ANISOTROPIC PLASMA AT STRONG COUPLING AND R-CHARGE FLUCTUATIONS*

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We briefly discuss our first attempt to describe an anisotropic quark–gluon plasma at strong coupling in the AdS/CFT correspondence framework. We constructed an exact dual gravity solution and found that despite the fact that it is singular it allows for a construction of natural incoming boundary conditions. A study of a small perturbation about this geometry shows that the dispersion relation depends strongly on the relative sign of the wave vector and the sign of the anisotropy. Yet, we did not encounter any instabilities.

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1. Introduction

What people observe during the high energy collisions in the RHIC seems to be strongly coupled quark–gluon plasma which is quite well described by hydrodynamics very soon after the collision. What is very interesting is the fact, that hydrodynamics crucially depends on the notion of isotropic energy-momentum tensor, which is definitely anisotropic just after the collision. Thus the mechanism of very fast isotropisation of the plasma remains intriguing. No complete explanation exists at the moment, but some models based on phenomena in weakly coupled plasma were proposed (see in particular [1]). It suggests that instabilities appearing in anisotropic plasma are responsible for the rapid isotropisation of the system. Since the study of real anisotropic evolving plasma is still beyond our reach, we use a simplified model of the anisotropic static plasma filling whole Minkowski space

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(detailed investigation of stability at weak coupling and real time isotropisation was done *i.e.* in [2,3]). The goal of investigating temporal evolution of the plasma is left for future investigation.

2. Motivation of plasma instabilities at weak coupling

The situation at weak coupling was investigated mostly in the case of an uniformly distributed anisotropic plasma filling the whole space. In this configuration one initially has an anisotropic distribution of momentum and thus pressure, which is described by the following energy-momentum tensor:

$$T_{\mu\nu} = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & p_{\rm L}(t) & 0 & 0 \\ 0 & 0 & p_{\rm T}(t) & 0 \\ 0 & 0 & 0 & p_{\rm T}(t) \end{pmatrix}, \tag{1}$$

where $\varepsilon = p_{\rm L} + 2p_{\rm T}$, and one is interested in dependence of the pressure (of soft modes). Those computations are mostly numeric but one can also consider simplified analytical method of computing the poles of gluon propagator in the anisotropic medium. Then, one studies how does longitudinal and transverse modes behave. It appears that this in turn depend crucially on the sign of the anisotropy defined as

$$\xi = \frac{p_{\rm T}}{p_{\rm L}} - 1\,. \tag{2}$$

For positive anisotropy, the longitudinal modes become unstable while transverse remain stable. In the opposite case the situation is reversed.

Those results can be thought of as an initial situation for the mentioned earlier numerical simulations. In the first, linear period, instabilities indicate initial direction of evolution. Later with the transition to the nonlinear regime the evolution is no longer exponential.

We would like to see if a similar analysis is possible in the strong coupling case. It is interesting to see if investigating fluctuations of static anisotropic plasma can give some indication of the direction of time evolution of the plasma.

3. Some tools from the AdS/CFT framework

In the AdS/CFT [4] framework we have various dualities between fields in the bulk and operators on the boundary. One of the most important is the duality between the metric of the anti de Sitter space-time and the expectation value of the energy-momentum tensor on the 4-dimensional boundary, where the $N = 4$ SYM fields live. We assume that the only nonvanishing expectation value on the boundary is precisely that of energy-momentum tensor and we have that [5]:

$$\langle T_{\mu\nu}(x^\mu)\rangle = \frac{N_c^2}{2\pi} g^{(4)}_{\mu\nu}(x^\mu)\,. \tag{3}$$

Here, the metric element is the coefficient in the expansion in z of the full AdS metric, which is the solution to vacuum Einstein's equations with cosmological constant $\Lambda = -6$. General form of this solutions in Fefferman–Graham coordinates reads:

$$ds^2 = \frac{g_{\mu\nu}(x^\mu, z)dx^\mu dx^\nu + dz^2}{z^2}\,. \tag{4}$$

Thus, we see that having some initial energy-momentum profile, we can solve for the metric in the bulk and in turn, read off how does the energy-momentum tensor evolve in the future. If we consider static case, we can analyze how does the metric look like for some particular model of energy-momentum tensor. This on the other hand can tell us something on the boundary plasma. Imposing some conditions on the metric, such as non-singularity of the geometry [6] can help in determining what is the physical configuration of the plasma system.

4. An exact dual geometry of a static anisotropic plasma system

We search for the solution of Einstein equations corresponding to uniform, static and anisotropic energy-momentum tensor:

$$\langle T_{\mu\nu}\rangle = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & p_{\rm L} & 0 & 0 \\ 0 & 0 & p_{\rm T} & 0 \\ 0 & 0 & 0 & p_{\rm T} \end{pmatrix} \tag{5}$$

with $\varepsilon = p_{\rm L} + 2p_{\rm T}$. Now the momenta are time independent. An ansatz for the metric is:

$$ds^2 = \frac{1}{z^2}\left(-a(z)dt^2 + b(z)dx_{\rm L}^2 + c(z)dx_{\rm T}^2 + dz^2\right). \tag{6}$$

The boundary condition for the above functions are such that they have to vanish for $z = 0$ and their z-expansion must conform to the mentioned earlier relation between VEV of energy-momentum tensor and metric expansion coefficient.

The most general solution of Einstein's equations subject to those conditions is:

$$a(z) = (1+A^2z^4)^{\frac{1}{2}-\frac{1}{4}\sqrt{36-2B^2}}(1-A^2z^4)^{\frac{1}{2}+\frac{1}{4}\sqrt{36-2B^2}}\,, \tag{7}$$
$$b(z) = (1+A^2z^4)^{\frac{1}{2}-\frac{B}{3}+\frac{1}{12}\sqrt{36-2B^2}}(1-A^2z^4)^{\frac{1}{2}+\frac{B}{3}-\frac{1}{12}\sqrt{36-2B^2}}\,, \tag{8}$$
$$c(z) = (1+A^2z^4)^{\frac{1}{2}+\frac{B}{6}+\frac{1}{12}\sqrt{36-2B^2}}(1-A^2z^4)^{\frac{1}{2}-\frac{B}{6}-\frac{1}{12}\sqrt{36-2B^2}}\,, \tag{9}$$

where A and B parameters are related to the energy density and pressure:

$$\varepsilon = \tfrac{1}{2}A^2\sqrt{36-B^2}\,, \tag{10}$$
$$p_{\rm L} = \tfrac{1}{6}A^2\sqrt{36-B^2} - \frac{2}{3}A^2B\,, \tag{11}$$
$$p_{\rm T} = \tfrac{1}{6}A^2\sqrt{36-B^2} + \frac{1}{3}A^2B\,. \tag{12}$$

To link our results to the previous weak coupling discussion let us relate the B parameter to the anisotropy parameter ξ:

$$B = \frac{6\xi}{\sqrt{18\xi^2+48\xi+36}}\,. \tag{13}$$

One clearly sees that for isotropic case $B = 0$ we get the standard AdS black hole solution.

5. The singularity

Having obtained the metric we realized that it is singular. The singularity in the bulk geometry corresponding to the anisotropic plasma may indicate that such a configuration is unstable or can not exist at all. We would like to think of it as one of the earliest stages in the evolution of the full time dependent metric.

On the other hand, fluctuations of the electromagnetic field in such a background could be compared to those in the weak coupling case, where the instabilities depend on the anisotropy in a well described manner. This way or another that could indicate that there could be something interesting about the anisotropic plasma.

6. Boundary conditions

Although the space-time seems to be pathological, the level of singularity is not as bad as for instance in the case of negative mass Schwarzschild black hole [7, 8], which we took as a toy model for comparison. In our case we were able to construct an analog of incoming boundary conditions at the boundary. In order to do so, we consider scalar equation in the background of our metric, and using separation of variables

$$\Phi = \phi(z) e^{-i\omega t + ik_1 x^1 + ik_3 x^3} \tag{14}$$

and the change of variable

$$x = \frac{1}{4} \operatorname{arctanh} z^4 \tag{15}$$

we transform the equation to the standard form

$$\frac{d^2\phi}{dx^2} + \frac{8}{(e^{16x}-1)^{\frac{3}{2}}} \Big(\omega^2 e^{2(6+\sqrt{36-2B^2})x} - k_{\mathrm{L}}^2 e^{2(6+\frac{4B}{3}-\frac{1}{3}\sqrt{36-2B^2})x} - k_{\mathrm{T}}^2 e^{2(6-\frac{2B}{3}-\frac{1}{3}\sqrt{36-2B^2})x} \Big) \phi = 0 \,. \tag{16}$$

Close to the singularity, which is located at $x = \infty$ we see that the equation takes the form

$$\frac{d^2\phi}{dx^2} + 8\omega^2 e^{-2(6-\sqrt{36-2B^2})x} \phi = 0 \tag{17}$$

from which we can obtain that in isotropic case $B = 0$ we obtain standard asymptotic behavior of incoming and outgoing waves,

$$e^{-i\sqrt{8}\omega x}\,, \qquad e^{+i\sqrt{8}\omega x}\,. \tag{18}$$

Thus, for very small B we obtain a horizon, for which we can construct boundary conditions. In our case we choose the incoming ones since we expect that no information can leave the singularity. As mentioned before, for negative mass black hole no such construction would be possible.

7. R-charge fluctuation modes

We are interested in studying hydrodynamic modes of electromagnetic field in the present background. We thus turn to the smallest negative imaginary mode. For simplicity we set $A = 1$. If we consider the isotropic case, $B = 0$, be get for the dispersion relation that

$$\omega = -i\frac{k^2}{2\sqrt{2}} + \dots \,. \tag{19}$$

Our main goal now is to see how does the anisotropy modify this dispersion relation.

Thus, we are considering the equation of motion of the U(1) gauge field,

$$\partial_\alpha \left(\sqrt{-g} F^{\alpha\beta}\right) = 0 \tag{20}$$

in the background geometry (6). Due to the specific geometry which is a result of the plasma anisotropy we have one distinguished direction, and we call it y. Moreover, we perform Fourier decomposition and adopt some special momentum choice, with only one nonvanishing component. Then we obtain two sets of modes, namely longitudinal and transverse ones:

$$\mathrm{L}: \; k = (k_\mathrm{L}, 0, 0)\,, \qquad \mathrm{T}: q = (0, 0, k_\mathrm{T})\,. \tag{21}$$

Moreover, since since we are considering vector fields, which introduces further combinations of polarizations and wave vector directions, which after introducing gauge invariant field variables can be summarized as

Longitudinal modes:

$$\text{(L–L)} \qquad E_y(k_\mathrm{L}, z) = \omega A_y(k_\mathrm{L}, z) + k_\mathrm{L} A_t(k_\mathrm{L}, z),\; k_\mathrm{L}||E_y\,, \tag{22}$$

$$\text{(L–T)} \qquad E_1(k_\mathrm{L}, z) = \omega A_1(k_\mathrm{L}, z),\; E_2(k_\mathrm{L}, z) = \omega A_2(k_\mathrm{L}, z)\,. \tag{23}$$

Transverse modes:

$$\text{(T–T)} \qquad E_1(k_\mathrm{T}, z) = \omega A_1(k_\mathrm{T}, z) + k_\mathrm{T} A_t(k, z)\,,\; k_\mathrm{T}||E_1\,, \tag{24}$$

$$\text{(T–L)} \qquad E_y(k_\mathrm{T}, z) = \omega A_y(k_\mathrm{T}, z),\; E_2(k_\mathrm{T}, z) = \omega A_2(k_\mathrm{T}, z)\,. \tag{25}$$

From now on, we focus on the analysis of small anisotropy $B \sim 0$ which simplifies the resulting equations. We can now try to impose our incoming boundary conditions mentioned earlier by expanding around the boundary, $z = 1$. We find that it is possible and get:

$$E_y \sim (1-z)^{-i\frac{\omega}{2\sqrt{2}} + \frac{B}{6}}\,, \qquad E_1 \sim (1-z)^{-i\frac{\omega}{2\sqrt{2}} - \frac{B}{12}}\,. \tag{26}$$

If we focus on the E_y mode, we can solve the corresponding equation by imposing a decomposition onto boundary term (coming from asymptotic analysis and boundary condition) and the bulk (we are interested in small frequencies and wave vectors so we rescale $\omega \to \varepsilon\omega$ and $k_\mathrm{L} \to \varepsilon k_\mathrm{L}$ and we set $u = z^2$):

$$E_y(z) = (1-z)^{-i\frac{\omega}{2\sqrt{2}} + \frac{B}{6}} g(u) \tag{27}$$

and

$$g(u) = 1 + \varepsilon g_0^a(u) + \varepsilon^2 g_0^b(u) + B(g_1^a(u) + \varepsilon g_1^b(u) + \ldots) + \ldots\,. \tag{28}$$

This equation can be solved perturbatively and we find the following dispersion relation:

$$\omega = -i\frac{k_{\rm L}^2 + \sqrt{k_{\rm L}^4 - \frac{16}{3}A^{\frac{1}{4}}Bk_{\rm L}^2}}{4\sqrt{2}A^{\frac{1}{8}}}\,. \tag{29}$$

By expanding with respect to momentum, we can find that for $B > 0$ this longitudinal mode is in linear regime,

$$\omega = \sqrt{\frac{B}{6}}k_{\rm L} + \ldots \tag{30}$$

while for $B < 0$ E_y is strongly damped:

$$\omega = -i\sqrt{\frac{-B}{6}}k_{\rm L} + \ldots\,. \tag{31}$$

This can be compared to the situation in the weak coupling where the longitudinal modes for positive anisotropy were unstable, while for negative one they remained stable.

For transverse modes the situation is somehow reversed, the dispersion reads

$$\omega = -i\frac{k_{\rm T}^2 + \sqrt{k_{\rm T}^4 + \frac{8}{3}BA^{\frac{1}{4}}k_{\rm T}^2}}{4\sqrt{2}A^{\frac{1}{8}}} \tag{32}$$

and now we see that the dependence on the sign of the anisotropy is opposite to the longitudinal case.

8. Summary

We managed to find an exact geometry dual to the anisotropic plasma system. The geometry appeared to be singular, yet it is possible to have some notion of physical boundary conditions on the singularity. Moreover, we analyzed fluctuations of the gauge field in this singular background and hoped to find some instabilities which could suggest that the plasma is unstable, but we did not found any, at least to this order of perturbation. We only found a change in the behavior of modes depending on the sign of anisotropy. Similar result was obtained with the aid of numerical simulations with large collision rate (corresponding to strong coupling) in [9]. This work can be though as a step towards the description of evolving plasma in which case one has to consider time dependent anisotropic energy-momentum tensor and thus time dependent metric. We plan to do so in the future [10].

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