Vol. 39 (2008) ACTA PHYSICA POLONICA B No 12

FINITE SIZE GIANT MAGNONS AND INTERACTIONS*

Olof Ohlsson Sax

Department of Physics and Astronomy, Uppsala University
Box 803, 751 08 Uppsala, Sweden

(Received October 15, 2008)

Magnon interactions give important contributions to the wrapping interactions of the $\mathcal{N} = 4$ spin-chain. Similar effects are expected for the finite size corrections to the giant magnon energy in $\mathrm{AdS}_5 \times \mathrm{S}^5$. In this paper I review the finite gap description of giant magnons and the leading order calculation of the finite size corrections to the giant magnon dispersion relation for multi-magnon states.

PACS numbers: 11.25.Tq, 11.55.Fv

1. Gauge invariant operators and spin-chains

A major test of the large N AdS/CFT correspondence [1] is the matching of the spectrum of the scaling dimensions of single trace operators in $\mathcal{N} = 4$ super Yang–Mills (SYM) and the energies of single strings propagating in $\mathrm{AdS}_5 \times \mathrm{S}^5$.

Many features of the spectrum of single trace operators can be understood by treating it as a lattice model. This is most easily understood in the SU(2) sector, which consists of operators built out of the scalars Z and Y. Here the operators can then be mapped to simple spin-chains. As realized by Minahan and Zarembo [2], the one-loop dilation operator in this sector equals the Hamiltonian of the Heisenberg SU(2) spin-chain. If we let Z represent spin up and Y represent spin down, a ferromagnetic ground state of a chain of length J is given by

$$|\Omega\rangle = \mathrm{Tr}\, Z^J = |\uparrow\uparrow\,\cdots\,\uparrow\uparrow\rangle\,. \tag{1}$$

This is a BPS state with classical dimension $E = J$ and vanishing anomalous dimension.

* Seminar talk presented at the XLVIII Cracow School of Theoretical Physics, "Aspects of Duality", Zakopane, Poland, June 13–22, 2008.

Non-BPS operators can be built out of this ground state by changing some of the Z's into Y's. In the spin-chain picture each such impurity corresponds to a magnon, *i.e.* a single fundamental excitation traveling along the chain with a certain momentum.

For an infinitely long chain, *i.e.* for $J \to \infty$, the spectrum consists of states with any number of magnons having arbitrary momenta. The contribution of each magnon to the scaling dimension, *i.e.* the dispersion relation of the magnons, is[1] [3]

$$\mathcal{E} = E - J = \sqrt{1 + 16g^2 \sin^2 \frac{p}{2}} \approx 1 + 8g^2 \sin^2 \frac{p}{2} \,. \tag{2}$$

For finite J, we need to take into account the periodic boundary conditions. In an integrable system such as the Heisenberg spin-chain, this leads to quantization of the momenta, which now satisfy Bethe equations of the form

$$e^{ip_i J} = \prod_{j \neq i} S^{-1}(p_j, p_i) \,, \tag{3}$$

where $S(p_j, p_i)$ is the two-body S-matrix. Hence the finite J spectrum is highly dependent on the magnon interactions.

To get a gauge independent operator from a spin-chain state, we need to take a trace. Taking into account the cyclicity of the trace, we will consider only spin-chain states that are symmetric under shifts of the spin sites. This leads to the level-matching condition

$$\sum p_i = 2\pi m \,, \qquad m \in \mathbb{Z} \,. \tag{4}$$

The success of this Bethe ansatz for the SU(2) sector relies on the similarity between the one-loop dilation operator in this sector and the integrable Heisenberg Hamiltonian. Miraculously there are many indications that even the all-loop dilatation operator of the full large N $\mathcal{N} = 4$ SYM is integrable. Hence it is possible to describe the full spectrum of single trace gauge invariant operators by the Bethe ansatz [4].

However, the Bethe ansatz is only valid for asymptotically long operators. The range of the interaction terms in the dilatation operator grows with the loop order. For short operators finite size corrections are expected to appear in the form of wrapping interactions. Ambjorn *et al.* [5] analyzed wrapping effects using the Thermodynamic Bethe Ansatz (TBA). They found that for an operator of length L, wrapping effects will generically appear at L loops.

[1] The coupling constant g is related to the 't Hooft coupling λ by $g^2 = \frac{\lambda}{16\pi^2}$.

Recently much work has been put into calculating these corrections. The simplest operator with a non-zero anomalous dimension is the Konishi operator. The dimension of this operator has been calculated to four loop order directly from the gauge theory [6] as well as using TBA [7], and the results of these calculations were found to agree perfectly.

2. The spinning point-like string and the giant magnon

By considering string states in $\mathbb{R} \times S^2 \subset \mathrm{AdS}_5 \times \mathrm{S}^5$ we find the duals of operators in the SU(2) sector of the gauge theory. The R-charge J now measures the angular momentum around the sphere. The simplest solution describes a point-like string spinning around a great circle. This supersymmetric state with $E = J$ is the dual of the spin-chain ground state.

Having identified the dual of the ground state the next step is to search for excitations. Berenstein *et al.* [8] considered the limit where $J, g \to \infty$ with g/J fixed, and showed that the semi-classical fluctuations around the ground state reproduce the leading order gauge theory spectrum in this limit.

Another possibility would be to let $J \to \infty$ while keeping g large but fixed. Hofman and Maldacena [9] looked for classical solutions such that the difference $E-J$ remains finite. They found a solution given by a world-sheet soliton. In space-time, the solution describes a string with end-points fixed on the equator with a constant angular separation $\Delta\varphi$, which is interpreted as the momentum p of the excitation. Using a conformal gauge where the J density is constant on the world-sheet, the world-sheet is infinitely large. The energy of the excited state is

$$\mathcal{E} = E - J = 4g \sin \frac{p}{2}\,, \tag{5}$$

which agrees with the large g limit of the gauge theory result (2). The situation is thus very similar to that of small fluctuations around an infinitely long spin chain.

The giant magnon solution seems to describe an open string in a closed string theory. However, a single magnon does not describe a physical configuration. Like in the gauge theory, physical states satisfy the level-matching condition (4). With an infinite world-sheet we can however relax this condition and consider a single magnon.

2.1. The giant magnon as a finite gap solution

One approach to deriving the giant magnon dispersion relation is by constructing a finite gap solution [10–12]. A classical solution to the equations of motion of the string is described by a meromorphic function $P(x)$, called the *quasi-momentum*, defined on a two-sheeted Riemann surface. The charges of the string depend on the analytical structure of $P(x)$. The possible singularities are square root branch cuts $\mathcal{C}_k$ and logarithmic branch cuts $\mathcal{B}_k$, conventionally referred to as *condensates*. If $P(x)$ has a square root branch cut along $\mathcal{C}_k$ it satisfies

$$P(x+i\epsilon)+P(x-i\epsilon)=2\pi n_k\,,\qquad x\in\mathcal{C}_k\,,\tag{6}$$

where $P(x)$ is to be evaluated once on each side of the cut.

The quantum numbers of a particular string configuration can be read off from the asymptotic behavior of the corresponding quasi-momentum[2]

$$\begin{aligned} P(x) &= \frac{E}{4g}\frac{1}{x\pm 1}+\cdots\,, \qquad (x\to\mp 1)\,, &(7)\\ P(x) &= \frac{J-Q}{2gx}+\cdots\,, \qquad (x\to\infty)\,, &(8)\\ P(x) &= p-\frac{J+Q}{2g}x+\cdots\,, \qquad (x\to 0)\,. &(9)\end{aligned}$$

Let us now consider a configuration consisting of a single condensate and make the ansatz

$$P(x)=\frac{E}{2g}\frac{x}{x^2-1}+G(x)\,,\qquad G(x)=-i\log\frac{x-X^+}{x-X^-}\,.\tag{10}$$

$P(x)$ has a single logarithmic cut between X^+ and X^-. The poles at ± 1 already have the correct residues. To get the right asymptotic behavior as $x\to 0,\infty$ we have to solve the equations

$$\begin{aligned} E-J+Q &= -2ig(X^+-X^-)\,, &(11)\\ E-J-Q &= 2ig\left(\frac{1}{X^+}-\frac{1}{X^-}\right)\,, &(12)\\ p &= -i\log\frac{X^+}{X^-}\,. &(13)\end{aligned}$$

Solving these equations for $X^\pm$ gives

$$X^\pm=\frac{Q+\sqrt{Q^2+16g^2\sin^2\frac{p}{2}}}{4g}e^{\pm i\frac{p}{2}}\csc\frac{p}{2}\,.\tag{14}$$

[2] The finite gap solutions considered here really correspond to strings on $\mathbb{R}\times S^3$. The conserved charges J and Q correspond to the isometries of a three-sphere. For solutions made up of giant magnons, Q counts the number of magnon constituents, with $Q=1$ corresponding to a fundamental magnon and larger Q corresponding to magnon bound states.

Plugging this back, solving for $E - J$ and setting $Q = 1$ gives the dispersion relation

$$\mathcal{E} = \sqrt{1 + 16g^2 \sin^2 \frac{p}{2}} \approx 4g \sin \frac{p}{2} \,. \tag{15}$$

To better understand how we arrived at this result, we consider an alternative finite gap derivation due to Vicedo [12]. In this formalism a solution to the classical equations of motion is encoded in terms of a meromorphic differential on a Riemann surface. Consider an elliptic, or two gap, solution, *i.e.* a solution with a genus one Riemann surface. This surface can be described as a two sheeted surface with two square root branch cuts connecting the points X^+ and X^-, and Y^+ and Y^-.

Defining the periods A_i and B as in Fig. 1, the differential dp will be subject to the following periodicity constraints

$$\int\limits_{A_i} dp = 0 \,, \qquad \int\limits_{B} dp = 2\pi n \,, \qquad n \in \mathbb{Z} \,. \tag{16}$$

In the singular limit where $Y^+ \to X^+$, the B period encircles the endpoint X^+ once. To satisfy the periodicity condition, dp has to have a simple pole of residue $-in$ at X^+. The expressions for the global charges in terms of dp now easily gives the giant magnon dispersion relation [13]. Thus a single giant magnon can be seen as a singular limit of an elliptic string state.

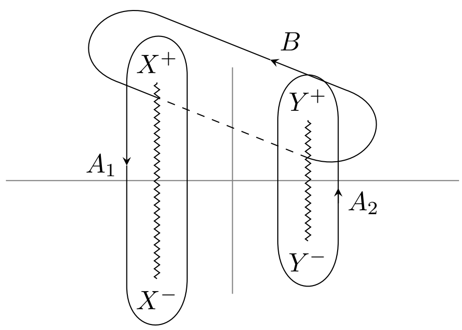

Fig. 1. The periods A_1, A_2 and B.

Hence we have two configurations corresponding to a giant magnon — a single condensate or two cuts sharing endpoints. As noted by Vicedo, the two descriptions are related by $\mathrm{SL}(2, \mathbb{Z})$ transformations, which exchange how the square root branch points are connected to form cuts. Performing such a transformation on a general finite gap solution gives a new solution, which corresponds to the same solution of the equations of motion for the string, provided the A- and B-periods of dp are preserved. To ensure this we may need to add extra condensates to the transformed solution.

3. Finite size giant magnons

In the previous section, we considered string solutions with one spin infinitely large, corresponding, in conformal gauge, to an infinitely long world sheet. For states with finite spin we expect the energy to receive finite size corrections, analogous to the wrapping interactions in gauge theory. At large coupling these corrections are expected to be exponentially suppressed in the string length J [5]. This was confirmed for the magnon by Arutyunov *et al.* [14] who derived an explicit generalization of the giant magnon solution. They found corrections to the dispersion relation of the form

$$\delta\mathcal{E} = -\frac{16g}{e^2}\sin^3\frac{p}{2}\exp\left(-2\frac{J}{4g\sin\frac{p}{2}}\right) + \cdots . \tag{17}$$

This result was later confirmed by Janik and Łukowski [15], using the Thermodynamic Bethe Ansatz (TBA) and the related Lüscher formulas for finite size corrections in two-dimensional quantum field theories.

4. Interacting magnons

We can also imagine string states consisting of several magnons. Hofman and Maldacena [9] showed that for infinite J, magnons can scatter against each other with a two-particle S-matrix which agrees with the SU(2|2) S-matrix of the gauge theory. The energy of these multi-magnon states is however simply the sum of the individual magnon energies. In the gauge theory we saw that for finite size states, interaction between the magnons affected both the allowed sets of momenta and the total anomalous dimension. Hence it would be interesting to see how the magnon interactions effect the spectrum on the string side.

4.1. Interacting finite size magnons as finite gap solutions

The description of interacting string states is in general a very complicated problem. From the point of view of the finite gap equations, we noted previously that a single giant magnon is a two gap solution. A state of n giant magnons corresponds to a solution with $2n$ cuts, and will hence be given as a function on a Riemann surface of genus $2n-1$, which generally has to be expressed in terms of hyperelliptic functions.

As noted above, the giant magnon in the infinite spin limit can be regarded as a finite gap solution consisting of a single condensate, resulting in a logarithmic resolvent, instead of the expected elliptic form of a generic two gap solution. A state of several magnons is now given by a number of such condensates. To see how this effects the solution, we note that the only equation that induces interactions between different cuts is Eq. (6), which describes the discontinuity of $P(x)$ along a square root branch cut. But this equation only involves square root cuts, and hence the many magnon solution is simply a sum of the individual solutions, resulting in an additive total energy, $E - J = \sum_i \mathcal{E}_i$.

A giant magnon could also be described as a singular limit of a two-cut solution. In this picture we expect finite size corrections to arise when we let the endpoints of the two cuts be very close to each other. Hence we will consider two square root branch cuts with a separation of length δ between the endpoints. Performing an $\mathrm{SL}(2,\mathbb{Z})$ transformation on this configuration we end up with a single condensate $\mathcal{B}_i$ with square root branch cuts $\mathcal{C}_i$ and $\bar{\mathcal{C}}_i$ of length δ attached at each end [16], as depicted in Fig. 2.

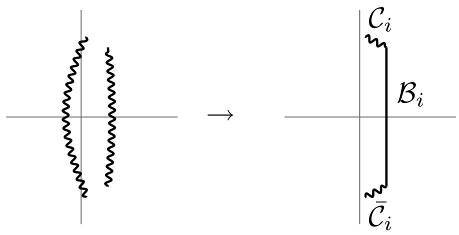

Fig. 2. Finite gap configurations for a finite J magnon as a two cut solution (left), and as a condensate with cuts at the ends (right). The two configurations are related by an $\mathrm{SL}(2,\mathbb{Z})$ transformation.

As an ansatz for the resolvent $G(x)$ of a set of finite size magnons we write[3]

$$G(x) = \sum_i G_i(x)\,, \qquad G_i(x) = -2i \log \frac{\sqrt{x - X_i^+} + \sqrt{x - Y_i^+}}{\sqrt{x - X_i^-} + \sqrt{x - Y_i^-}}\,. \tag{18}$$

The square root cuts of G_i are such that the relative sign in the numerator (denominator) changes when we cross $\mathcal{C}_i$ ($\bar{\mathcal{C}}_i$). As a simple check of this ansatz we can let $Y_i^\pm \to X_i^\pm$ to recover the previous giant magnon resolvent.

Since we are interested in the leading order corrections, we make an expansion by setting

$$Y_i^\pm = X_i^\pm \pm i\delta_i e^{\pm i\phi_i} + \cdots\,, \tag{19}$$

where $\delta_i \ll 1$ is real. We also need to take into account the back-reaction of $X_i^\pm$ by expanding it around δ_i. Calculating the asymptotic behavior as $x \to \infty$ and $x \to 0$ and solving the resulting equations iteratively to the

[3] In [16] the density corresponding to this resolvent was derived using the integral equations from [11]. The same resolvent has also been used to calculate finite-size corrections to giant magnons in $\mathrm{AdS}_4 \times \mathrm{CP}^3$ [17].

second order in δ_i by requiring that p_i and Q_i receive no corrections, we get the correction[4]

$$\delta\mathcal{E}_i = -\frac{g\delta_i^2}{4}\cos\left(p - 2\phi_i\right)\sin\frac{p}{2} + \cdots . \tag{20}$$

So far there has been no contribution from the interactions between the magnons. However, we still have to determine the parameters δ_i and ϕ_i. We will do that by ensuring that the cuts really correspond to square root branch cuts, requiring that they satisfy (6). Thus we need to calculate $P(x + i\epsilon) + P(x - i\epsilon)$ for x on a cut $\mathcal{C}_i$ to the leading order in δ_i. Solving the resulting equation for δ_i, we get

$$i\delta_i e^{i\phi_i} = 4\left(X_i^+ - X_i^-\right) e^{-\frac{iE}{2g}\frac{X_i^+}{\left(X_i^+\right)^2 - 1} + i\pi m} \prod_{k\neq i} \frac{X_i^+ - X_k^-}{X_i^+ - X_k^+} . \tag{21}$$

Inserting this into the above expression for the correction to the dispersion relation we get the final result

$$\delta\mathcal{E}_i = -\frac{16g}{e^2}\sin^3\frac{p_i}{2}\, e^{-2\frac{J}{\mathcal{E}_i}} \prod_{k\neq i} \frac{\sin^2\frac{p_i+p_k}{4}}{\sin^2\frac{p_i-p_k}{4}} e^{-2\frac{\mathcal{E}_k}{\mathcal{E}_i}} . \tag{22}$$

The exponential suppression, as well as the pre-factor, agrees with the one-magnon result in (17). The correction is changed by the magnon interactions by a multiplicative factor of order one, which is very similar in form to the two magnon scattering phase in [9].

4.2. Interacting finite size magnons from sine-Gordon and TBA

Interacting magnons can also be studied using the correspondence between the equations of motions for an $\mathbb{R} \times S^2$ sigma model and those of the sine-Gordon model. Hofman and Maldacena used this to calculate the magnon S-matrix. Klose and McLoughlin [19] considered periodic two-phase solutions to the sine-Gordon equation. These solutions can be interpreted as interacting finite two-magnon solutions. The resulting solutions turned out to be elliptic, rather than hyperelliptic, which considerably simplified the problem. The corrections to the dispersion relation perfectly agrees with the one calculated in the previous sections.

The finite size corrections to the multi-magnon energy has also been calculated using a generalization of the Lüscher rules [20]. Again the result agrees with the finite size calculation.

[4] Since we want to consider only fundamental magnons, Q_i/g will be put to zero. Dyonic magnons with $Q_i \sim g$ were treated in the same fashion in [16]. This case was also independently treated by Hatsuda and Suzuki [18].

5. Conclusions

The spectrum of long operators in $\mathcal{N} = 4$ SYM and for large spin states in the dual string theory is by now well known. The next step in finding the full spectra is the understanding of the finite size corrections to these states.

The giant magnon can be seen as a fundamental excitation on the string world-sheet, dual to impurities propagating in the spin-chain picture. At infinite J the spectra of both theories consist of states with magnons with arbitrary momenta (up to level-matching). The magnons may scatter against each other, but the total energy is given as a sum of the energies of the corresponding free magnons.

For finite J the dispersion relation of the spin-chain impurities receives wrapping corrections due to virtual excitations travelling around the chain. In addition the allowed momenta become quantized through a Bethe equation. The interaction between the magnons gives essential contributions to both of these corrections.

The dispersion relation of a single giant magnon gets exponential corrections at finite J [14]. Again these corrections stem from virtual excitations wrapping the world-sheet [15]. The energy of a multi-magnon state is however not the sum of the individual energies. In this paper, the finite-gap calculation of the leading order contribution from magnon interactions has been reviewed. The resulting order-one factors are related to the two magnon S-matrix.

As in the gauge theory, we expect the momenta of finite J giant magnons to be quantized in terms of the magnon S-matrix. The exact nature of this quantization remains an unsolved problem. The solution would deepen the understanding of the relation between the perturbative gauge and string theories.

I would like to thank J. Minahan and V. Giangreco Marotta Puletti for their valuable comments and discussions.

REFERENCES

[1] J.M. Maldacena, *Adv. Theor. Math. Phys.* **2**, 231 (1998) [*Int. J. Theor. Phys.* **38**, 1113 (1999)] [arXiv:hep-th/9711200].

[2] J.A. Minahan, K. Zarembo, *J. High Energy Phys.* **0303**, 013 (2003) [arXiv:hep-th/0212208].

[3] N. Beisert, *Adv. Theor. Math. Phys.* **12**, 945 (2008) [arXiv:hep-th/0511082].

[4] N. Beisert, M. Staudacher, *Nucl. Phys.* **B727**, 1 (2005) [arXiv:hep-th/0504190].

[5] J. Ambjorn, R.A. Janik, C. Kristjansen, *Nucl. Phys.* **B736**, 288 (2006) [arXiv:hep-th/0510171].

[6] F. Fiamberti, A. Santambrogio, C. Sieg, D. Zanon, *Phys. Lett.* **B666**, 100 (2008) [arXiv:0712.3522 [hep-th]]; F. Fiamberti, A. Santambrogio, C. Sieg, D. Zanon, *Nucl. Phys.* **B805**, 231 (2008) [arXiv:0806.2095 [hep-th]]; V.N. Velizhanin, arXiv:0808.3832 [hep-th].

[7] Z. Bajnok, R.A. Janik, arXiv:0807.0399 [hep-th].

[8] D.E. Berenstein, J.M. Maldacena, H.S. Nastase, *J. High Energy Phys.* **0204**, 013 (2002) [arXiv:hep-th/0202021].

[9] D.M. Hofman, J.M. Maldacena, *J. Phys.* **A39**, 13095 (2006) [arXiv:hep-th/0604135].

[10] V.A. Kazakov, A. Marshakov, J.A. Minahan, K. Zarembo, *J. High Energy Phys.* **0405**, 024 (2004) [arXiv:hep-th/0402207].

[11] J.A. Minahan, A. Tirziu, A.A. Tseytlin, *J. High Energy Phys.* **0608**, 049 (2006) [arXiv:hep-th/0606145].

[12] B. Vicedo, *J. High Energy Phys.* **0712**, 078 (2007) [arXiv:hep-th/0703180].

[13] K. Okamura, arXiv:0803.3999 [hep-th].

[14] G. Arutyunov, S. Frolov, M. Zamaklar, *Nucl. Phys.* **B778**, 1 (2007) [arXiv:hep-th/0606126].

[15] R.A. Janik, T. Lukowski, *Phys. Rev.* **D76**, 126008 (2007) [arXiv:0708.2208 [hep-th]].

[16] J.A. Minahan, O. Ohlsson Sax, *Nucl. Phys.* **B801**, 97 (2008) [arXiv:0801.2064 [hep-th]].

[17] T. Lukowski, O. Ohlsson Sax, arXiv:0810.1246 [hep-th].

[18] Y. Hatsuda, R. Suzuki, *Nucl. Phys.* **B800**, 349 (2008) [arXiv:0801.0747 [hep-th]].

[19] T. Klose, T. McLoughlin, *J. Phys.* **A41**, 285401 (2008) [arXiv:0803.2324 [hep-th]].

[20] Y. Hatsuda, R. Suzuki, *J. High Energy Phys.* **0809**, 025 (2008) [arXiv:0807.0643 [hep-th]].