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HYDRODYNAMICS AND GAUGE/GRAVITY DUALITY*

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We review recent applications of gauge/gravity duality to finite-temperature quantum field theory. In particular we describe how the shear viscosity can be computed from gravity duals.

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1. Introduction

Recently, there is a lot of interest in the application of AdS/CFT correspondence [1–3] in hydrodynamics. The motivation for these studies are experiments with collisions of heavy nuclei. At the Relativistic Heavy Ion Collider (RHIC) at Brookhaven, gold nuclei are collided with center of mass energy of 200 GeV/nucleon. In terms of the total energy, this corresponds to 200×197 GeV per collision. At the LHC heavy ion experiments will be done with lead nuclei. The center of mass energy will be 5.5 TeV per nucleon, almost 30 times more than the energy at RHIC.

The collision event can be visualized as follows. Accelerated to the RHIC energy, the nuclei are Lorentz-contracted pancakes with thickness less than 0.1 fm. The two nuclei passed through each other during a very brief moment. Most of the baryons go right through each other, but particles are created between the two receding nuclei. The "stuff" that is left between the two receding nuclei is what will thermalize into a quark gluon plasma (QGP). The plasma expands, cools down, and disintegrates into particles, which fly into the detectors.

While this picture still lacks many details, we now know that the medium created during the heavy ion collisions does behave like a medium, *i.e.*, in a collective manner. One piece of evidence comes from the so-called "elliptic

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flow." In order to see this effect, one selects events where there is a non-zero impact parameter. In this case region of the QGP is of an almond shape. The region then expands due to the pressure gradient: the medium is hotter, and has larger pressure at the center of the almond then at the edge. When the region expands, it does so more quickly along the shorter axis of the almond, where the pressure gradient is larger, compared to the longer axis, where the pressure gradient is smaller. As the result the detector will see an anisotropic distribution of particles: those flying along the shorter axis of the almond are in average more energetic than those moving along the longer axis. This anisotropy is quantitatively characterized by a parameter called v_2 (the index 2 refers to the fact that it is the coefficient of the $\cos 2\phi$ Fourier component of a function of the azimuthal angle ϕ).

2. Relativistic hydrodynamics

One of the simplest, but very successful, model of the heavy ion collisions is that of a liquid drop which evolves according to the equations of relativistic hydrodynamics. For the hydrodynamic equation to apply, it is necessary that the characteristic length scale of the problem is much larger than the mean free path. In other words, hydrodynamics is an effective theory which captures the low-frequency dynamics of modes with $\omega, q \ll \ell_{\text{mfp}}^{-1}$. As the size of the nucleus is about 6–7 fermi, and the mean free path is perhaps a fraction of a fermi (see later), the hydrodynamic approximation is expected to work reasonably well.

Let us recall the equations of hydrodynamics. For simplicity, let us consider a plasma with no conserved charge. Only energy and momentum are conserved, which is encoded in the equation

$$\partial_\mu T^{\mu\nu} = 0\,. \tag{1}$$

At zeroth order in derivatives (or ω and q, which will be our expansion parameters) we have

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + P g^{\mu\nu}\,, \tag{2}$$

where ϵ is the energy density. This form of the stress-energy tensor can be obtained by taking the stress-energy tensor in the local rest frame (*i.e.*, the frame where $u^\mu = (1,0,0,0)$), $T^{\mu\nu} = \text{diag}(\epsilon, P, P, P)$, and boosting it.

One can see that Eqs. (1) and (2) define a system of four equations for four unknowns functions of space and time. The unknowns are the three independent components of the velocity $u^\mu(t, \boldsymbol{x})$ (recall the constraint $u^\mu u_\mu = -1$) and the temperature $T(t, \boldsymbol{x})$. The pressure P and the energy density ϵ depends on T through the equation of state. As a reminder, we have the following basic equations: $dP = s dT$, $d\epsilon = T dS$, $\epsilon + P = Ts$.

To next order, we have:

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + Pg^{\mu\nu} + \text{terms with one derivative} + \dots\,, \tag{3}$$

where the terms with one derivative give the viscous part of the stress energy tensor $\Pi^{\mu\nu}$. We assume an expansion in derivatives, which requires variations being smooth, $\partial_\mu \ll \ell^{-1}_{\rm mfp}$. We can write down the following expressions containing one derivative

$$\begin{aligned} &\partial^\nu u^\mu + \partial^\mu u^\nu\,, && u^\mu \partial^\nu T + u^\nu \partial^\mu T\,, \\ &g^{\mu\nu}(\partial\cdot u)\,, && g^{\mu\nu}(u\cdot\partial T)\,, \\ &u^{\{\mu}u^{\nu\}}(\partial\cdot u)\,, && u^{\{\mu}u^{\nu\}}(u\cdot\partial T)\,. \end{aligned} \tag{4}$$

It is possible to get rid of some of these terms by performing a shift:

$$u^m u \to \tilde{u}^\mu + \#\partial^\mu T\,. \tag{5}$$

We can use this freedom to impose a "gauge choice" $u_\mu \Pi^{\mu\nu} = 0$. In the local rest frame where $u^\mu = (1, \bar{0})$, this gives us $\Pi^{00} = \Pi^{0i} = 0$. With this constraint, the most general form of $\Pi^{\mu\nu}$ is

$$\Pi^{\mu\nu} = -\eta P^{\mu\alpha}P^{\nu\beta}\left(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{3}g_{\alpha\beta}\partial\cdot u\right) - \zeta P^{\mu\nu}(\partial\cdot u)\,. \tag{6}$$

The first, traceless part has a numerical coefficient η which is called the shear viscosity. The trace part is proportional to the bulk viscosity ζ. The bulk viscosity encodes the resistance of the system to uniform expansion. Both η and ζ are functions of the temperature T.

If the theory is conformal, then the energy-momentum tensor should be traceless in flat space,

$$T^\mu_\mu = \epsilon - 3P + \zeta(\partial\cdot u)\,. \tag{7}$$

Since this identity should be satisfied for all solutions, we find $\epsilon = 3P$ and $\zeta = 0$.

3. Hydrodynamic modes

Now, let us consider small fluctuations around the thermal equilibrium, corresponding to a fluid with uniform temperature and at rest,

$$T(x) = T_0\,, \qquad u^\mu = (1, \vec{0})\,. \tag{8}$$

We deal with small, linearized perturbations (the perturbations do not interact with each other).

$$
\begin{aligned}
u^i &\ll 1\,,\\
u^0 &= 1+\mathcal{O}(\bar{u}^2)\,,\\
T &= T_0+\delta T\,,\\
\epsilon &= \epsilon_0+\delta\epsilon\,,\\
p &= p_0+\delta p\,.
\end{aligned}
$$

Plugging this back into the first order expression for the energy momentum tensor, we have:

$$
\begin{aligned}
T^{00} &= \epsilon_0+\delta\epsilon+\ldots\,,\\
T^{0i} &= (\epsilon_0+P_0)u^i\,,\\
T^{ij} &= (P_0+\delta P)\delta^{ij}-\eta\left(\partial_i u_j+\partial_j u_i-\frac{2}{3}\delta_{ij}\partial_k u_k\right)-\zeta(\partial_k u_k)\delta^{ij}\,. \qquad (9)
\end{aligned}
$$

We now insert these expressions into the conservation laws

$$
\begin{aligned}
0 &= \partial_0 T^{00}+\partial_i T^{0i}\,,\\
0 &= \partial_0 T^{0i}+\partial_j T^{ij}\,. \qquad (10)
\end{aligned}
$$

We can perform Fourier transforms of the temperature and the velocity vector. Let us look at the component proportional to $\exp(-i\omega t+i\vec{q}\cdot\vec{x})$. We split the velocity vectors into a transverse part and a longitudinal part, $\vec{u}=\vec{u}_{\rm T}+\vec{u}_{\rm L}$, where $\vec{q}\cdot\vec{u}_{\rm T}=0$, $\vec{u}_{\rm L}\parallel\vec{q}$. For the transverse mode we get

$$
\left[(\epsilon_0+P_0)\omega+i\eta\vec{q}^{\,2}\right]\vec{u}_{\rm T}=0\,, \qquad (11)
$$

and for the longitudinal mode, there is a coupling between the fluctuations in the energy density and the longitudinal velocity component,

$$
\begin{pmatrix} \omega & -(\epsilon_0+p_0) \\ -q\left(\frac{\partial p}{\partial\epsilon}\right) & (\epsilon_0+p_0)\omega+i\left(\zeta+\frac{4}{3}\eta\right)q^2 \end{pmatrix}\begin{pmatrix} \delta\epsilon \\ u_p \end{pmatrix}=0\,. \qquad (12)
$$

For the transverse mode, we find the dispersion relation $\omega=-iDq^2$, which corresponds to an overdamped mode. The dispersion relation of the longitudinal modes is obtained by diagonalizing the matrix and we find that:

$$
\omega=\pm c_{\rm s}q-i\frac{\Gamma}{2}q^2\,, \qquad (13)
$$

where $c_{\rm s}$ is the speed of sound,

$$c_{\rm s} = \sqrt{\frac{\partial P}{\partial \epsilon}}\,, \tag{14}$$

and

$$\Gamma = \frac{1}{\epsilon_0 + p_0}\left(\zeta + \frac{4}{3}\eta\right)\,. \tag{15}$$

The imaginary part of (13) is much smaller than the real part in the limit $q \to 0$.

3.1. Linear response theory and Kubo's formula

The hydrodynamic equations can be viewed as a low-energy effective theory. As such, it is capable of making prediction for the low-momentum behavior of correlation functions. We shall extract one prediction, namely, the Kubo formula that relate the shear viscosity with a thermal correlation function of stress-energy tensor.

Let us remind ourselves the linear response theory. Consider a theory with an action S. We perturb the theory by introducing a source J coupled to some operator $\mathcal{O}$:

$$S \to S + \int dx J(x)\mathcal{O}(x)\,. \tag{16}$$

Let us assume that the expectation value of $\mathcal{O}$ is zero in the absence of the source. If the source is small, then the average of $\mathcal{O}$ is

$$\langle \mathcal{O}(x)\rangle = -\int dy G_{\rm R}(x-y)J(y)\,, \tag{17}$$

where $G_{\rm R}$ is the retarded Green's function of $\mathcal{O}$,

$$iG_{\rm R}(x-y) = \theta\left(x^0 - y^0\right)\langle[\mathcal{O}(x),\, \mathcal{O}(y)]\rangle\ . \tag{18}$$

We are interested in the two point function of the stress energy tensor, $\left\langle T^{\mu\nu}(x)T^{\alpha\beta}(y)\right\rangle$. Since the source of the stress-energy tensor in the metric, we therefore need to consider small perturbations of the metric:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,, \tag{19}$$

where $h_{\mu\nu} \ll 1$. We can therefore write that:

$$\langle T^{\mu\nu}(x)\rangle \sim \int dy\, \left\langle T^{\mu\nu}(x)T^{\alpha\beta}(y)\right\rangle_{\rm R} h_{\alpha\beta}(y)\,. \tag{20}$$

If $h_{\mu\nu}$ varies over slowly in space and time, its influence on the fluid can be captured within hydrodynamics. To quantify this effect, we need to generalize the hydrodynamic equations to curved space. That can be done easily by replacing derivatives by covariant derivatives. The conservation law becomes

$$\nabla_\mu T^{\mu\nu} = 0\,, \tag{21}$$

and the constitutive equation becomes

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + P g^{\mu\nu} - \eta P^{\mu\alpha} P^{\nu\beta} \left(\nabla_\alpha u^\beta + \nabla_\beta u^\alpha - \frac{2}{3} g_{\alpha\beta} \nabla \cdot u \right) \tag{22}$$

(we set the bulk viscosity $\zeta = 0$). Now, let us consider a perturbation with only one component:

$$h_{xy} = h_{xy}(t) \tag{23}$$

with every other $h_{\mu\nu} = 0$. Recall that $h_{\mu\nu}$ is an external source and as such does not need to satisfy the Einstein equations. With respect to the spatial O(3) group, (23) is a spin-two perturbation and hence, to linear order, cannot excite any fluctuation of the velocity (which is a vector) or the temperature (a scalar). Therefore $u^x = u^y = u^z = 0$ and $T = T_0$ to linear order. Now let us look in more detail at the xy component of the stress-energy tensor,

$$T_{xy} = P g_{xy} - \eta \left(\nabla_x u_y + \nabla_y u_x \right) = P h_{xy} + \eta \partial_0 h_{xy} = -\int dy G_{\mathrm{R}}(x-y) h_{xy}(y)\,, \tag{24}$$

where we have used that:

$$\nabla_\mu u_\nu = \partial_\mu u_\nu - \Gamma^\lambda_{\mu\nu} u_\lambda \tag{25}$$

but because all but u_0 are zero, we are left with:

$$\nabla_x u_y = -\Gamma^0_{xy} = \tfrac{1}{2} \partial_0 h_{xy}\,, \tag{26}$$

and so for the two point function in the zero spatial momentum limit, we obtain:

$$\langle T^{xy} T^{xy} \rangle (\omega; \bar{q} \to 0) = P - i\eta\omega + \mathcal{O}\left(\omega^2\right) \tag{27}$$

and all other two point functions involving $T^{\mu\nu}$ vanish. Note that the first term in the series is a contact term. We are then left with Kubo's formula relating the viscosity with limiting behavior of the Green's function.

$$\eta = -\lim_{\omega \to 0} \frac{1}{\omega} G_{\mathrm{R}}^{xy.xy}\left(\omega, \bar{0}\right)\,. \tag{28}$$

4. AdS/CFT calculation of the viscosity

We are now interested in calculating the retarded Greens function which is related to the two point function of the stress energy tensor. Using the AdS/CFT correspondence, this calculation is very simple. We know that the supergravity field which couples to the stress energy tensor is the metric, so we will be interested in looking at the propagation of gravitons in the supergravity background. We start by showing how to calculate this at $T = 0$ and then go to the finite temperature case which is of relevance for the calculation of the shear viscosity from the Kubo relation.

We write the $T = 0$ AdS metric as:

$$ds^2 = \frac{R^2}{z^2}\left(-dt^2 + d\bar{x}^2 + dz^2\right) . \tag{29}$$

We then wish to calculate the equation of motion for fluctuations on top of the metric. We are only interested fluctuations of a single component of the metric. In order to extract the T_{xy} two point function we will define $\phi = h^x{}_y$ and obtain the equation of motion for ϕ:

$$\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\right) = 0 \tag{30}$$

which, for the AdS metric, reduces to a simple equation:

$$\phi'' - \frac{3}{2}\phi' - q^2\phi = 0\,, \tag{31}$$

where we have made the plane wave ansatz $\phi = \phi(z)e^{iq.x}$. The solution to this can be written (after Fourier transforming) as:

$$\phi(q,z) = f_q(z)\phi_0(q) = \tfrac{1}{2}(qz)^2K_2(qz)\phi_0(q)\,, \tag{32}$$

where $\phi_0(q)$ is the Fourier transform of $\phi_0(x) = \phi(x,z)|_{z=0}$ and K_2 is the modified Bessel function which diverges in the IR ($z \to 0$) and goes to zero in the UV ($z \to \infty$). Taking the solution and putting it back into the classical action yields only a surface term:

$$\begin{aligned} S_d[\phi] &= \int dz d^4x\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = \int d^4x\sqrt{-g}g^{zz}\phi\partial_z\phi|_{z=0}^{z=\infty} \\ &= \int d^4q\frac{f_{-q}(z)\partial_z f_q(z)}{z^3}\phi_0(-q)\phi(q)\,. \end{aligned} \tag{33}$$

In order to calculate the correlator of two elements of the stress energy tensor we simply need to take two derivatives of the generating function with respect to the source of the stress energy tensor, which is the boundary value of ϕ. This gives us:

$$\langle T^{xy}T^{xy}\rangle \sim \frac{\delta^2 S_{\rm cl}}{\delta\phi_0\delta\phi_0} = \frac{f_{-q}(z)\partial_z f_q(z)}{z^3}|_{z=\epsilon} = \#\frac{q^2}{\epsilon^2} + \#q^4\ln q^2\,. \tag{34}$$

There is a contact term proportional to ϵ^{-2}. To treat this term one needs to know more about the procedure of holographic renormalization. In our case, we will just be interested in the imaginary part of the above object, which is $\sim q^4$.

Now we can perform the same calculation on the AdS black hole metric

$$ds^2 = \frac{r^2}{R^2}\left(-f(r)dt^2 + d\bar{x}^2\right) + \frac{R^2}{r^2 f(r)}dr^2\,, \tag{35}$$

where $f(r) = r_0^4/r^4$ and $T = r_0/\pi R^2$. Performing a change of coordinates $u = (r/r_0)^2$, we can again derive the equation of motion for the xy component of the metric fluctuation, ϕ:

$$\phi'' - \frac{1+u^2}{uf}\phi' + \frac{\omega^2 - q^2 f}{uf^2}\phi = 0\,, \tag{36}$$

where we have rescaled ω and q of $2\pi T$ to make them dimensionless. Close to the horizon the equation is:

$$\phi'' - \frac{\phi'}{1-u} + \frac{\omega^2}{4(1-u^2)}\phi = 0 \tag{37}$$

which has two solutions $\phi = (1-u)^{i\omega/2}$. The appropriate boundary condition at the horizon for the retarded Greens function is the incoming wave boundary conditions, which picks up the asymptotics $\phi = (1-u)^{-i\omega/2}$. The solution to the mode equation can be written as

$$F_q(u) = (1-u)^{\frac{-i\omega}{2}}G(u)\,, \tag{38}$$

where G is regular at the horizon and tends to 1 at the boundary. The equation for $G(u)$ can be easily obtained by substituting (38) into Eq. (36), but we will not write it down. The solution can be found as a series in ω and q. Just as in the case of the zero temperature calculation we are now asked to take:

$$\langle T^{xy}T^{xy}\rangle = \#\left.\frac{F_{-q}(u)F'_q(u)}{u}\right|_{u\to 0}\,. \tag{39}$$

Again, the imaginary part of the correlator is finite, and it is what needed for the Kubo formula for the shear viscosity. The result for η can be presented in a simple form by dividing it to the volume entropy density s, which can be found from black hole thermodynamics. Using thermodynamics it is then very easy to calculate the entropy density and we can take the ratio of these two quantities to get:

$$\frac{\eta}{s} = \frac{1}{4\pi} \,. \tag{40}$$

We recall that this result was obtained for the $\mathcal{N} = 4$ supersymmetric Yang–Mills theory.

More surprisingly, the result (40) holds for all theories with Einstein gravity duals. These include theories with different dimensions, including in the presence of finite chemical potential, with broken conformal symmetry, with fundamental matter *etc.* There are now several proofs of the universality of η/s in theories with gravity duals. The most intuitive argument is based on the identification of the shear viscosity with the cross-section of graviton absorption on a black hole in the zero frequency limit [4],

$$\eta \sim \lim_{\omega \to 0} \sigma_{\rm abs}(\omega) \,. \tag{41}$$

Within Einstein gravity, one can show that $\sigma_{\rm abs}$ approaches the geometric area of the horizon in the limit $\omega \to 0$. On the other hand, the entropy is also proportional to the area of the horizon. When one takes the ratio η/s, the factors of the horizon area cancel out and one is left with the universal value $\hbar/4\pi$.

Within kinetic theory, the ratio η/s proportional to the ratio of the mean free path and the de Broglie wavelength of quasiparticles. Thus in theories with gravity duals the mean free path is of the same order as the de Broglie wavelength, which is consistent with the fact that they are strongly coupled. From the field-theoretical point of view it is completely mysterious why η/s is constant in all theories of this class.

4.1. Second-order hydrodynamics

It is possible to go one order beyond the first-order hydrodynamics and take into account second-order corrections in the stress-energy tensor. Conformal symmetry imposes strong restriction on the possible form of the second-order terms. It was found that there are five additional independent kinetic coefficients at this level. All coefficients have been found for the $\mathcal{N} = 4$ supersymmetric Yang–Mills theory [5, 6].

5. Conclusion

We have seen that gauge/gravity duality can be generalized to finite temperature and used for the computation of real time quantities, like the kinetic coefficients including the shear viscosity. Such applications of gauge/gravity duality have revealed deep connections between thermal field theory, hydrodynamics, and black hole physics.

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