following we will use the explicit choice $(s^0, s^1, s^2) = (-i\sigma_3, \sigma_1, -\sigma_2)$ where σ_i are the usual Pauli matrices.

The Noether charge corresponding to right multiplication is,

$$Q_{\mathrm{R}} = \frac{1}{2}\eta_{AB} Q^A_{\mathrm{R}} s^B \;=\; \frac{\sqrt{\lambda}}{4\pi}\int_0^{2\pi} d\sigma\, j_\tau \in \mathrm{su}(1,1)\,.$$

The corresponding Cartan generator is,

$$Q^0_{\mathrm{R}} = \Delta + S \;=\; \frac{\sqrt{\lambda}}{4\pi}\int_0^{2\pi} d\sigma\, j^0_\tau\,. \tag{46}$$

We will focus on states of highest weight for which $Q^1_{\mathrm{R}} = Q^2_{\mathrm{R}} = 0$.

At fixed worldsheet time, we will assume that our solution has K spikes at the points $\sigma = \sigma_j \in [0, 2\pi]$ with $j = 1, \ldots, K$. At these points the σ-derivatives of all world-sheet fields vanish and thus,

$$j_\pm(\sigma = \sigma_j, \tau) \;=\; j_\tau(\sigma = \sigma_j, \tau)$$

for all j. To understand the behaviour of the charge density near the spikes, we consider the simplest two-spike solution: the GKP folded string [4]. This describes a folded string rotating around its midpoint in AdS_3. In global coordinates it has the form $t = \tilde{\tau}$, $\psi = \psi_0 + \omega\tilde{\tau}$ (with $\omega \geq 1$) and $\rho = \rho(\sigma) = \mathrm{am}[i\tilde{\sigma}|\sqrt{1-\omega^2}]$ where,

$$\tilde{\sigma} = \frac{L}{2\pi}\,\sigma\,, \qquad \tilde{\tau} = \frac{L}{2\pi}\,\tau \quad \text{with} \quad L = \frac{4}{\omega}\,\mathbb{K}\left(\frac{1}{\omega}\right)\,.$$

This is a two-parameter family of solutions labelled by ω (which determines S) and ψ_0. The spikes are located at the points $\sigma=\sigma_1=\pi/2$ and $\sigma=\sigma_2=3\pi/2$. One may obtain the following explicit form for the conserved current,

$$\begin{aligned} j^0_\tau(\sigma,\tau) &= \frac{L}{\pi}\frac{\left[1+\frac{1}{\omega}\mathrm{sn}^2\left(\omega\tilde{\sigma}\,\middle|\,\frac{1}{\omega}\right)\right]}{\mathrm{dn}^2\left(\omega\tilde{\sigma}\,\middle|\,\frac{1}{\omega}\right)}\,, \\ j^1_\tau(\sigma,\tau)+i\,j^2_\tau(\sigma,\tau) &= i\,\frac{L}{\pi}\frac{\omega+1}{\omega}\,\exp\left(i\psi_0+i(\omega-1)\tilde{\tau}\right)\frac{\mathrm{sn}\left(\omega\tilde{\sigma}\,\middle|\,\frac{1}{\omega}\right)}{\mathrm{dn}^2\left(\omega\tilde{\sigma}\,\middle|\,\frac{1}{\omega}\right)}\,. \end{aligned} \tag{47}$$

Conventions for elliptic integrals and functions are as in [32].

The two spikes approach the boundary in the limit $\omega\to 1$. In this limit the conserved charges of the solution scale as,

$$S\simeq\frac{\sqrt{\lambda}}{\pi}\frac{1}{\omega-1}+\ldots\,,\qquad \Delta-S\simeq\frac{\sqrt{\lambda}}{\pi}\log\frac{1}{\omega-1}+\ldots\simeq\frac{\sqrt{\lambda}}{\pi}\log S+\ldots\ .$$

Thus the limit $\omega\to 1$ implies $S\to\infty$. By inspection the current $j^0_\tau(\sigma,\tau)$, which is the density of the conserved charge $\Delta+S\simeq 2S$, diverges as $S\to\infty$. We define a normalised charge density,

$$\mu^A(\sigma,\tau) = \lim_{S\to\infty}\left[\frac{\sqrt{\lambda}}{8\pi S}\,j^A_\tau(\sigma,\tau)\right]\,,$$

which remains finite and obeys,

$$\int_0^{2\pi} d\sigma\vec{\mu}(\sigma,\tau) = \begin{pmatrix}1\\0\\0\end{pmatrix}\,,$$

for highest-weight states.

Expanding around the spike point σ_1 we set, $\sigma=\sigma_1+\hat{\sigma}$ with $\hat{\sigma}\ll 1$ we find,

$$\begin{aligned} \mu^0(\sigma,\tau) &\simeq \lim_{\kappa\to 0}\left[\frac{1}{2\pi\kappa}\,\mathrm{sech}^2\left(\frac{2\hat{\sigma}}{\kappa\pi}\right)\right] \\ &= \frac{1}{2}\,\delta(\hat{\sigma}) \end{aligned} \tag{48}$$

and

$$\begin{aligned} \mu^1(\sigma,\tau)+i\,\mu^2(\sigma,\tau) &\simeq \lim_{\kappa\to 0}\left[-\frac{e^{i\psi_0}}{2\pi\kappa}\,\mathrm{sech}^2\left(\frac{2\hat{\sigma}}{\kappa\pi}\right)\right] \\ &= -\frac{e^{i\psi_0}}{2}\,\delta(\hat{\sigma}) \end{aligned} \tag{49}$$

with $\kappa^{-1} = \log(1/\sqrt{\omega - 1})$. Here we have used the standard definition of the Dirac δ-function,

$$\delta(x) \;=\; \lim_{\varepsilon\to 0}\left[\frac{1}{2\varepsilon}\mathrm{sech}^2\left(\frac{x}{\varepsilon}\right)\right] .$$

The $\kappa \to 0$ limit leading to (48), (49) focuses on the region of the string near the first spike. The full charge density is obtained by including a similar contribution from the second spike at $\sigma = \sigma_2$,

$$\begin{aligned}\mu^0(\sigma,\tau) \;&=\; \frac{1}{2}\delta(\sigma-\sigma_1) + \frac{1}{2}\delta(\sigma-\sigma_2)\,,\\ \mu^1(\sigma,\tau) + i\,\mu^2(\sigma,\tau) \;&=\; i\,\frac{e^{i\psi_0}}{2}\delta(\sigma-\sigma_1) - i\,\frac{e^{i\psi_0}}{2}\delta(\sigma-\sigma_2)\,.\end{aligned}$$

Equivalently we can write the large-S limit of the current as,

$$\lim_{S\to\infty}\left[j_\tau^A(\sigma,\tau)\right] \;=\; \frac{8\pi}{\sqrt{\lambda}}\sum_{k=1}^{2} L_k^A\,\delta\left(\sigma-\sigma_k\right)\,, \tag{50}$$

with

$$\vec{L}_1 = \frac{S}{2}\left(\begin{array}{c} 1 \\ -\sin\psi_0 \\ \cos\psi_0 \end{array}\right)\,, \qquad \vec{L}_2 = \frac{S}{2}\left(\begin{array}{c} 1 \\ \sin\psi_0 \\ \cos\psi_0 \end{array}\right)\,. \tag{51}$$

One can easily check that the highest weight conditions $Q^1_{\mathrm{R}} = Q^2_{\mathrm{R}} = 0$ and the normalisation condition (46) are satisfied.

The key feature of the above result is that the charge density j_τ becomes δ-function localised at the spikes in the limit they approach the boundary. We do not have much explicit information about solutions for $K > 2$ except in the $\mathbb{Z}_K$ symmetric case considered in [19]. Recently the symmetric solution of [19] has been analysed [33] (see also [34]) in the same conformal gauge as we have just used to describe the GKP string. The behaviour in the vicinity of each spike is similar to that at the folds of the GKP string and in particular δ-function localisation of the charge density will occur at each spike as it approaches the boundary. We will assume that the same is true for generic solutions with K spikes and thus we propose the obvious generalisation of (50),

$$\lim_{S\to\infty}\left[j_\tau^A(\sigma,\tau)\right] \;=\; \frac{8\pi}{\sqrt{\lambda}}\sum_{k=1}^{K} L_k^A\,\delta\left(\sigma-\sigma_j\right)\,, \tag{52}$$

where L_k^A are undetermined functions of the worldsheet time. The above expression is also subject to the Virasoro constraint which implies that,

$$\lim_{\sigma\to\sigma_k}\left[\frac{1}{2}\mathrm{Tr}_2[j_\pm^2(\sigma,\tau)]\right] = \lim_{\sigma\to\sigma_k}\left[\frac{1}{2}\mathrm{Tr}_2[j_\tau^2(\sigma,\tau)]\right] = \frac{J^2}{\sqrt{\lambda}}\,, \tag{53}$$

where we have used the fact that the space-like component of the current vanishes at the spike. The above constraint can only be obeyed in (52) if,

$$\eta_{AB}L_k^A L_k^B = 0 \tag{54}$$

for each value of k. Evaluating the total charge by integrating over the string, the highest-weight condition becomes,

$$\sum_{k=1}^{K}\vec{L}_k = \begin{pmatrix} S \\ 0 \\ 0 \end{pmatrix}. \tag{55}$$

As a check, for $K=2$ we can solve the conditions (54), (55) and recover (51) as the general solution. In the general case there are $2K-2$ remaining free parameters (including S), as expected from the finite gap construction.

We will now treat the unknown quantities L_k^A as dynamical variables. We choose a cyclic ordering for the K spikes; $0<\sigma_1<\sigma_2<\ldots<\sigma_K<2\pi$ and introduce K arbitrary points on the string $\mu_k\in(0,2\pi)$ with, $\mu_k<\sigma_k<\mu_{k+1}$ for $j=1,\ldots,K$ with the convention that $\mu_{K+1}=\mu_1$. We can then write,

$$L_k^A = \lim_{S\to\infty}\left[\frac{\sqrt{\lambda}}{8\pi}\int_{\mu_k}^{\mu_{k+1}} d\sigma\, j_\tau^A(\sigma,\tau)\right]. \tag{56}$$

In the Hamiltonian formalism for the Principal Chiral Model, the τ-component of the Noether current has Poisson brackets,

$$\{j_\tau^A(\sigma,\tau),\, j_\tau^B(\sigma',\tau)\} = -\frac{4\pi}{\sqrt{\lambda}}\,2\varepsilon^{ABC}j_{\tau\,C}(\sigma.\tau)\,\delta(\sigma-\sigma')\,. \tag{57}$$

Substituting (56) for j_τ^A in (57) we obtain the brackets,

$$\{L_j^A, L_k^B\} = -\varepsilon^{ABC}\delta_{jk}L_{C\,k} \tag{58}$$

for the variables L_k^A. These steps certainly involve dynamical assumptions and it remains to be shown that the L_k^A are not subject to additional constraints. Another question is whether we should also include dynamics for the locations, σ_j, of the spikes. These issues could be addressed by reconstructing actual string solutions as in [23, 24] and then taking the large-S

limit. For the moment we will rely on the consistent outcome of this analysis to provide some retrospective justification for the assumptions made.

Now we are ready to consider the scaling limit of the monodromy matrix,

$$\Omega\,[x;\tau] \;=\; \mathcal{P}\exp\left[\frac{1}{2}\int_0^{2\pi} d\sigma\left(\frac{j_+}{x-1}+\frac{j_-}{x+1}\right)\right].$$

As above we have $j_\pm^A \sim S$ and we also scale the spectral parameter as $x \sim S$ as $S \to \infty$. The monodromy matrix becomes,

$$\Omega\,[x;\tau] \;\simeq\; \mathcal{P}\exp\left[\frac{1}{x}\int_0^{2\pi} d\sigma j_\tau\right].$$

Using the limit form (52) for $j_\tau(\sigma,\tau)$ we obtain,

$$\begin{aligned}\Omega\,[x;\tau] \;&\simeq\; \prod_{k=1}^{K}\exp\left[\frac{4\pi}{\sqrt{\lambda}}\frac{1}{x}\,\eta_{AB}L_k^A s^B\right]\\ &=\; \frac{1}{u^K}\prod_{k=1}^{K}\mathbb{L}_k(u)\,,\end{aligned}\tag{59}$$

where we set $u=\sqrt{\lambda}x/4\pi$ and identify,

$$\begin{aligned}\mathbb{L}_k(u) \;&=\; \left[u\mathbb{I}+\eta_{AB}L_k^A s^B\right]\\ &=\; \begin{pmatrix} u+iL_k^0 & L_k^1+iL_k^2\\ L_k^1-iL_k^2 & u-iL_k^0\end{pmatrix},\end{aligned}$$

where we have used the explicit choice form of the generators given above. Notice that the last equality in (59) is exact because the Taylor expansion of the exponential truncates after two terms by virtue of the relation (54). Finally, the above expression for Ω coincides (up to an irrelevant overall factor) with the monodromy (10) of the classical $\mathrm{SL}(2,\mathbb{R})$ spin chain if we identify $\mathcal{L}_k^0 = L_k^0$ and $i\mathcal{L}_k^\pm = L_k^1 \pm iL_k^2$ as the classical spin at the k-th site. With this identification we also reproduce the Poisson brackets (4), quadratic Casimir condition (5) and highest-weight condition (9) of the spin chain.

The above analysis indicates that the motion of the spikes is governed by the same finite-dimensional complex integrable system as the gauge theory spins. In particular the evolution of the spikes in global AdS time should be generated by the Hamiltonian $H_K = \log q_K$. It is not quite clear if the relevant trajectories are literally the same as the depends also on reality conditions for the initial data. It would be interesting to investigate this further and construct some explicit trajectories of the spikes using the methods of [35].

7. Conclusion

In this paper we have argued that the dynamics of the K gap solution of classical string theory on $\mathrm{AdS}_3 \times S^1$ is effectively described by a classical spin chain of length K in the limit of large angular momentum, $S \to \infty$. Thus the continuous string effectively gives rise a finite-dimensional lattice system in the large-S limit. This is the opposite of the usual situation where a continuous system arises as the thermodynamic or continuum limit of a discrete one.

Building on the ideas of [19], we have argued that this new phenomenon can be understood in terms of the localisation of the worldsheet fields at K special points or spikes. Another point of view is provided by the degeneration of the spectral curve shown in Fig. 4. The moduli of the degenerate curve $\tilde{\Sigma}_1$ correspond to the the K lowest modes of the string[13]. The remaining modes of the string correspond to the double points mentioned in Section 3 where the quasi-momentum $p(x)$ attains a value $n\pi$ for some $n \in \mathbb{Z}$. On the initial curve Σ these double points accumulate at the four singular points $\pm 1^{\pm}$. In the limit $S \to \infty$, the singular points and all the double points end up on the genus zero curve $\tilde{\Sigma}_2$. This has a simple interpretation: the lowest K modes of the string effectively decouple from the infinite tower of higher modes as $S \to \infty$ and become an isolated finite dimensional system.

Another mysterious aspect of the results presented above is the precise matching between one-loop gauge theory and semiclassical string theory up to a single universal function of the 't Hooft coupling. The decoupling described in the previous paragraph throws some light on this. Consider the one-loop correction to the semiclassical large-S spectrum in the string σ-model. This is calculated by summing the small fluctuation frequencies for all the worlsheet fields (including fluctuations of all the $\mathrm{AdS}_5 \times S^5$ worldsheet fields). These frequencies are in turn determined by evaluating a particular abelian integral $q(x)$ (the quasi-energy) at each of the double points mentioned above [36]. Because of the factorisation of Σ into two disjoint components $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$, it is easy to see that the frequencies which only depend on data determined by $\tilde{\Sigma}_2$ are independent of the moduli of $\tilde{\Sigma}_1$.

[13] As mentioned above we are exciting only modes of the string which carry positive angular momentum S.

Similar considerations should also apply to the fluctuations of the string outside $\mathrm{AdS}_3 \times S^1$. It follows that the one-loop correction will be the same for all states in the spectrum (6). The agreement we have found suggests that this argument might extend to all σ-model loops.

It would also be interesting to understand these results in more detail from the point of view of the planar $\mathcal{N} = 4$ theory. Spikes near the boundary are dual to localised excitations on S^3 with the same quantum numbers as an elementary gluon (or other adjoint field). It would be interesting to investigate possible connections with the gluon scattering amplitudes discussed in [37]. Finally we note that one-loop, large-spin operator spectrum (3) is essentially universal to all planar four-dimensional gauge theories. This suggests that the limit of semiclassical string theory studied in this paper may have applications to large-N QCD.

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Appendix A

Matching condition

On the original spectral curve

$$\Sigma: \qquad y^2 \;=\; (x-b)\,(x+b) \prod_{i=1}^{K-1} \left(x - a_+^{(i)}\right) \left(x - a_-^{(i)}\right) ,$$

the extra A-cycle condition can be written as,

$$\oint_{\mathcal{A}^+_{K/2}} dp \;=\; 2I_1 + 2I_2 = 0\,, \tag{A.1}$$

where,

$$I_1 \;=\; \int_b^{a_+^{(1)}} dp_1\,, \qquad I_2 = \int_b^{a_+^{(1)}} dp_2\,,$$

with the explict expressions for dp_1 and dp_2 given in Eq. (22).

The first integral can be treated using the change of variables $x = \rho\tilde{x}$ which gives,

$$y^2 \;=\; \rho^{2K}\left(\tilde{x}^2 - \frac{b^2}{\rho^2}\right)\tilde{y}_1^2\,,$$

where $\tilde{y}_1$ is the hyperelliptic coordinate on the curve $\tilde{\Sigma}_1$,

$$\tilde{\Sigma}_1: \qquad \tilde{y}_1^2 \;=\; \prod_{i=1}^{K-1}\left(\tilde{x} - \tilde{a}_+^{(i)}\right)\left(\tilde{x} - \tilde{a}_-^{(i)}\right).$$

Then we have,

$$I_1 \;=\; -\int_{\epsilon}^{\tilde{a}_+^{(1)}} \frac{\left(\sum_{\ell=0}^{K-2}\tilde{C}_\ell\,\tilde{x}^\ell\right)}{\tilde{y}_1}\,\frac{d\tilde{x}}{\sqrt{\tilde{x}^2-\epsilon^2}}\,,$$

where $\epsilon = b/\rho$. We need to find the leading behaviour of this integral as $\epsilon \to 0$. For this purpose it is convenient to write,

$$I_1 = \frac{1}{\epsilon}\frac{\partial}{\partial\epsilon}\hat{I}(\epsilon)\,, \qquad \text{where} \qquad \hat{I}(\epsilon) = \int_{\epsilon}^{\tilde{a}_+^{(1)}} \sqrt{\tilde{x}^2-\epsilon^2}\,\frac{\left(\sum_{\ell=0}^{K-2}\tilde{C}_\ell\,\tilde{x}^\ell\right)}{\tilde{y}_1}\,d\tilde{x}\,,$$

and expand the square root in the integrand in powers of ϵ^2.

$$\hat{I}(\epsilon) = \sum_{k=0}^{\infty}\epsilon^{2k}\hat{I}_k(\epsilon)\,,$$

with,

$$\hat{I}_k(\epsilon) \;=\; (-1)^k\begin{pmatrix}\frac{1}{2}\\ k\end{pmatrix}\int_{\epsilon}^{\tilde{a}_+^{(1)}} \frac{\tilde{x}^{1-2k}\left(\sum_{\ell=0}^{K-2}\tilde{C}_\ell\,\tilde{x}^\ell\right)}{\tilde{y}_1}\,d\tilde{x}\,.$$

Each term $\hat{I}_k(\epsilon)$, with $k \neq 1$, is analytic in ϵ the leading contribution as $\epsilon \to 0$ is proportional to $\tilde{C}_0/\tilde{Q} = K$ ($\tilde{Q}$ is defined in Eq. (29) above). As a result each of these terms only gives rise to a moduli independent constant in the $\epsilon \to 0$ limit. The leading moduli-dependence comes from the remaining term $\hat{I}_1(\epsilon)$ which is non-analytic at $\epsilon = 0$. The resulting contribution to I_1 is,

$$I_1 \simeq \frac{1}{\epsilon}\frac{\partial}{\partial\epsilon}\epsilon^2\,\hat{I}_1(\epsilon) \;\simeq\; -\int_{\epsilon}^{\tilde{a}_+^{(1)}} \frac{\left(\sum_{\ell=0}^{K-2}\tilde{C}_\ell\,\tilde{x}^{\ell-1}\right)}{\tilde{y}_1}\,d\tilde{x}\,.$$

The remaining integral can be then expressed as a contour integral on the curve $\tilde{\Sigma}_1$,

$$I_1 \simeq \frac{1}{2}\int\limits_{\epsilon^-}^{\epsilon^+} d\tilde{p}_1 = \tilde{p}_1(\tilde{x}=\epsilon)\,, \tag{A.2}$$

with $\epsilon = b/\rho$, where $\epsilon^{\pm}$ are the points above $\tilde{x}$ on $\tilde{\Sigma}_1$. Using the explicit formula $\tilde{p}_1(\tilde{x}) = \cos^{-1}(f/2)$ with f given as given in Eq. (35),

$$I_1 \simeq \frac{1}{i}\log\left(\frac{\hat{q}_K\rho^K}{b^K}\right) + \dots\ ,$$

where the dots denote subleading terms.

The second integral I_2, in the period condition (A.A.1) has limiting behavior,

$$\begin{aligned} I_2 \;=\; & \int\limits_b^\infty d\tilde{p}_2 = -\frac{i\pi J}{\sqrt{\lambda}} \\ & \times \int\limits_b^\infty \left[\sqrt{b^2-1}\left(\frac{1}{(x-1)^2}+\frac{1}{(x+1)^2}\right)-\frac{1}{\sqrt{b^2-1}}\left(\frac{1}{(x-1)}-\frac{1}{(x+1)}\right)\right]\frac{dx}{\tilde{y}_2} \end{aligned}$$

with,

$$\tilde{y}_2^2 \;=\; x^2 - b^2\,.$$

Anticipating the fact that $b \to 1$ as $\rho \to 0$, the leading piece is,

$$I_2 \simeq \frac{2\pi J i}{\sqrt{\lambda}}\frac{1}{\sqrt{b^2-1}} + \dots\ ,$$

where the dots denote subleading terms.

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