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CUSP ANOMALOUS DIMENSION IN MAXIMALLY SUPERSYMMETRIC YANG–MILLS THEORY*

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The main features of the cusp anomalous dimension in $\mathcal{N}=4$ supersymmetric Yang–Mills theory are reviewed. Moreover, the strong coupling expansion of the cusp derived in B. Basso, G.P. Korchemsky, J. Kotanski, *Phys. Rev. Lett.* **100**, 091601 (2008) is presented.

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1. The cusp and AdS/CFT correspondence

This talk is mainly based on the work [1] performed in collaboration with B. Basso[1] and G. Korchemsky[1], where the method of strong coupling expansion of the cusp anomalous dimension was found. The cusp anomalous dimension [2], called also the cusp, is a physical observable, which appears in many branches of particle physics. It is related to the logarithmic growth of the anomalous dimensions of high-spin Wilson operators and to the gluon Regge trajectory; it governs behavior of the Sudakov form factors as well as it defines infrared singularities of on-shell scattering amplitude.

In order to define the cusp anomalous dimension in $\mathcal{N}=4$ supersymmetric Yang–Mills (MSYM) theory one can consider local operators composed of L scalar fields and S covariant derivatives, $\langle D^{s_1}XD^{s_2}X\dots D^{s_L}X\rangle$, with the spin $S=\sum_{k=1}^{L}s_k$. In the high spin limit the anomalous dimensions of these operators read as

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$$\gamma_S^{(L=2)}(g) = 2\varGamma_{\rm cusp}(g)\ln S + \dots\,, \tag{1}$$

where the leading coefficient, $\varGamma_{\rm cusp}(g)$ is called cusp anomalous dimension. In $\mathcal{N}=4$ SYM theory the cusp is a fundamental quantity and it depends neither on the twist L[2] nor on composed fields.

In 1998 the AdS/CFT correspondence was proposed by Maldacena, Polyakov, Klebanov, Gubser and Witten [3]. It relates MSYM operators to the IIB string theory observable, *i.e.* the cusp anomalous dimension in MSYM theory corresponds to the energy of folded strings rotating in AdS_3 [4]. Due to dual properties of the AdS/CFT conjecture, calculations of the strong coupling limit in the SYM theory are equivalent to the semi-classical expansion of the string theory. Moreover, if the physical quantity like the cusp can be calculated in $\mathcal{N}=4$ SYM theory both in the weak as well as in the strong coupling expansion, *e.g.* making use of integrable methods [5], an additional aspect of investigation arises. In this special case one may try to test validity of the AdS/CFT correspondence.

From the string theory side the cusp anomalous dimension

$$\varGamma_{\rm cusp}(g) = 2g - \frac{3\ln 2}{2\pi} + \mathcal{O}(1/g)\,, \qquad g = \frac{\sqrt{\lambda}}{4\pi}\,, \tag{2}$$

with $\lambda = g_{\rm YM}^2 N_{\rm c}$ being 't Hooft coupling was calculated in the 1-loop string perturbation calculation [4, 6] and from the string Bethe ansatz [7]. The question is if this result can be also obtained from $\mathcal{N}=4$ SYM theory.

2. Beisert–Eden–Staudacher equation

In Ref. [8] Beisert, Eden and Staudacher derived from all-loop Bethe ansatz [9] the equation

$$\widehat{\sigma}_g(t) = \frac{t}{e^t-1}\left[K_g(2gt,0) - 4g^2\int\limits_0^\infty dt' K_g\left(2gt,2gt'\right)\widehat{\sigma}_g\left(t'\right)\right]\,, \tag{3}$$

with a complicated kernel, $K_g(t,t') = \sum_{n,m=1}^\infty z_{nm}(g)\frac{J_n(t)J_m(t')}{tt'}$, defined in Ref. [8]. The fluctuation density $\widehat{\sigma}_g(t)$ is related to the Fourier transform of the Bethe roots distribution and it predicts the cusp anomalous dimension for arbitrary values of the coupling constant $\varGamma_{\rm cusp}(g) = 8g^2\widehat{\sigma}_g(0)$. The weak coupling expansion obtained from the BES equation is given as

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[2] For higher twists there are additional degrees of freedom and the cusp defines the high spin asymptotic of the minimal anomalous dimension.

$$
\begin{aligned}
2\Gamma_{\rm cusp}(g) \;=\; & 8g^2-\tfrac{8}{3}\pi^2g^4+\tfrac{88}{45}\pi^4g^6-16\left(\tfrac{73}{630}\pi^6+4\zeta(3)^2\right)g^8\\
&+32\left(\tfrac{887}{14175}\pi^8+\tfrac{4}{3}\pi^2\zeta(3)^2+40\zeta(3)\zeta(5)\right)g^{10}\\
&-64\left(\tfrac{136883}{3742200}\pi^{10}+\tfrac{8}{15}\pi^4\,\zeta(3)^2+\tfrac{40}{3}\pi^2\zeta(3)\zeta(5)\right.\\
&\left.+210\zeta(3)\zeta(7)+102\zeta(5)^2\right)g^{12}+\dots\,, \qquad (4)
\end{aligned}
$$

a sign alternating series with the convergence radius $\frac{1}{4}$. The three loop result was also obtained from QCD using the maximal tanscendentality principle [11] and the four loop value was checked in perturbation calculations [12].

3. Strong coupling expansion in numerical approach

In order to solve the BES equation numerically [13] one can expand the fluctuation density over Bessel functions and truncate the Bessel series $\sigma_g(t) = \frac{t}{e^t-1}\sum_{n=1}^{M} s_n(g)\frac{J_n(2gt)}{2gt}$ with $s_{n\geq M+1} = 0$. The integral equation becomes a matrix equation

$$
s(g) = \frac{1}{1+K(g)}\cdot h \qquad \text{and} \qquad \Gamma_{\rm cusp}(g) = 4g^2 s_1(g)\,, \qquad (5)
$$

with $K(g)$ being a complicated g-dependent matrix $(M\times M)$ and $h = (1,0,0,0,0,\dots)$ is a boundary condition vector. The numerical solution to Eq. (5) is presented in Fig. 1.

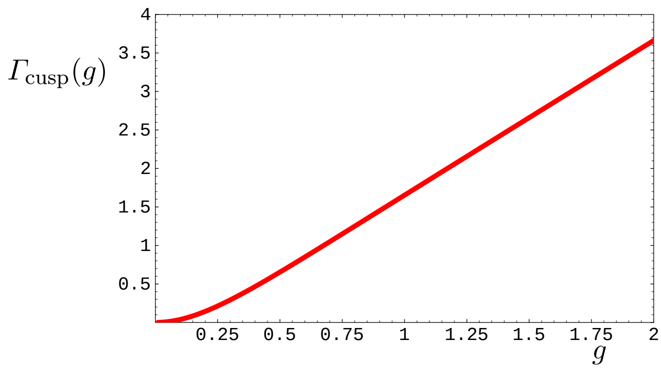

Fig. 1. The cusp anomalous dimension as a function of coupling constant g.

The first few coefficients of the strong coupling expansion were fitted in Ref. [13]:

$$
2\Gamma_{\rm cusp}(g) = 4.000000g - 0.661907 - 0.0232g^{-1} + \dots\,. \qquad (6)
$$

One can see that the first two coefficients agree with string theory, for instance $0.661907 \approx \frac{3\ln 2}{\pi}$. From the numerical results two questions arise: if it is possible to calculate these coefficients from $\mathcal{N}=4$ SYM theory analytically and if the third coefficient is consistent with the string theory[3].

4. Ordinary strong coupling expansion

One can try to perform the strong coupling expansion analytically using the infinite Bessel series, *i.e.* working in infinite-dimension matrices [14]:

$$[1+K(g)]\cdot s(g)=h\,. \tag{7}$$

Expanding the matrix $K(g)$ and the solution $s(g)$ in powers of $1/g$ gives:

$$s_n(g)=g^{-1}\sum_{j=0} g^{-j}s_n^{(j)}\,,\qquad K(g)=g\sum_{j=0} g^{-j}K^{(j)}\,. \tag{8}$$

The leading order solution, $s^{(0)}=[K]^{-1}\cdot h+[\text{zero modes}]$, is defined up to zero modes. This ambiguity is fixed in Ref. [14] by the constraint from numerics $s_{2k-1}^{(0)}=s_{2k}^{(0)}$, $s_{2k-1}^{(0)}=s_{2k}^{(0)}=(-1)^{k+1}\frac{\Gamma(k+\frac{1}{2})}{\Gamma(k)\Gamma(\frac{1}{2})}$. For the next-to-leading order constraints, zero modes fixing is more complicated and cannot by easily extracted from numerics.

5. Our main idea

Let us define the even and odd unknowns:

$$s_n^{\pm}(g)=\tfrac{1}{2}(s_{2n-1}(g)\pm s_{2n}(g))\,. \tag{9}$$

From numerical calculations in Fig. 2 one can see that the strong coupling expansion works well for $n\sim 1$ while it fails for large n.

The approach proposed in Ref. [1] reads as follows:

1. construct the solution for $s_n^{\pm}(g)$ in the region $n\sim 1$ and parameterize the contribution of (zero modes) by yet unknown coefficients $c_p^{\pm}(g)$,
2. construct the asymptotic solution for $s_n^{\pm}(g)$ in the region $n\gg 1$,
3. sew two asymptotic expressions for $s_n^{\pm}(g)$ in the intermediate region $n\sim g^{1/2}$ and determine the infinite set of zero mode coefficients $c_p^{\pm}(g)$.

One can do this without knowing the exact solution, performing the scaling limit $n,g\to\infty$ where $x=(n-\frac{1}{4})^2=$ fixed.

[3] Early calculations of A. Tseytlin *et al.* disagree with this value.

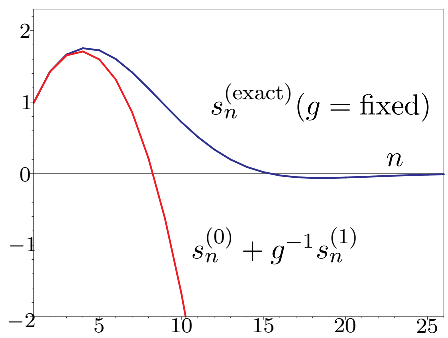

Fig. 2. The n-dependence of the $s_n(g)$ coefficients and its comparison to first few terms of strong coupling expansion. Both lines are plotted for large fixed g.

6. Fixing zero mode coefficients

Changing variables properly [1] one can find a solution of the BES equation:

$$\Gamma_{\text{cusp}}(g) = 2g + \sum_{p=1}^{\infty} \frac{1}{g^{p-1}} \left[\frac{2c_p^-(g)}{\sqrt{\pi}} \Gamma\left(2p - \tfrac{3}{2}\right) + \frac{2c_p^+(g)}{\sqrt{\pi}} \Gamma\left(2p - \tfrac{1}{2}\right) \right], \quad (10)$$

as a function of yet unknown $c_p^{\pm}(g) = \sum_{r\geq 0} g^{-r} c_{p,r}^{\pm}$. The expansion coefficients $s_m^{\pm}(g)$ should have corrected the scaling behavior in the scaling limit

$$m, g \to \infty \qquad \text{for} \qquad x = (m - \frac{1}{2})^2/g = \text{fixed}\,, \quad (11)$$

so that

$$s_m^{\pm}(g) = \frac{(gx)^{-1/4}}{g\sqrt{\pi}} \left[\gamma_{\pm}^{(0)}(x) + + \frac{\gamma_{\pm}^{(1)}(x)}{gx} + \mathcal{O}\left(1/g^2\right) \right], \quad (12)$$

where the expansion of $\gamma_{\pm}^{(r)}(x)$ runs over integer positive powers of x and $\gamma_{\pm}^{(r)}(x)$ should have a faster-than-power-law decrease at large x, or equivalently, its Laplace transform should be an analytical function.

7. Quantization conditions

Let us consider the leading order of $1/g$. Performing the Laplace transform from $\gamma_{\pm}^{(0)}(x)$ to $\tilde{\gamma}_{\pm}^{(0)}(s)$ one can obtain the analyticity condition

$$\sum_{p\geq 0} s^p\, c_{p,0}^{+} \Gamma(p - \tfrac{1}{4}) = 2\left[\Gamma\left(\tfrac{3}{4}\right)\right]^2 \frac{\Gamma\left(1 - \frac{s}{2\pi}\right)}{\Gamma\left(\frac{3}{4} - \frac{s}{2\pi}\right)}, \qquad c_{0,0}^{+} = -\tfrac{1}{2}, \quad (13)$$

corresponding to cancellation of roots and poles of $\tilde{\gamma}_+^{(0)}(s)$. A similar condition for $\tilde{\gamma}_-^{(0)}(s)$ reads as follows

$$\sum_{p\geq 0} s^p \left[c^-_{p,0}\Gamma\left(p-\tfrac{3}{4}\right)+2c^+_{p,0}(p-\tfrac{1}{4})\Gamma\left(p+\tfrac{1}{4}\right)\right]=\frac{\left[\Gamma\left(\frac{1}{4}\right)\right]^2}{4}\frac{\Gamma\left(1-\frac{s}{2\pi}\right)}{\Gamma\left(\frac{1}{4}-\frac{s}{2\pi}\right)}\,, \tag{14}$$

where $c^-_{0,0}=0$. The expansion of Eqs. (13) and (14) in s gives

$$c^+_{1,0}=-\frac{3\ln 2}{\pi}+\frac{1}{2}+\mathcal{O}(1/g)\,,\qquad c^-_{1,0}=\frac{3\ln 2}{4\pi}-\frac{1}{4}+\mathcal{O}(1/g)\,. \tag{15}$$

Substituting (15) to (10) one gets

$$\Gamma_{\rm cusp}(g)=2g-\frac{3\ln 2}{2\pi}+\mathcal{O}\left(g^{-1}\right)\,. \tag{16}$$

Using this method one can continue the calculations for higher orders of $1/g$ series. Finally, one gets

$$\begin{aligned}\Gamma_{\rm cusp}\,(g{+}c_1) \;=\; 2g\Big[&1-c_2g^{-2}-c_3g^{-3}-\left(c_4+2\,c_2^2\right)g^{-4}\\
&-(c_5+23\,c_2c_3)g^{-5}-\left(c_6+\tfrac{166}{7}\,c_2c_4+54\,c_3^2+25\,c_2^3\right)g^{-6}\\
&-\left(c_7+\tfrac{1721}{29}\,c_2c_5+\tfrac{1431}{7}\,c_3c_4+457\,c_2^2c_3\right)g^{-7}\\
&-\left(c_8+\tfrac{6352}{107}\,c_2c_6+\tfrac{12606}{29}\,c_3c_5+\tfrac{7916}{49}\,c_4^2+\tfrac{6864}{7}\,c_2^2c_4\right.\\
&\left.+4563\,c_2c_3^2+374\,c_2^4\right)g^{-8}\\
&-\left(c_9+\tfrac{30943}{277}c_7c_2+\tfrac{72089}{107}c_6c_3+\tfrac{216437}{203}c_5c_4\right.\\
&\left.+\tfrac{71712}{29}c_5c_2^2+17016c_4c_3c_2+13131c_3^3+16904c_3c_2^3\right)g^{-9}\\
&-\left(c_{10}+\tfrac{464314}{4183}c_8c_2+\tfrac{308416}{277}c_7c_3+\tfrac{154466}{107}c_6c_4\right.\\
&+\tfrac{455267}{107}c_6c_2^2+\tfrac{1296500}{841}c_5^2+\tfrac{1818888}{29}c_5c_3c_2+\tfrac{1137597}{49}c_4^2c_2\\
&\left.+\tfrac{756936}{7}c_4c_3^2+\tfrac{254648}{7}c_4c_2^3+253728c_3^2c_2^2+10666c_2^5\right)g^{-10}\\
&+O\left(g^{-11}\right)\Big]\,,\end{aligned} \tag{17}$$

where the expansion coefficients are given by

$$\begin{aligned}&c_1=\tfrac{3\ln 2}{4\pi},\quad c_2=\tfrac{1}{16\pi^2}K,\quad c_3=\tfrac{27}{2^{11}\pi^3}\zeta(3),\quad c_4=\tfrac{21}{2^{10}\pi^4}\beta(4),\\
&c_5=\tfrac{43065}{2^{21}\pi^5}\zeta(5),\quad c_6=\tfrac{1605}{2^{15}\pi^6}\beta(6),\quad c_7=\tfrac{101303055}{2^{30}\pi^7}\zeta(7),\\
&c_8=\tfrac{1317645}{2^{22}\pi^8}\beta(8),\quad c_9=\tfrac{1991809466325}{2^{41}\pi^9}\zeta(9),\quad c_{10}=\tfrac{524012895}{2^{27}\pi^{10}}\beta(10),\end{aligned} \tag{18}$$

with the Riemann zeta function, $\zeta(x) = \sum_{n\geq 1} n^{-x}$, the Dirichlet beta function, $\beta(x) = \sum_{n\geq 0} (-1)^n (2n+1)^{-x}$ and $K = \beta(2)$ the Catalan's constant. To simplify the formula, the cusp in Eq. (17) is shifted by c_1. One has to notice that all expansion coefficients, except the first one, are negative and they decrease with the series order.

8. Asymptotic expansion $\mathcal{O}(g^{-40})$

Using the above method one can evaluate numerically the strong coupling expansion coefficients to rather high order and find that the asymptotic expansion is not Borel summable

$$\Gamma_{\rm cusp}(g) \sim -g \sum_k \frac{\Gamma\left(k-\frac{1}{2}\right)}{(2\pi g)^k} = g \int_0^\infty \frac{du\, u^{-1/2} {\rm e}^{-u}}{u - 2\pi g}\,, \tag{19}$$

where the Borel transform has a pole at $u = 2\pi g$.

Ambiguity due to different prescriptions to integrate over the pole is for large g $\delta\Gamma_{\rm cusp}(g) \sim g^{1/2} \exp(-2\pi g)$. Similar corrections appear in the solution of the FRS equation [15]

$$\gamma_S^{(L)}(g) = 2(\Gamma_{\rm cusp}(g) + \epsilon(g, L)) \ln S + \ldots\,. \tag{20}$$

This result agrees with the $O(6)$ sigma model from the string theory side [16].

9. Conclusions

The above calculations of the cusp anomalous dimension of $\mathcal{N} = 4$ SYM theory shows that the integrability provides us strong methods for solving complicated problems. Both, the weak and strong coupling expansion of the cusp can be found to an arbitrary order[4]. The results agree with result from the string theory[5] confirming validity of the AdS/CFT correspondence [17]. Moreover, it seems than on MSYM theory side it is easier also to calculate strong coupling expansion of the cusp anomalous dimension. Therefore, confirmation of the above results from the string theory will be a challenging task.

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[4] The calculations are limited only by ability of programs for symbolic calculations.

[5] After our work A. Tseytlin *et al.* corrected their two loop string calculation, which now agrees with out results.

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