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HYDRODYNAMIC FLOW OF THE QUARK–GLUON PLASMA AND GAUGE/GRAVITY CORRESPONDENCE*

MICHAL P. HELLER†, ROMUALD A. JANIK‡

M. Smoluchowski Institute of Physics, Jagellonian University
Reymonta 4, 30-059 Krakow, Poland

R. PESCHANSKI§

Institut de Physique Théorique, URA 2306, Unité de Recherche associée au CNRS
CEA-Saclay, 91191 Gif/Yvette Cedex, France

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The contribution presents a summary of the Gauge/Gravity approach to the study of hydrodynamic flow of the quark–gluon plasma formed in heavy-ion collisions. Considering the ideal case of a supersymmetric Yang–Mills theory for which the AdS/CFT correspondence gives a precise form of the Gauge/Gravity duality, the properties of the strongly coupled expanding plasma are put in one-to-one correspondence with the metric of a 5-dimensional black hole moving away in the 5th dimension and its deformations consistent with the relevant Einstein equations. Several recently studied aspects of this framework are recalled and put in perspective. This paper is a written version of the four lectures given by the authors on that subject.

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1. Hydrodynamics are relevant for heavy-ion collisions

One of the most striking lessons one may draw [1, 2] from experiments on heavy-ion collisions at high energy (*e.g.* at the RHIC accelerator, Brookhaven) is that fluid hydrodynamics seems to be relevant for understanding

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† `heller@th.if.uj.edu.pl`

‡ `ufrjanik@th.if.uj.edu.pl`

§ `robi.peschanski@cea.fr`

the dynamics of the reaction. Indeed, the elliptic flow [3] describing the anisotropy of the low-$p_{\rm T}$ particles produced in a collision at non zero impact parameter implies the existence of a collective flow of the particles following a hydrodynamical pressure gradient due to the initial eccentricity in the collision. Moreover, most hydrodynamical simulations which are successful to describe this elliptic flow are consistent with an almost "perfect fluid" behaviour, *i.e.* a small "viscosity over entropy" ratio η/s (see, for instance, the reviews [2]).

The validity of a hydrodynamical description assuming a quasi-perfect fluid behaviour has been nicely anticipated in Ref. [4]. The so-called *Bjorken flow* is based on the hypothesis of an intermediate stage of the reaction process, namely a boost-invariant[1] quark–gluon plasma phase as a relativistic expanding fluid. It is formed after a (quite rapid) thermalization period and finally decays into hadrons, see Fig. 1. The boost-invariance can be justified in the central region of the collision since the observed distribution of particles is flat, in agreement with the prediction of hydrodynamical boost-invariance, where (space-time) fluid and (energy-momentum) particle rapidities are proved to be equal [4], see Sec. 3.

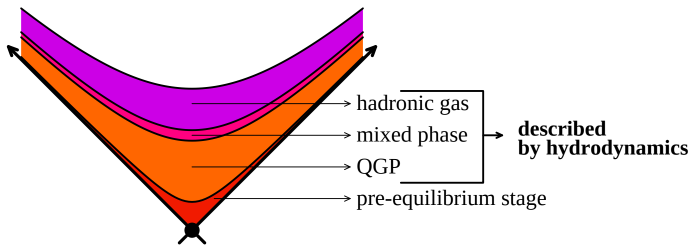

Fig. 1. Description of QGP formation in heavy ion collisions. The kinematic landscape is defined by $\tau = \sqrt{x_0^2 - x_1^2}$; $\eta = \frac{1}{2}\log\frac{x_0+x_1}{x_0-x_1}$; $x_{\rm T} = \{x_2, x_3\}$, where the coordinates along the light-cone are $x_0 \pm x_1$, the transverse ones are $\{x_2, x_3\}$ and τ is the proper time, η the "space-time rapidity".

The Bjorken flow was instrumental for deriving many qualitative and even quantitative features of the quark–gluon plasma formation in heavy-ion reactions. However, as inherent to the hydrodynamic approach, it says only little on the relation with the microscopic gauge field theory, *i.e.* Quantum Chromodynamics (QCD). Some important questions remain unsolved, such as the reason why the fluid behaves like a perfect fluid, what is the

[1] The introduction of hydrodynamics in the description of high-energy hadronic collisions has been proposed by Landau [5], assuming "full stopping" initial conditions which result in a non boost-invariant solution or *Landau flow* (see [6] for a unified description of Bjorken and Landau flows). We will comment later on the relevance of the Landau flow for AdS/CFT.

small amount of viscosity it may require, why and how fast thermalization proceeds, *etc.* The problem is made even more difficult by the strong coupling regime of QCD which is very probably required, since a perturbative description leads in general to a high η/s. Indeed, the mean free path induced by the gauge theory should be small (hence the coupling strong) in order to damp the near-by force transversal to the flow, measuring the shear viscosity.

It is thus interesting to use our modern (but still largely in progress) knowledge of non perturbative methods in quantum field theory to fill the gap between the macroscopic and microscopic descriptions of the quark–gluon plasma produced in heavy-ion collisions. Lattice gauge theory methods are very useful to analyze the static properties of the quark–gluon plasma, but there are still powerless to describe the plasma in collision. Hence we are led to rely upon the new tools offered by the Gauge/Gravity correspondence and in particular the one which is the most studied and well-known namely the AdS/CFT duality [7] between the $\mathcal{N}=4$ supersymmetric Yang–Mills theory and the type IIB superstring in the large $N_{\rm c}$ approximation. The features of the gauge theory on the (physical) Minkowski space in $3+1$ dimensions at strong coupling are in one-to-one relation with corresponding ones in the bulk of the target space of the 10-dimensional string and in particular in the 5-dimensional metric of the AdS space, the boundary of which can be identified with the 4-dimensional Minkowski space.

One should be aware when using the AdS/CFT tools that there does not yet exist a gravity dual construction for QCD. However, the nice feature of the quark–gluon plasma problems is that it is a deconfined phase of QCD, characterized by collective degrees of freedom and thus one may expect to get useful information from AdS/CFT duality. This has been already proved when describing static geometries by an evaluation of η/s [8]. The subject of the present lectures is the investigation of the Gauge/Gravity correspondence, in particular the AdS/CFT duality, in a dynamical setting corresponding to a collision.

2. Relativistic hydrodynamics and Bjorken flow

On theoretical grounds, there are quite appealing features for applying hydrodynamic concepts to high-energy heavy-ion reactions. Such concepts have been already introduced some time ago [4,5] and find a plausible realization nowadays. The fact that a rather dense interacting medium is created in the first stage of the collision allows one to admit that the individual partonic or hadronic degrees of freedom are not relevant during the early evolution of the medium and justifies its treatment as a fluid. For the same reason local equilibrium is a plausible assumption. Moreover, the high quantum occupation numbers allow one to use a classical picture and to assume that the "pieces of fluid" may follow quasi-classical trajectories in space-time, expressed as an in-out cascade [9] with the straight-line trajectories starting at the origin (see Fig. 2), with

$$y = \eta\,, \tag{1}$$

where

$$y = \frac{1}{2}\log\left(\frac{E+p}{E-p}\right)\,, \qquad \eta = \frac{1}{2}\log\left(\frac{x_0+x_1}{x_0-x_1}\right) \tag{2}$$

are, respectively, the rapidity and "space-time rapidity" of the piece of the fluid[2].

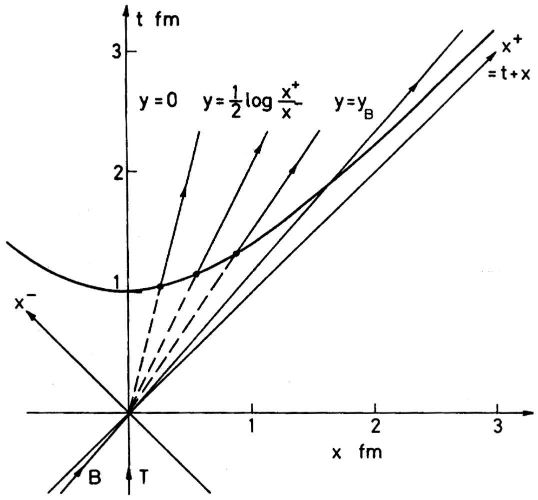

Fig. 2. In–Out cascade. The "piece of fluid" with space-time rapidity η gives rise to hadrons at rapidity $y \equiv \eta$, after crossing the "freeze-out" hyperbola at fixed proper-time τ.

Note that (1) can be rewritten in the form

$$2y = \log u^+ - \log u^- = \log x^+ - \log x^-\,, \tag{3}$$

where $u^\pm = e^{\pm y}$ are the light-cone components of the fluid (four-)velocity and $x^\pm = x_0 \pm x_1$ are the light-cone kinematical variables.

Taking (1) as the starting point and using the perfect fluid hydrodynamics, Bjorken developed in his seminal paper [4] a suggestive (and very useful in many applications) physical picture of the central rapidity region of

[2] We keep the conventional notation η, not to be confused with viscosity. The difference is clear enough to avoid mistakes.

highly relativistic collisions of heavy ions. In this picture the condition (1) leads to a boost-invariant geometry of the expanding fluid and thus to the central plateau in the distribution of particles.

Let us introduce the relativistic hydrodynamic equations in light-cone variables. We consider the "perfect fluid" approximation for which the energy-momentum tensor is

$$T^{\mu\nu} = (\epsilon + p)u^\mu u^\nu - p\eta^{\mu\nu}\,, \tag{4}$$

where ϵ is the energy density, p is the pressure and u^μ is the 4-velocity. We assume that the energy density and pressure are related by the equation of state:

$$\epsilon = gp\,, \tag{5}$$

where $1/\sqrt{g}$ is the sound velocity in the liquid. For the "conformal case" $T^{\mu\mu} = 0$ and thus $g \equiv 3$.

Using

$$u^\pm \equiv u^0 \pm u^1 = e^{\pm y}\,, \tag{6}$$

and introducing

$$x^\pm = x^0 \pm x^1 = \tau e^{\pm\eta} \to \left(\frac{\partial}{\partial x^0} \pm \frac{\partial}{\partial x^1}\right) = \frac{1}{2}\frac{\partial}{\partial x^\pm} \equiv \frac{1}{2}\partial_\pm\,, \tag{7}$$

where $\tau = \sqrt{x^+x^-}$ is the proper time and η is the spatial rapidity of the fluid, the hydrodynamic equations

$$\partial_\mu T^{\mu\nu} = 0 \tag{8}$$

take the form

$$\partial_\pm T^{01} + \frac{1}{2}\partial_+(T^{11} \pm T^{00}) - \frac{1}{2}\partial_-(T^{11} \mp T^{00}) = 0\,. \tag{9}$$

Using now (4) and the equation of state (5) we deduce from this

$$\begin{aligned} g\partial_+[\log p] &= -\frac{(1+g)^2}{2}\partial_+ y - \frac{g^2-1}{2}e^{-2y}\partial_- y\,, \\ g\partial_-[\log p] &= \frac{(1+g)^2}{2}\partial_- y + \frac{g^2-1}{2}e^{2y}\partial_+ y\,. \end{aligned} \tag{10}$$

These are two equations for two unknowns which describe the state of the liquid: the pressure p and the rapidity y. They should be expressed in terms of the positions x^+, x^- in the liquid. Other thermodynamic quantities can be obtained from the equation of state (5) and the standard thermodynamical identities:

$$p+\epsilon = Ts\,, \qquad d\epsilon = Tds\,, \tag{11}$$

where we have assumed for simplicity vanishing chemical potential.

The result is

$$\epsilon = gp = \epsilon_0 T^{g+1}\,, \qquad s = s_0 T^g \to s \sim \epsilon^{g/(g+1)}\,. \tag{12}$$

The simplest possibility to describe the expansion of the fluid was suggested by Bjorken [4] who proposed to use the Ansatz (1) in the hydrodynamical context. Introducing (1) into (10) we obtain

$$g\partial_+[\log p] = -\frac{1+g}{2x^+}\,, \qquad g\partial_-[\log p] = -\frac{g+1}{2x^-} \tag{13}$$

from which we deduce

$$p = \epsilon\; g^{-1} = p_0\; (x^+x^-)^{-(g+1)/2g} = p_0\; \tau^{-(g+1)/g}\,, \tag{14}$$

where p_0 is a constant, and thus specifically

$$p = \frac{\epsilon}{3} = p_0\; (x^+x^-)^{-2/3} = p_0\; \tau^{-4/3} \propto T^4 \tag{15}$$

for the conformal case.

Thus the system is boost-invariant: the pressure does not depend neither on η nor on y. So are ϵ, s and T, given by (12). It is interesting to note that the Landau flow corresponds asymptotically only to a logarithmic correction of relation 1, namely

$$u^\pm \sim x^\pm \sqrt{\log x^\pm}\,, \tag{16}$$

which gives finally rise (as already noticed in [5], and for instance recently discussed in [10]) to a gaussian shape in the y distribution of the entropy, revealing a non boost-invariant picture, at least at some distance from central rapidity.

3. Interest of AdS/QCD duality

In the previous sections we mentioned the ubiquity of hydrodynamic methods in the description of QGP produced at RHIC. Yet, despite their success in describing data, we have to keep in mind that they are used as a phenomenological model without a real derivation from gauge theory. This is quite understandable since almost perfect fluid hydrodynamics is intrinsically a strong coupling phenomenon — for which one lacks a purely gauge theoretical method[3].

[3] Lattice QCD methods do not work well here as this would require analytical continuation to Minkowski signature which is nontrivial in this context.

On the other hand, there exists a wide class of gauge theories, which can be studied analytically at strong coupling. These are superconformal field theories with gravity duals. String theory methods (namely AdS/CFT correspondence) maps gauge theory dynamics (CFT) at strong coupling and large number of colors into solving Einstein equations in asymptotically anti-de Sitter space (AdS). The theories with gravity duals can differ substantially from real world QCD at zero temperature. The best known example of such theory — $\mathcal{N} = 4$ super Yang–Mills (SYM) — is a superconformal field theory with matter in the adjoint representation of the gauge group $\mathrm{SU}(N_\mathrm{c})$. Because of the conformal symmetry at the quantum level this theory does not exhibit confinement. On the other hand differences between $\mathcal{N} = 4$ SYM and QCD are less significant above QCD's critical temperature, when quarks and gluons are in the deconfined phase. Moreover, it was observed on the lattice that QCD exhibits a quasi-conformal window in the certain range of temperatures, where the equation of state is well-approximated by $\epsilon = 3p$. The above observations together with experimental results suggesting that quark–gluon plasma is a strongly coupled medium is an incentive to use the AdS/CFT correspondence as a tool to get insight into the non-perturbative dynamics.

4. AdS/CFT setup

We will now describe how to set up an AdS/CFT computation for determining the spacetime behaviour of the energy-momentum tensor [11]. This method does not make any underlying assumptions about local equilibrium or hydrodynamical behavior. We will obtain hydrodynamic expansion as a generic late time behaviour of the expanding strongly coupled plasma.

Suppose that we consider some macroscopic state of the plasma characterized by a spacetime profile of the energy-momentum tensor

$$T_{\mu\nu}(x^\rho)\,. \qquad (17)$$

Then, since the AdS/CFT correspondence asserts the exact equivalence of gauge and string theory, such a state should have its counterpart on the string side of the correspondence. Typically it will be given by a modification of the geometry of the original $\mathrm{AdS}_5\times\mathrm{S}^5$ metric. This comes from the fact that operators in gauge theory correspond to fields in supergravity (or string theory). When we consider a state with a nonzero expectation value of an operator, the *dual* gravity background will have the corresponding field modified from its "vacuum" $\mathrm{AdS}_5\times\mathrm{S}^5$ value. In the case of the energy momentum tensor the corresponding field is just the 5-dimensional metric. One then has to assume that the geometry is well defined *i.e.* it does not have a naked singularity — a singularity not hidden by an event horizon. This principle will select the allowed physical spacetime profiles of gauge theory energy-momentum tensor. Thus together with the Einstein equations this becomes the main dynamical mechanism for the strongly coupled gauge theory.

The simplest way to formulate the precise correspondence between the expectation value of the energy-momentum tensor and bulk geometry is to use the Fefferman–Graham system of coordinates [12] for the latter:

$$ds^2 = \frac{g_{\mu\nu}(x^\rho, z)dx^\mu dx^\nu + dz^2}{z^2}\,. \tag{18}$$

This metric has to be a solution of 5-dimensional Einstein's equation with negative cosmological constant[4]:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - 6\, g_{\mu\nu} = 0\,. \tag{19}$$

The expectation value of the energy momentum tensor may be easily recovered by expanding the metric near the boundary $z = 0$, following the "holographic renormalization" procedure [13],

$$g_{\mu\nu}(x^\rho, z) = \eta_{\mu\nu} + z^4 g^{(4)}_{\mu\nu}(x^\rho) + \dots\,. \tag{20}$$

Then

$$\langle T_{\mu\nu}(x^\rho)\rangle = \frac{N_c^2}{2\pi^2}\, g^{(4)}_{\mu\nu}(x^\rho)\,. \tag{21}$$

This relation can be used in two ways. Firstly, given a solution of Einstein equations we may read off the corresponding gauge theoretical energy-momentum tensor. Secondly, given a traceless and conserved energy-momentum profile one may integrate Einstein equations into the bulk in order to obtain the dual geometry[5]. Then the criterion of nonsingularity of the geometry obtained in this way will determine the allowed spacetime evolution of the plasma. Let us note that this formulation is in fact quite far away from a conventional initial value problem.

Before we move to the case of expanding plasma, it is convenient to consider the simple situation of a static uniform plasma with a constant energy momentum tensor. Then the Einstein's equations can be solved analytically and we find [11] that the exact dual geometry of such a system is

$$ds^2 = -\frac{(1 - z^4/z_0^4)^2}{(1 + z^4/z_0^4)z^2}\, dt^2 + (1 + z^4/z_0^4)\frac{dx^2}{z^2} + \frac{dz^2}{z^2}\,. \tag{22}$$

[4] One can show that such solutions lift to 10-dimensional solutions of ten dimensional type IIB supergravity. The effective 5-dimensional negative cosmological constant comes from the 5-form field in 10-dimensional supergravity.

[5] This can be done order by order in z^2, which is a near-boundary expansion. However, potential singularities are hidden deep in the bulk, thus this power series needs to be resummed.

This metric may look at first glance unfamiliar, but a change of coordinates

$$\tilde{z} = \frac{z}{\sqrt{1 + \frac{z^4}{z_0^4}}} \tag{23}$$

transforms it to the standard AdS Schwarzschild static black hole

$$ds^2 = -\frac{1 - \tilde{z}^4/\tilde{z}_0^4}{\tilde{z}^2} dt^2 + \frac{dx^2}{\tilde{z}^2} + \frac{1}{1 - \tilde{z}^4/\tilde{z}_0^4} \frac{d\tilde{z}^2}{\tilde{z}^2} \tag{24}$$

with $\tilde{z}_0 = z_0/\sqrt{2}$ being the location of the horizon. Before we proceed further, let us note here one crucial thing: the fact, that the dual geometry of a gauge theory system with constant energy density is a black hole was *not* an assumption, but rather an outcome of a computation.

The Hawking temperature

$$T = \frac{1}{\pi \tilde{z}_0} \equiv \frac{\sqrt{2}}{\pi z_0} \tag{25}$$

is then identified with the gauge theory temperature, and the entropy with the Bekenstein–Hawking black hole entropy

$$S = \frac{N_c^2}{2\pi \tilde{z}_0^3} = \frac{\pi^2}{2} N_c^2 T^3 \tag{26}$$

which is 3/4 of the entropy at zero coupling. To finish our discussion of the static black hole, we note that the Fefferman–Graham coordinates cover only the part of spacetime lying outside the horizon.

5. Boost invariant flow

Let us now apply the above procedure to a generic boost-invariant flow, in view of making contact with the hydrodynamical Bjorken flow described in Sec. 2. However, we do not want to make any preassumptions on the dynamics, since we would like to recover the hydrodynamic behaviour as an outcome of an AdS/CFT computation. To this end let us consider the most general gauge theory energy-momentum tensor which is boost-invariant and does not depend on transverse coordinates (see Fig. 1). Then conservation of energy-momentum $\partial_\mu T^{\mu\nu} = 0$ and tracelessness $T^\mu_\mu = 0$ allow to express all nonvanishing components of $T_{\mu\nu}$ in terms of a single function $\varepsilon(\tau)$ — the energy density at mid rapidity:

$$T_{\mu\nu} = \begin{pmatrix} \varepsilon(\tau) & 0 & 0 & 0 \\ 0 & -\tau^3 \frac{d}{d\tau}\varepsilon(\tau) - \tau^2 \varepsilon(\tau) & 0 & 0 \\ 0 & 0 & \varepsilon(\tau) + \frac{1}{2}\tau \frac{d}{d\tau}\varepsilon(\tau) & 0 \\ 0 & 0 & 0 & \varepsilon(\tau) + \frac{1}{2}\tau \frac{d}{d\tau}\varepsilon(\tau) \end{pmatrix} . \tag{27}$$

Let us concentrate, following [11] on the late time asymptotics of this function *i.e.*

$$\varepsilon(\tau) \sim \frac{1}{\tau^s} + \ldots \tag{28}$$

for $\tau \to \infty$. Energy positivity requires that $0 \leq s \leq 4$. We will consider sharp inequalities here[6]. The most general metric consistent with the symmetry assumptions is

$$ds^2 = \frac{-e^{a(\tau,z)}d\tau^2 + \tau^2 e^{b(\tau,z)}dy^2 + e^{c(\tau,z)}dx_\perp^2}{z^2} + \frac{dz^2}{z^2}\,. \tag{29}$$

In order to find the late time form of the solution corresponding to $\varepsilon(\tau) = 1/\tau^s$ we may solve the Einstein equations in a power series for the metric coefficients

$$a(\tau,z) = \sum_{n=0}^{N} a_n(\tau) z^{4+2n}\,, \tag{30}$$

where $a_0(\tau) = -\varepsilon(\tau) = -1/\tau^s$. Then from each coefficient $a_n(\tau)$ we may extract the leading large τ behaviour and neglect the subleading terms. It turns out that this procedure is exactly equivalent to introducing a scaling variable

$$v = \frac{z}{\tau^{s/4}}\,, \tag{31}$$

and assuming the metric coefficients to be just functions of v *e.g.* $a(z,\tau) = a(v)$ in the large proper time limit (namely $\tau \to \infty$, $z \to \infty$ with v kept fixed). In this limit Einstein's equations become just ordinary differential equations and may be solved analytically. The singularity of these geometries can then be tested by computing the scalar curvature invariant

$$\mathfrak{R}^2 = R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}\,. \tag{32}$$

Since our solutions are defined only in the large proper time limit $\tau \to \infty$ with the scaling variable v kept fixed, we have to evaluate $\mathfrak{R}^2$ in the same manner[7].

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[6] Recently the case $s = 4$ has been considered in [14].

[7] It should be stressed, however, that this condition is really a condition of regularity of the expansion of the curvature invariant. It is safe to do as long as each term in the large proper-time expansion is regular. On the other hand any singularity present in this expansion might be either a genuine curvature singularity or a singularity of the expansion, see a detailed discussion in Sec. 8.

This procedure is described in detail in [11]. The result is that

- for generic s the resulting solution is singular,
- the only nonsingular solution corresponds to $s = 4/3$ which is just the hydrodynamic Bjorken expansion (see 15, Sec. 2),
- the resulting metric takes the form

$$ds^2 = \frac{1}{z^2}\left[-\frac{\left(1-\frac{e_0}{3}\frac{z^4}{\tau^{4/3}}\right)^2}{1+\frac{e_0}{3}\frac{z^4}{\tau^{4/3}}}d\tau^2 + \left(1+\frac{e_0}{3}\frac{z^4}{\tau^{4/3}}\right)\left(\tau^2 dy^2 + dx_\perp^2\right)\right] + \frac{dz^2}{z^2}\,, \quad (33)$$

where we reinstated the dimensionful parameter e_0 so that

$$\varepsilon(\tau) = e_0/\tau^{4/3}\,. \quad (34)$$

Let us note some salient features of this result. The geometry (33) bears striking resemblance to the AdS black hole geometry (22) but with the position of the "effective horizon" being time dependent

$$z_0 = \sqrt[4]{\frac{3}{e_0}}\,\tau^{1/3}\,. \quad (35)$$

Then assuming similar relations as for the black hole case one gets the Bjorken scaling of the temperature and entropy

$$\begin{aligned} T &= \frac{\sqrt{2}}{\pi z_0} = \frac{2^{1/2}e_0^{1/4}}{\pi 3^{1/4}}\tau^{-1/3}\,, \\ S &\propto \frac{\tau}{z_0^3} = \text{const}\,. \end{aligned} \quad (36)$$

We see that the "movement" of the horizon into the bulk of AdS corresponds physically to cooling of the expanding gauge theory plasma system.

A significant fact that has to be kept in mind is that the geometry (33), in contrast to (22), is not an exact solution of Einstein's equation. It is valid only for large times. For smaller times it has to be modified. We will now discuss this issue in more detail as it reflects important physical properties of the gauge theory plasma.

6. Beyond perfect fluid

The geometry (33) is only a solution of Einstein equations in the scaling limit. However, with some effort, one can get also the first subleading corrections to the metric *i.e.*

$$a(z,\tau) = a_0(v) + \frac{1}{\tau^{4/3}}\, a_2(v) + \dots \,. \tag{37}$$

Then after evaluating $\mathfrak{R}^2$, keeping track of subleading terms we find

$$\mathfrak{R}^2 = R_0(v) + \frac{1}{\tau^{4/3}} R_2(v) + \dots\,, \tag{38}$$

where $R_0(v)$ is finite, but $R_2(v)$ develops a 4th order pole singularity. The physical meaning of this behaviour is indeed quite clear. The geometry (37) is dual to a state in gauge theory which undergoes expansion according to exact *perfect fluid* hydrodynamics. Yet we know that gauge theory plasma has nonzero viscosity and hence the perfect fluid behaviour

$$\varepsilon(\tau) = \frac{1}{\tau^{4/3}} \tag{39}$$

is not exact but, if it would be described by viscous Bjorken expansion (viscous hydrodynamics), it would be modified to

$$\varepsilon(\tau) = \frac{1}{\tau^{4/3}} \left(1 - \frac{2\eta_0}{\tau^{2/3}} + \dots\right) \tag{40}$$

where η_0 is related to the shear viscosity through $\eta = \eta_0/\tau$ (which follows from the scaling $\eta \propto T^3$).

Let us show how this arises using the AdS/CFT methods. We will not presuppose a specific form of subleading correction but will start from

$$\varepsilon(\tau) = \frac{1}{\tau^{4/3}} \left(1 - \frac{2\eta_0}{\tau^{r}} + \dots\right) \tag{41}$$

with a generic r. In order to verify that plasma expansion follows viscous hydrodynamics we will have to first show that $r = 2/3$. The metric coefficients will now have an additional piece scaling as $\frac{1}{\tau^r} a_r(v)$. It turns out that the curvature scalar $\mathfrak{R}^2$ is always *nonsingular* at that order[8]. Hence we have to go one order further *i.e.* find all coefficients appearing in the following expansion

$$a(z,\tau) = a_0(v) + \frac{1}{\tau^r} a_r(v) + \frac{1}{\tau^{2r}} a_{2r}(v) + \frac{1}{\tau^{4/3}} a_2(v) + \dots\,, \tag{42}$$

[8] This was first observed for $r = 2/3$ in [15].

Then the curvature scalar has the form

$$\mathfrak{R}^2 = R_0(v) + \frac{1}{\tau^r} R_r(v) + \frac{1}{\tau^{2r}} R_{2r}(v) + \frac{1}{\tau^{4/3}} R_2(v) + \ldots \tag{43}$$

with $R_0(v)$ and $R_r(v)$ being nonsingular, while *both* $R_{2r}(v)$ and $R_2(v)$ turn out to have 4th order pole singularities. In order for them to have a chance to cancel we have to have

$$r = \frac{2}{3} \tag{44}$$

which is exactly the scaling of a viscosity correction to Bjorken flow. Moreover, cancellation occurs only when the shear viscosity coefficient has the value[9]

$$\eta_0 = 2^{-1/2} 3^{-3/4} \tag{45}$$

which is equivalent to $\eta/s = 1/4\pi$ (for details see [16]). In a similar manner one can go one order higher and determine a coefficient of second order hydrodynamics. However at that order, it turns out that there remains a leftover logarithmic singularity. We will show, in Sec. 8, that the logarithmic singularity arises due to a pathology of the Fefferman–Graham expansion and can be avoided when one makes a different late time expansion.

Finally let us comment on why it is interesting to verify the viscous hydrodynamic behaviour with the specific viscosity coefficient for the expanding plasma. Already before, there have been studies of *linearized* perturbations around the uniform plasma which demonstrated that hydrodynamic behaviour appears for small fluctuations and the value of viscosity was obtained from the Kubo formula [8]. It was interesting to verify whether hydrodynamics also applies in its fully nonlinear regime. The agreement of the resulting value of the viscosity coefficient is thus a nontrivial consistency check.

Another motivation for developing an AdS/CFT framework for studying such time-dependent phenomena is the fact that some of the most interesting and puzzling phenomena in heavy ion collisions are definitely very far from equilibrium. We will mention some examples in Sec. 9.

7. Beyond boost-invariance

The calculations presented in the previous sections were performed for systems with boost invariance symmetry and full translational and rotational symmetry in the transverse plane. This allowed us to perform explicit computations as the symmetry assumptions effectively reduced the calculation to systems of ordinary differential equations. In this manner we obtained

[9] We set here $e_0 = 1$.

directly the solution for gauge theory energy density $\varepsilon(\tau)$. Then, in order to find the link with hydrodynamics, we found that this solution is a solution of hydrodynamic equations with specific values for the transport coefficients.

This approach has both an advantage and a drawback. The advantage is that one does not presuppose any kind of dynamics and one may try to apply it in contexts very far from equilibrium, where hydrodynamic description does not apply. The drawback is that the appearance of hydrodynamic equations is not transparent and it is difficult to relax the symmetry assumptions due to the complexity of solving nonlinear Einstein's equations.

Recently the latter drawback was addressed and it was shown in general how the equations of hydrodynamics arise from the gravity side [17]. Here we will briefly review this approach.

Let us start from the static black hole (22) and (24) but written in yet another coordinate system — the incoming Eddington–Finkelstein coordinates:

$$ds^2 = -2dtdr - r^2\left(1 - \frac{T^4}{\pi^4 r^4}\right)dt^2 + r^2\eta_{ij}dx^i dx^j\,. \tag{46}$$

Here T is the temperature, $r = \infty$ corresponds to the boundary. $x^\mu = \mathrm{const}$ are null curves going from the boundary into the black hole. The advantage of this coordinate system is that it is well defined on the horizon and extends all the way from the boundary to the singularity at the center of the black hole.

The geometry given above corresponds to a uniform plasma at rest (*i.e.* with the 4-velocity $u^\mu = (1,0,0,0)$) and given temperature T. We may now perform a boost (and perform a dilatation) to obtain the dual geometry to a moving plasma system with uniform 4-velocity u^μ and temperature T:

$$ds^2 = -2u_\mu dx^\mu dr - r^2\left(1 - \frac{T^4}{\pi^4 r^4}\right)u_\mu u_\nu dx^\mu dx^\nu + r^2(\eta_{\mu\nu} + u_\mu u_\nu)dx^\mu dx^\nu\,. \tag{47}$$

The idea of Ref. [17] is to allow u^μ and T to be (slowly-varying) functions of the spacetime coordinates. Once this is done the geometry (47) ceases to be an exact solution of Einstein equation because of nonvanishing gradients of the parameters u^μ and T. This suggests to perform an expansion of the solution in terms of gradients which has been carried out in [17] up to second order in derivatives. The integration constants arising at each order are again fixed by requiring regularity of the metric at the horizon. The resulting metric is expressed in terms of 4-velocities and temperatures and their derivatives, so when one extracts the energy-momentum tensor it will be given directly in terms of those quantities. Up to first order the expression is

$$T^{\mu\nu} = \frac{N_c^2}{8\pi^2}\left\{(\pi T)^4(\eta^{\mu\nu} + 4u^\mu u^\nu) - 2(\pi T)^3\sigma^{\mu\nu}_{\mathrm{shear}}\right\}\,. \tag{48}$$

The first term is just the perfect fluid energy momentum tensor, while the second term involves the shear viscosity. This result essentially demonstrates how general hydrodynamic equations arise from gravity in AdS/CFT. Indeed, once it is shown that the general form of the gauge theory energy-momentum tensor has the form (48), then conservation of energy momentum $\partial_\mu T^{\mu\nu} = 0$ is equivalent, by definition, to conformal relativistic Navier–Stokes equations. As a byproduct, the above construction also gives a map from solutions of (viscous) hydrodynamics to gravity solutions. However, this setup requires that the starting point is not far off from equilibrium. For processes which do not admit a hydrodynamic description (like the early stage of a heavy-ion collision) one has to resort to different methods.

8. Reduction of singularities

The leftover logarithmic singularity found in the third order of the square of the Riemann tensor [18] (as well as in the higher curvature invariants [19]) might be the signal of either genuine curvature singularity or the singularity of the chosen expansion scheme[10]. If the first is true, this means that the whole framework is inconsistent and either one needs to include additional degrees of freedom to cure it or the boost-invariant flow is unphysical[11]. The results presented in [19] show that no supergravity field can fix the problem, which led to conjectures, that boost-invariant flow cannot be realized within the supergravity framework [20]. On the other hand, the gravity dual of general fluid flow up to the second order in derivatives was shown to be regular and it was hard to imagine how possible singularities could form in the third order [17, 21]. The resolution of this puzzle was presented in [22] (see also [23]). It turns out that there exists a singular coordinate transformation from Fefferman–Graham coordinates to Eddington–Finkelstein ones, which yields a completely regular and smooth metric from the boundary up to the standard black-brane singularity. This leads to regular (apart from the standard black-brane singularity) large proper-time expansion of curvature invariants. The metric Ansatz in Eddington–Finkelstein coordinates takes the form

$$ds^2 = 2d\tilde{\tau}\, dr - \tilde{A}\,(\tilde{\tau}, r)\, d\tilde{\tau}^2 + (1 + r\,\tilde{\tau})^2\, e^{\tilde{b}(\tilde{\tau},r)} dy^2 + e^{\tilde{c}(\tilde{\tau},r)} dx_\perp^2 \tag{49}$$

and was motivated by the boosted black-brane metric (47) with a boost and dilatation parameters $u = 1\,\partial_{\tilde{\tau}}$ and $T \sim \tilde{\tau}^{-4/3}$. The functions $\tilde{A}\,(\tilde{\tau}, r)$,

[10] As it was stressed before, the large-proper time expansion of curvature invariants is not diffeomorphism-invariant. The encountered singularities are physical only if there is no coordinate transformation which removes them.

[11] Since it corresponds to the naked singularity on the gravity side.

$\tilde{b}(\tilde{\tau},r)$ and $\tilde{c}(\tilde{\tau},r)$ are expanded in a large-proper time expansion analogously as it was in the Fefferman–Graham case, *i.e.*

$$\tilde{A}(\tilde{\tau},r) = \tilde{A}_0\left(r\,\tilde{\tau}^{1/3}\right) + \frac{1}{\tilde{\tau}^{2/3}}\tilde{A}_1\left(r\,\tilde{\tau}^{1/3}\right) + \frac{1}{\tilde{\tau}^{4/3}}\tilde{A}_2\left(r\,\tilde{\tau}^{1/3}\right) + \dots \,. \tag{50}$$

This form of expansion can also be justified by [24]. The terms damped by inverse power of proper time come from the gradient expansion. The boundary metric in proper-time-rapidity coordinates has non-vanishing Christoffel symbols $\Gamma \sim \tilde{\tau}^{-1}$, thus the four velocity gradient ∇u (which is constant in these coordinates) gives a factor of $\tilde{\tau}^{-1}$. On the other hand the expansion parameter multiplying each term in gradient expansion is the inverse power of the temperature T(see [17]). Because $T \sim \tilde{\tau}^{-1/3}$, the overall damping is indeed $\tilde{\tau}^{-2/3}$ — a fact derived in [18] from the non-singularity argument.

The non-perturbative[12] piece in the metric at dy^2 introduced in [23] is responsible for a correct limit energy density $\to 0$. It also becomes important if one wants to solve the problem of early-time dynamics [27] using Eddington–Finkelstein coordinates.

The integration constants[13] are fixed by requiring the regularity of the metric functions $\tilde{A}_i(\tilde{v})$, $\tilde{b}_i(\tilde{v})$ and $\tilde{c}_i(\tilde{v})$ at each order i. This is justified since the Eddington–Finkelstein are valid for $\tilde{\tau} > 0$ and $0 < \tilde{v} = r\,\tilde{\tau} < \infty$. The singular coordinate transformation which takes the metric from Eddington–Finkelstein coordinates to Fefferman–Graham ones is given order by order in the gradient expansion by

$$\tilde{\tau}(\tau,z) = \tau\left\{T_0\left(z\,\tau^{-1/3}\right) + \frac{1}{\tau^{2/3}}T_1\left(z\,\tau^{-1/3}\right) + \dots\right\}, \tag{51}$$

$$r(\tau,z) = \frac{1}{z}\left\{R_0\left(z\,\tau^{-1/3}\right) + \frac{1}{\tau^{2/3}}R_1\left(z\,\tau^{-1/3}\right) + \dots\right\}. \tag{52}$$

The leading-order solutions (corresponding to the perfect fluid on the gauge theory side) are related by

$$\begin{aligned}\tilde{\tau} &\to \tau\left\{1 - \frac{1}{\tau^{2/3}}\left[\frac{3^{1/4}\pi}{4\sqrt{2}} + \frac{3^{1/4}}{2\sqrt{2}}\tan^{-1}\left(\frac{3^{1/4}}{\sqrt{2}}r\,\tau^{1/3}\right)\right.\right.\\ &\quad\left.\left. + \frac{3^{1/4}}{4\sqrt{2}}\log\left(\frac{r\,\tau^{1/3} - \frac{\sqrt{2}}{3^{1/4}}}{r\,\tau^{1/3} + \frac{\sqrt{2}}{3^{1/4}}}\right)\right]\right\},\\ r &\to \frac{1}{z}\sqrt{1 + \frac{z^4}{3\,\tau^{4/3}}}\,.\end{aligned} \tag{53}$$

[12] In the sense of large-proper time expansion.

[13] Not all of them — there is a remaining gauge freedom (coordinate transformation), which leaves the metric Ansatz unchanged: $r \to r + f(\tilde{\tau})$, where $f(\tilde{\tau})$ is an arbitrary function.

The transformation is singular at $z = 3^{1/4}\tau^{1/3}$, which is precisely the locus of the logarithmic singularity encountered in [18]. Formulas for higher order transformation coefficients are too long to be presented here and can be found in [28]. The energy-momentum tensor extracted from the solution in Eddington–Finkelstein coordinates reproduces the energy momentum tensor obtained in [18].

9. Beyond hydrodynamics

Gauge-gravity duality has already proven to be an invaluable tool in describing properties of static or near-equilibrium (hydrodynamics) strongly coupled gauge theory systems. Noticeable achievements in that direction are the viscosity evaluation bound [8] and the consistent formulation of the second order conformal hydrodynamics [17, 29]. These successes came from the holographic understanding of hydrodynamics. On the other hand there is much more interesting and nontrivial dynamics than hydro. Far from equilibrium behavior of gauge theories is a fascinating and pretty much open problem of experimental importance, like the early universe or initial stages of heavy ion collisions[14]. The AdS/CFT correspondence is surely capable to shed new light on these problems, or even be understood as a formulation of far from equilibrium gauge theory.

In the context of heavy-ion collisions the most important and probably the most difficult questions concern the issues of early time dynamics [27] and the transition to an isotropic [34] and thermalized medium. One of the puzzles here is the short time in which nuclear matter approach local equilibrium. Experimental data fitting well with hydrodynamical simulations with small viscosity justified applications of the AdS/CFT correspondence at strong coupling for the late stages of heavy-ion collisions. It is not clear to what extent early time dynamics is driven by non-perturbative effects and whether the lessons learned from AdS/CFT might be directly applied to the nuclear matter in the early stages of the evolution. Approaching equilibrium is also of an interest from the General Relativity point of view. Isotropic and thermalized matter on the gauge theory side corresponds to a black hole in AdS, whereas thermalization and approach to local equilibrium should be governed by the dynamics of gravitational collapse.

Perhaps some of these questions might be answered by studying collisions of shock-waves in AdS. The geometry corresponding to a projectile in $3 + 1$

[14] In the late stages of heavy ion collision, strongly coupled quark–gluon plasma forms and holographic technics at strong gauge coupling are better justified then just after the collision (running of the coupling). Nevertheless, applying AdS/CFT correspondence to describe far from equilibrium processes in gauge theories is an interesting problem even from a purely theoretical point of view.

dimensions was constructed in [11] using holographic renormalization. The metric

$$ds^2 = \frac{1}{z^2}\left\{-2dx^+dx^- + f\left(x^-\right)\, z^4 \left(dx^-\right)^2 + dx_\perp^2 + dz^2\right\} \tag{54}$$

with an arbitrary function $f\left(x^-\right)$ corresponds to the situation when

- the dynamics is one-dimensional (*i.e.* no dependence on transverse coordinates),
- the energy-momentum tensor depends only on a single light-cone variable (here chosen to be $x^- = x^0 - x^1$).

Traceless and conserved energy momentum tensor satisfying the above assumptions takes a particularly simple form — its only non-zero component is $T^{--} = f\left(x^-\right)$. Choosing $f\left(x^-\right) = M\delta\left(x^-\right)$ leads to a shock-wave — infinitely thin plane of matter moving at the speed of light, which is a toy-model for highly boosted nucleus. The idea is to collide two such projectiles and single out the physical behavior of the plasma from the regularity of the dual geometry. This is a difficult problem, because of the broken boost-invariance, which leads to solving Einstein equations in 3 variables (x^+, x^- and z or equivalently τ, y and z). The geometry before the collision ($x^+ + x^- < 0$) is known — it is simply the superposition of two incoming shock-waves

$$ds^2=\frac{1}{z^2}\left\{-2dx^+dx^-+M\delta\left(x^-\right)z^4\left(dx^-\right)^2+M\delta\left(x^+\right)z^4\left(dx^+\right)^2+dx_\perp^2+dz^2\right\}\,. \tag{55}$$

Shock-waves collide at $x^+ = x^- = 0$ and from now on the dynamics of the system must be deduced from Einstein equations. The first attempt to address this issue in [31] focused on the simpler setup than presented so far — a shock-waves collision in 1+1 dimensions. The energy-momentum tensor for such a system before the collision is given by $T_{++} = f\left(x^-\right)$ and $T_{--} = g\left(x^+\right)$ with vanishing off-diagonal components. This is at the same time the most general form of the energy-momentum tensor for a $1+1$ dimensional CFT. A nice feature is that the dual geometry for the *whole* collision process can be constructed here exactly. However, in this low dimensional context, the projectiles pass each other unaffected (or propelled back-to-back [31]), so the physics of plasma production and thermalization is absent here. On the other hand the problem of genuine interest — collision of shock-waves in $3+1$ dimensions — requires some approximation scheme in which Einstein equations become tractable. Up to now, two proposals have been made. The first one [32] treats proper time as a small parameter but suffers from a negative energy density in some regions due to the conditions imposed at the light-cone. The second one [33] solves Einstein equations perturbatively in M leading to the prediction that shock-waves stop almost immediately after the collision (reminding of the *full stopping* condition of the Landau flow, see Sec. 2.). This problems surely deserve further studies.

There are also other studies of dynamical processes in an evolving plasma system which go beyond hydrodynamics. One example is the problem of thermalization of small perturbations around the expanding plasma (some first investigations has been performed in [11]). Another use of the evolving geometries is to study other physical processes in the presence of the evolving plasma system like *e.g.* the physics of mesons and flavours studied through embedding D7 branes in the time-dependent geometries [35]. Finally one may study isotropization of anisotropic plasma. The first investigations have been performed in [34], see also [36].

10. Summary

The Gauge/Gravity approach to the formation and evolution of a quark–gluon plasma in heavy-ion collisions described above has the interest of casting an exploratory bridge between the rigorous results of string theory and some pending questions raised by the experiments on quark–gluon plasma. These questions cannot yet be raised in the framework of strongly coupled QCD, for which we do not possess the adequate tools, but they can be addressed for the first time in a quantitative and rigorous way in the supersymmetric case of the AdS/CFT correspondence. It is thus a novel and valuable approach and can serve as a model for further studies.

Let us summarize some aspects of this approach, being aware (and with apologies for those not quoted or mentioned) that this subject is in constant development which will force us to mention only a few of them.

Starting with the experimental evidence that hydrodynamics is relevant in the formation and evolution of a quark–gluon plasma in heavy-ion collisions and in particular of the "Bjorken flow" description, we present the AdS/CFT setup allowing to describe the dynamics of the plasma (in the AdS/CFT case). We show that it is possible to derive the geometry dual to the asymptotic evolution of the plasma in terms of an expansion in a scaling variable. the nonsingularity requirement on the Gravity side gives a set of "selection rules" on the Gauge side: the perfect fluid at first order, the $\eta/s = 1/4\pi$ property and other transport coefficients at higher orders ...

Beyond the boost-invariant Bjorken flow, there exists an intriguing but rigorous one-to-one correspondence between the complete (and even completed using AdS/CFT!) hydrodynamic equations and the solutions of the Einstein equations in the bulk of the 5-dimensional space. Some apparent obstacles, such as the appearance of logarithmic singularities in the asymptotic expansion of the 5-dimensional metric, have been shown to be a mere artefact of the choice of the expansion parameter and have been cured.

Beyond the hydrodynamical description of the transient plasma, one goal is now to explore the dynamical aspects far from equilibrium. We describe some very recent attempts, which even though not conclusive yet, show the interest of the extension of Gauge/Gravity correspondence to attack some down-to-earth problems, such as the short thermalization time, probably observed by the heavy-ion phenomenology and more generally the effect of the initial conditions on the whole process.

It is quite interesting to see that some complex aspects of Gauge field theory dynamics can find unexpected answers from Gravity. It would be intriguing that some nontrivial aspects of Gravity (such as the dynamics of moving black holes) also could gain some new insight from the correspondence with some aspects of heavy-ion collisions.

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REFERENCES

[1] E.V. Shuryak, *Nucl. Phys.* **A750**, 64 (2005) [arXiv: hep-ph/0405066].

[2] P. Huovinen, P.V. Ruuskanen, arXiv: nucl-th/0605008; P.F. Kolb, U.W. Heinz, nucl-th/0305084.

[3] J.Y. Ollitrault, *Phys. Rev.* **D46**, 229 (1992); See also the recent reviews: J.Y. Ollitrault, *Eur. J. Phys.* **29**, 275 (2008); T. Hirano, N. van der Kolk, A. Bilandzic, arXiv:0808.2684 [nucl-th].

[4] J.D. Bjorken, *Phys. Rev.* **D27**, 140 (1983).

[5] L.D. Landau, *Izv. Akad. Nauk Ser. Fiz.* **17**, 51 (1953) (in Russian) [English translation, *Collected Papers of L.D. Landau*, ed. D. ter Haar, Gordon and Breach, New York 1968].

[6] A. Bialas, R.A. Janik, R.B. Peschanski, *Phys. Rev.* **C76**, 054901 (2007).

[7] J.M. Maldacena, *Adv. Theor. Math. Phys.* **2**, 231 (1998) [*Int. J. Theor. Phys.* **38**, 1113 (1999)]; S.S. Gubser, I.R. Klebanov, A.M. Polyakov, *Phys. Lett.* **B428**, 105 (1998); E. Witten, *Adv. Theor. Math. Phys.* **2**, 253 (1998).

[8] G. Policastro, D.T. Son, A.O. Starinets, *Phys. Rev. Lett.* **87**, 081601 (2001); P. Kovtun, D.T. Son, A.O. Starinets, *Phys. Rev. Lett.* **94**, 111601 (2005).

[9] J.D. Bjorken, *Lect. Notes Phys.* **56**, 93 (1976); Lectures given at the Int. Summer Inst. in Theoretical Physics, Current Induced Reactions, Hamburg, Germany, Sept. 15–26, 1975; Also in Hamburg Theor. Inst. 1975:93 (QCD161:I83:1975); K. Gottfried, *Phys. Rev. Lett.* **32**, 957 (1974); F.E. Low, K. Gottfried, *Phys. Rev.* **D17**, 2487 (1978).

[10] G. Beuf, R. Peschanski, E.N. Saridakis, `arXiv:0808.1073 [nucl-th]`.

[11] R.A. Janik, R.B. Peschanski, *Phys. Rev.* **D73**, 045013 (2006); *Phys. Rev.* **D74**, 046007 (2006).

[12] C. Fefferman, C.R. Graham, *Conformal Invariants* in *Elie Cartan et les Mathématiques d'aujourd'hui*, Astérisque, Société Mathématique de France, Paris 1985, p. 95.

[13] S. de Haro, S.N. Solodukhin, K. Skenderis, *Commun. Math. Phys.* **217**, 595 (2001) `[hep-th/0002230]`; K. Skenderis, *Class. Quant. Grav.* **19**, 5849 (2002) `[hep-th/0209067]`.

[14] K. Kajantie, J. Louko, T. Tahkokallio, `arXiv:0809.5033 [hep-th]`.

[15] S. Nakamura, S.J. Sin, *J. High Energy Phys.* **0609**, 020 (2006).

[16] R.A. Janik, *Phys. Rev. Lett.* **98**, 022302 (2007).

[17] S. Bhattacharyya, V.E. Hubeny, S. Minwalla, M. Rangamani, *J. High Energy Phys.* **0802**, 045 (2008).

[18] M.P. Heller, R.A. Janik, *Phys. Rev.* **D76**, 025027 (2007).

[19] P. Benincasa, A. Buchel, M.P. Heller, R.A. Janik, *Phys. Rev.* **D77**, 046006 (2008).

[20] A. Buchel, `arXiv:0803.3421 [hep-th]`.

[21] S. Bhattacharyya *et al.*, *J. High Energy Phys.* **0806**, 055 (2008).

[22] M.P. Heller, P. Surowka, R. Loganayagam, M. Spalinski, S.E. Vazquez, `arXiv:0805.3774 [hep-th]`.

[23] S. Kinoshita, S. Mukohyama, S. Nakamura, K.-y. Oda, `arXiv:0807.3797 [hep-th]`.

[24] S. Bhattacharyya, R. Loganayagam, S. Minwalla, S. Nampuri, S.P. Trivedi, S.R. Wadia, `arXiv:0806.0006 [hep-th]`.

[25] A. Ashtekar, B. Krishnan, *Living Rev. Rel.* **7**, 10 (2004) `[gr-qc/0407042]`.

[26] W. Israel, J.M. Stewart, *Annals Phys.* **118**, 341 (1979).

[27] Y.V. Kovchegov, A. Taliotis, *Phys. Rev.* **C76**, 014905 (2007).

[28] A Mathematica file containing the results of [22] can be downloaded: `http://th.if.uj.edu.pl/ heller/boost.nb` and `http://th.if.uj.edu.pl/ heller/trafo.nb`

[29] R. Baier, P. Romatschke, D.T. Son, A.O. Starinets, M.A. Stephanov, *J. High Energy Phys.* **0804**, 100 (2008).

[30] P. Kovtun, D.T. Son, A.O. Starinets, *Phys. Rev. Lett.* **94**, 111601 (2005).

[31] K. Kajantie, J. Louko, T. Tahkokallio, *Phys. Rev.* **D77**, 066001 (2008).

[32] D. Grumiller, P. Romatschke, *J. High Energy Phys.* **0808**, 027 (2008).

[33] J.L. Albacete, Y.V. Kovchegov, A. Taliotis, *J. High Energy Phys.* **0807**, 100 (2008).

[34] R.A. Janik, P. Witaszczyk, *J. High Energy Phys.* **0809**, 026 (2008).

[35] J. Grosse, R.A. Janik, P. Surowka, *Phys. Rev.* **D77**, 066010 (2008).

[36] P. Witaszczyk, *Acta Phys. Pol. B* **39**, 3205 (2008), these proceedings.