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We review a recently discovered continuum limit for the one-matrix model which describes “causal” two-dimensional quantum gravity. The behaviour of the quantum geometry in this limit is different from the quantum geometry of Euclidean two-dimensional quantum gravity defined by taking the “standard” continuum limit of the one-matrix model. Geodesic distance and time scale with canonical dimensions in this new limit, contrary to the situation in Euclidean two-dimensional quantum gravity. Remarkably, whenever we compare, the known results of (generalized) causal dynamical triangulations are reproduced exactly by the one-matrix model. We complement previous results by giving a geometrical interpretation of the new model in terms of a generalization of the loop equation of Euclidean dynamical triangulations. In addition, we discuss the time evolution of the quantum geometry.

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1. Introduction

Two dimensional quantum gravity is an interesting playground for quantum geometry. General ideas for string theory and quantum gravity can be tested by exactly solvable models. Particularly for string theory the 2d gravity point of view has been rather fruitful. Following the seminal work of Polyakov *et al.* [1] powerful conformal field theory methods were developed. They allowed an exact solution for a select class of observables of 2d gravity (see *e.g.* [2]), including the coupling of simple matter models.

Starting earlier [3] and further developed in parallel with the continuum methods [4] is the method of dynamical triangulation (DT). Especially (generalized) enumeration of the DT random surfaces by matrix models proved fruitful, and random matrix models became an important tool in the study of 2d Euclidean quantum gravity coupled to certain conformal field theories. Moreover, the current understanding is that whenever the discrete and continuum methods can be compared the results coincide [5].

For a large class of observables the discrete methods have been proven to be more powerful. Correlators with surfaces of higher genus can be efficiently computed by matrix model techniques [6] and fixed geodesic distance (proptime) correlation functions can be extracted by loop equations [7], transfer matrices [8] or through bijections with random trees [9]. So far these results have eluded the continuum methods.

In 1998 a different theory of 2d gravity was introduced called causal dynamical triangulations (CDT) [10]. Using computer simulations the method has been successfully extended to 4d quantum gravity [11]. The results are very promising and indicate that four dimensional gravity might be non-perturbatively renormalizable. The origin of the renormalizability could be a nontrivial fixed point scenario as described by Weinberg [12].

Although similar in spirit to Euclidean DT, the continuum limit of 2d CDT is significantly different. The main cause that puts 2d CDT in a different universality class from non-critical string theory is that in CDT the topology of spatial slices is fixed. This makes generic triangulations in CDT much better behaved, since unlike in Euclidean DT the spatial topology fluctuations cannot dominate the continuum limit. Consequently, the scaling of time in Euclidean DT is non-canonical and the Hausdorff dimension of the quantum geometry is $d_H = 4$. The quantum geometry of CDT on the other hand has a canonically scaling time variable and a Hausdorff dimension of $d_H = 2$.

Recent developments have shown that spatial topology fluctuations *can* be included while preserving the appealing features of CDT. The main idea is to assign a scaling coupling constant to the spatial topology change process [13] and that this generalized CDT model can be described by a matrix model [14].

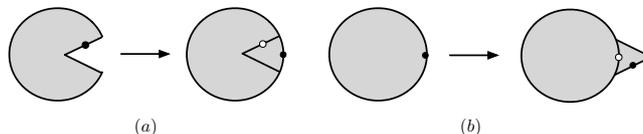


Fig. 1. Illustration of the two composition moves to add a triangle. The white dot on the right-hand-side shows the position of the mark before the triangle was added whereas the black dot shows the mark after the triangle was added.

Before coming to the matrix model we rederive the disc function of pure CDT, *i.e.* without spatial topology change, by a simple geometrical loop equation. Such an equation is known to be remarkably powerful. It allows one to compute time dependent correlators, where time is defined as the geodesic distance [7, 15]. At the end of this letter we derive the differential equation for the time dependent propagator and show that unlike in Euclidean DT, but typical for CDT, the scaling of time is canonical.

After discussing pure CDT we add a term to the loop equation that introduces spatial topology change. Only upon adding this term we can relate the loop equation to the Schwinger–Dyson equation of a one-matrix model with a linear term in the action. Remarkably, the linear term allows us to take a continuum limit that is very different from the well known limit of Euclidean DT but very similar to the continuum limit of CDT [14]. In fact, the continuum limit of the generalized loop equation reproduces the results of [13, 16]. Amazingly, the continuum limit of the matrix model can already be taken at the level of the matrix action, giving another matrix model that has a direct continuum interpretation [17].

2. Geometrical loop equations for 2D causal quantum gravity

In this section we compute the generating function for a set \mathcal{N} of triangulations which are similar to the original set of causal triangulations and which lead to the same continuum physics: Let n denote the number of triangles and l the number of links at the boundary (which has one marked link), and assume the topology is that of the disk and denote the generating function $\Phi(g, x)$:

$$\Phi(g, x) = \sum_{l,n=0}^{\infty} [\Phi(g, x)]_{n,l} = \sum_{l,n=0}^{\infty} \mathcal{N}(n, l) g^n x^l = \sum_{l=0}^{\infty} p_l(g) x^l. \quad (1)$$

The triangulations can be generated by recursively adding triangles. In our model there are two possible moves. Firstly, one can glue two edges of the additional triangle to the triangulation, one to the marked edge and the other one next to it in the clockwise direction (Fig. 1 (a)). Secondly,

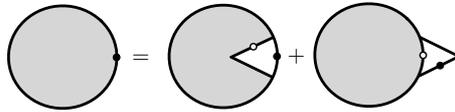


Fig. 2. Graphical representation of the loop equations (2) and (5).

one can add a triangle by simply gluing one of its edges to the marked edge of the triangulation and assigning the new mark to the new edge further clockwise (Fig. 1 (b)). Together, the two moves give the following generating equation for large n and l (see Fig. 2),

$$[\Phi(g, x)]_{n,l} = \frac{g}{x} [\Phi(g, x)]_{n,l} + gx[\Phi(g, x)]_{n,l}. \tag{2}$$

To keep the nice pictorial interpretation whilst making equation (2) exact, even for $n, l = 1$ and $n, l = 0$, one defines the following derivative operator, see *e.g.* [18],

$$\partial_x \sum_{l=0}^{\infty} c_l x^l = \sum_{l=1}^{\infty} c_l x^{l-1}, \tag{3}$$

$$\partial_x \Phi = \frac{1}{x} (\Phi - 1), \quad \partial_x^2 \Phi = \frac{1}{x^2} (\Phi - xp_1(g) - 1). \tag{4}$$

The exact generating equation can now be written as

$$\Phi(g, x) = 1 + gx + gx\partial_x^2 \Phi(g, x) + gx^2 \partial_x \Phi(g, x), \tag{5}$$

where x and ∂_x have the clear graphical interpretation of adding and removing boundary edges.

Equation (5) is a simple linear equation and the solution is given by

$$\Phi(g, x) = g \left(\frac{1 - (1/g - p_1(g)) x}{gx^2 - x + g} \right), \tag{6}$$

where $p_1(g)$ can be determined by demanding that the singularity structure of (6) does not change discontinuously near $g = 0$. The poles of (6) are located at

$$x_{\pm} = \frac{1 \pm \sqrt{1 - 4g^2}}{2g}, \quad \frac{1}{g} - x_- = x_+. \tag{7}$$

Since the expansion of $p_1(g)$ needs to be a power series we have that $p_1 = x_-$, hence the disc function is given by

$$\Phi(g, x) = \frac{1}{1 - p_1(g)x}, \quad p_l(g) = p_1(g)^l. \tag{8}$$

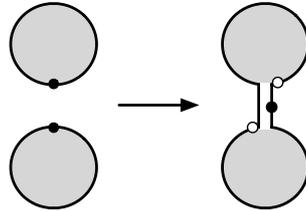


Fig. 3. Illustration of the composition moves to add a double link.

3. The continuum limit

Using the same scaling relations as in the transfer matrix formalism of CDT,

$$g = \frac{1}{2}e^{-a^2\Lambda/2}, \quad x = e^{-aX}, \quad (9)$$

we reproduce the continuum disc amplitude of causal quantum gravity [10]

$$W_\Lambda(X) = \frac{1}{X + \sqrt{\Lambda}}, \quad W_\Lambda(L) = e^{-\sqrt{\Lambda}L}. \quad (10)$$

4. A matrix model for generalized 2D causal quantum gravity

To include spatial topology change we introduce a quadratic term in the loop equation (5) (see Fig. 3)

$$\Phi_\beta(g, x) = 1 + gx + gx\partial_x^2\Phi_\beta(g, x) + gx^2\partial_x\Phi_\beta(g, x) + \beta x^2\Phi_\beta(g, x)^2, \quad (11)$$

where β is a coupling constant that determines the rate of the spatial topology fluctuations (see Fig. 4). To conform with matrix model conventions it is useful to introduce the following notation

$$w_\beta(g, z) = \frac{\Phi_\beta(g, x = 1/z)}{z}. \quad (12)$$

With these conventions the loop equation is given by

$$\beta w_\beta(g, z)^2 - v'(z)w_\beta(g, z) + q_\beta(g, z) = 0, \quad (13)$$

where

$$v(z) = -gz + \frac{1}{2}z^2 - \frac{1}{3}gz^3, \quad v'(z) = -g + z - gz^2, \quad (14)$$

and

$$q_\beta(g, z) = 1 - g(p_1(g, \beta) + z). \quad (15)$$

Written in the form of (13) it is easily seen that the loop equation corresponds to the Schwinger–Dyson equation of a simple one-matrix model [14],

$$Z_{\text{disc.}} = \int Dm \exp\left(-\frac{N}{\beta} [\text{Tr } v(m)]\right), \quad (16)$$

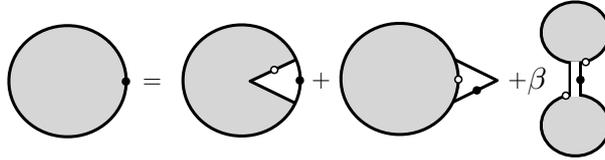


Fig. 4. Graphical representation of the loop equation (11).

where m is a Hermitian $N \times N$ -matrix and the functional form of the potential $v(m)$ is given by (14).

The solution of the loop equation (13) is of the following well-known form

$$w_\beta(g, z) = \frac{1}{2\beta} \left(v'(z) - \sqrt{v'(z)^2 - 4\beta q_\beta(g, z)} \right). \tag{17}$$

At this stage the solution of the disc function is still implicit since it depends on $p_1(g, \beta)$ through (15). Demanding the solution to have only one cut in the complex z plane gives the explicit result

$$w_\beta(g, z) = \frac{1}{2\beta} \left(-g + z - gz^2 + (gz - c)\sqrt{(z - c_+)(z - c_-)} \right), \tag{18}$$

where c is the solution of a third order polynomial,

$$2c^3 - 3c^2 + (2g^2 + 1)c = g^2(1 - 2\beta), \tag{19}$$

and

$$c_\pm = \frac{1 - c \pm \sqrt{2}\sqrt{(1 - c)c - g^2}}{g}. \tag{20}$$

5. The “causal” continuum limit

In the well known continuum limit of the one-matrix model with polynomial potential and generic coupling constants the critical value for the boundary cosmological constant z coincides with the critical value of c_+ only. As a result, this standard limit has the peculiar feature that it leaves a non-scaling term as a memory of the discrete theory, since

$$w_\lambda^{\text{Euc.}}(x) = w_{\text{ns}}(x) + a^{3/2} W_A^{\text{Euc.}}(X) + \mathcal{O}(a^2), \tag{21}$$

where $w_{\text{ns}}(x)$ is a non-scaling part and

$$W_A^{\text{Euc.}}(X) = \left(X - \frac{A}{2} \right) \sqrt{X + \sqrt{A}} \tag{22}$$

is the continuum disc function. Observe that the continuum disc function is even subleading in the lattice cutoff a .

In our specific model the matrix potential is such that the critical points of c_+ and c_- coincide. This leads us naturally to the universality class of two-dimensional CDT implying the same scaling relations as before (9), provided one also scales the coupling constant β [13]:

$$\beta = \frac{1}{2}g_s a^3, \quad c = \frac{1}{2}e^{aC}. \tag{23}$$

Contrary to the standard limit of the one matrix model our “causal” continuum limit is free from leading nonscaling contributions,

$$w_\beta(g, x) = \frac{1}{a} W_{\Lambda, g_s}(X) + \mathcal{O}(a^0), \tag{24}$$

where $W_{\Lambda, g_s}(X)$ is the continuum disc function previously derived with other methods [13, 16],

$$W_{\Lambda, g_s}(X) = \frac{1}{2g_s} \left(-(X^2 - \Lambda) + (X - C)\sqrt{(X + C)^2 - 2g_s/C} \right), \tag{25}$$

where C is the solution to a third order polynomial equation,

$$C^3 - \Lambda C + g_s = 0. \tag{26}$$

Observe that this equation is precisely the continuum limit of equation (19). Furthermore it is interesting to note that the structure of the discrete (18) and continuum (25) disc functions is very similar. This is not a coincidence since, as has been noticed recently, the continuum results can also be described by a matrix model [17],

$$Z_{\text{cont.}} = \int DM \exp \left(-\frac{N}{g_s} [\text{Tr } V(M)] \right), \tag{27}$$

with the following potential

$$V(M) = \Lambda M - \frac{1}{3}M^3. \tag{28}$$

In fact, it can be shown that the continuum matrix model (27) can be understood as the continuum limit of the matrix model (16) with a standard combinatorial interpretation [14]. While our continuum limit (9) and (23) is non-standard from the matrix model point of view it is very natural from a CDT perspective [13, 16].

6. Time evolution

To see that the non-critical string theory limit of the matrix model is really very different from our “causal” continuum limit, we briefly discuss the so-called fixed time (geodesic distance) two-loop amplitude $G_\beta(l_1, l_2; g; t)$. This amplitude is defined as the sum over all triangulations with initial boundary of length l_1 and final boundary of length l_2 at fixed geodesic distance t . With this in mind, it is natural to interpret the loop equation as a time dependent process [7, 15, 19], where each addition or subtraction of a triangle is a “ $1/l_1$ -th” part of a time step. For large l_1 we have

$$\begin{aligned} \frac{1}{l_1} \frac{\partial}{\partial t} G_\beta(l_1, l_2; g; t) &= gG_\beta(l_1 - 1, l_2; g; t) - G_\beta(l_1, l_2; g; t) \\ &+ gG_\beta(l_1 + 1, l_2; g; t) + 2\beta \sum_{l=0}^{\infty} p_l(g, \beta) G_\beta(l_1 - l - 2, l_2; g; t). \end{aligned} \quad (29)$$

After a “discrete Laplace transformation” this equation becomes

$$\frac{\partial}{\partial t} G_\beta(z, w; g; t) = \frac{\partial}{\partial z} [(-g + z - gz^2 - 2\beta w_\beta(g, z)) G_\beta(z, w; g; t)]. \quad (30)$$

The crucial difference between this equation and similar equations in non-critical string theory is the scaling of time in their continuum limits. Unlike in non-critical string theory the continuum limit of (30) involves a canonically scaling time parameter $T \sim at$, yielding

$$\frac{\partial}{\partial T} G_{\Lambda, g_s}(X, Y; T) = -\frac{\partial}{\partial X} [(X^2 - \Lambda + 2g_s W_{\Lambda, g_s}(X)) G_{\Lambda, g_s}(X, Y; T)]. \quad (31)$$

This is precisely the result of the propagator derived from generalized CDT [13, 16]. Already since the inception of CDT [10] it has been known that the scaling of the geodesic distance is intimately related to the Hausdorff dimension of the quantum geometry. One can argue that the “causal” continuum limit of the one-matrix model is better behaved since its Hausdorff dimension is $d_H = 2$ instead of $d_H = 4$ in non-critical string theory [10].

7. Conclusions

We have described a recently found “causal” continuum limit for the one-matrix model [14]. With the here described combinatorial interpretation this limit is naturally defined when inserting a linear term in the action with a specially chosen coefficient. We have shown here that this coefficient can naturally be interpreted as an additional way to add a triangle in the loop equations. The associated “causal” continuum limit is very different from

the “old” double scaling limit and exactly reproduces the known results of causal dynamical triangulations [10, 13]. An intriguing aspect of this new continuum limit is that the continuum results are also described by a matrix model. This matrix model has of course both an expansion in the coupling constants g_s and Λ , as well as a large N expansion in powers of $1/N^2$, which reorganizes the power expansions in g_s and Λ in convergent “subseries”. By comparing with the generalized causal dynamical triangulation model the powers of N^{-2h+2} in the large N expansion can be identified with the continuum causal dynamical surfaces of genus h [14, 17]. In this sense our “causal” continuum limit leads to a picture that is much closer in spirit to the original idea by ’t Hooft [20] for QCD.

As a complement to previous results we also derived the differential equation for the fixed geodesic distance two-loop amplitude. The observed canonical scaling of time is directly related to the fact that the quantum geometry obtained from our new “causal” continuum limit has Hausdorff dimension $d_H = 2$ instead of $d_H = 4$ for the “old” continuum limit of the one-matrix model.

Until now, only the transfer matrix formalism has been available for analytical computations in causal dynamical triangulations. Here we extended the available tools and presented new, more powerful, matrix model and loop equation methods. Importantly, these new methods allow one to analytically study simple matter models coupled to two-dimensional causal quantum gravity. Several of these models are currently under investigation.

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REFERENCES

- [1] A.M. Polyakov, *Phys. Lett.* **B103**, 207 (1981); A.A. Belavin, A.M. Polyakov, A.B. Zamolodchikov, *Nucl. Phys.* **B241**, 333 (1984); V.G. Knizhnik, A.M. Polyakov, A.B. Zamolodchikov, *Mod. Phys. Lett.* **A3**, 819 (1988).
- [2] V. Fateev, A.B. Zamolodchikov, A.B. Zamolodchikov, [hep-th/0001012](#).
- [3] W.T. Tutte, *Can. J. Math.* **14**, 21 (1962); E. Brezin, C. Itzykson, G. Parisi, J.B. Zuber, *Commun. Math. Phys.* **59**, 35 (1978).

- [4] J. Ambjørn, B. Durhuus, J. Fröhlich, *Nucl. Phys.* **B257**, 433 (1985); J. Ambjørn, B. Durhuus, J. Fröhlich, P. Orland, *Nucl. Phys.* **B270**, 457 (1986); F. David, *Nucl. Phys.* **B257**, 543 (1985); A. Billoire, F. David, *Nucl. Phys.* **B275**, 617 (1986); V.A. Kazakov, A.A. Migdal, I.K. Kostov, *Phys. Lett.* **B157**, 295 (1985).
- [5] E.J. Martinec, [hep-th/0305148](#).
- [6] J. Ambjørn, L. Chekhov, C.F. Kristjansen, Y. Makeenko, *Nucl. Phys.* **B404**, 127 (1993) [[hep-th/9302014](#)]; B. Eynard, N. Orantin, [math-ph/0702045](#).
- [7] Y. Watabiki, *Nucl. Phys.* **B441**, 119 (1995) [[hep-th/9401096](#)].
- [8] H. Kawai, N. Kawamoto, T. Mogami, Y. Watabiki, *Phys. Lett.* **B306**, 19 (1993) [[hep-th/9302133](#)].
- [9] G. Schaeffer, *Elec. J. Comb.* **4**, R20 (1997); J. Bouttier, P. Di Francesco, E. Guitter, *Nucl. Phys.* **B645**, 477 (2002) [[cond-mat/0207682](#)].
- [10] J. Ambjørn, R. Loll, *Nucl. Phys.* **B536**, 407 (1998) [[hep-th/9805108](#)].
- [11] J. Ambjørn, J. Jurkiewicz, R. Loll, *Phys. Lett.* **B607**, 205 (2005) [[hep-th/0411152](#)]; *Phys. Rev. Lett.* **95**, 171301 (2005) [[hep-th/0505113](#)]; *Phys. Rev.* **D72**, 064014 (2005) [[hep-th/0505154](#)]; J. Ambjørn, A. Görlich, J. Jurkiewicz, R. Loll, *Phys. Rev. Lett.* **100**, 091304 (2008) [[arXiv:0712.2485](#) [[hep-th](#)]].
- [12] S. Weinberg, in *General Relativity*, eds. S. Hawking, W. Israel, 1980, pp. 790–831.
- [13] J. Ambjørn, R. Loll, W. Westra, S. Zohren, *J. High Energy Phys.* **12**, 017 (2007) [[arXiv:0709.2784](#) [[gr-qc](#)]].
- [14] J. Ambjørn, R. Loll, Y. Watabiki, W. Westra, S. Zohren, Utrecht Preprint 2008.
- [15] J. Ambjørn, J. Correia, C. Kristjansen, R. Loll, *Phys. Lett.* **B475**, 24 (2000) [[hep-th/9912267](#)].
- [16] J. Ambjørn, R. Loll, Y. Watabiki, W. Westra, S. Zohren, *J. High Energy Phys.* **05**, 032 (2008) [[arXiv:0802.0719](#) [[hep-th](#)]]; [arXiv:0802.0896](#) [[hep-th](#)].
- [17] J. Ambjørn, R. Loll, Y. Watabiki, W. Westra, S. Zohren, *Phys. Lett.* **B665**, 252 (2008) [[arXiv:0804.0252](#)].
- [18] S.M. Carroll, M.E. Ortiz, W. Taylor, *Nucl. Phys.* **B468**, 383 (1996) [[hep-th/9510199](#)].
- [19] M. Arnsdorf, *Class. Quantum Grav.* **19**, 1065 (2002) [[gr-qc/0110026](#)].
- [20] G. 't Hooft, *Nucl. Phys.* **B72**, 461 (1974).