

STATISTICAL PHYSICS OF DYONS
AND CONFINEMENT*

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We present a semiclassical description of the $SU(N)$ Yang–Mills theory whose partition function at nonzero temperatures is approximated by a saddle point — an ensemble of an infinite number of interacting dyons of N kinds. The ensemble is governed by an exactly solvable 3D quantum field theory, allowing calculation of correlations functions relevant to confinement. We show that known criteria of confinement are satisfied in this semiclassical approximation: *(i)* the average Polyakov line is zero below some critical temperature, and nonzero above it, *(ii)* static quarks in any nonzero N -ality representation have linear rising potential energy, *(iii)* the average spatial Wilson loop falls off exponentially with the area, *(iv)* N^2 gluons are canceled out from the spectrum. The critical temperature is in good agreement with lattice data.

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1. Philosophy

Quantum Chromodynamics (QCD) is hardly an exactly solvable quantum field theory, even in the large N limit. Therefore, one can either do exact calculations in a theory that has more symmetries but is not our world, or work with QCD but make approximations. The first is useful as a theoretical laboratory, the second is necessary to understand semi-quantitatively the key phenomena, to explain experimental data, and to make predictions.

An approximation is considered to be legitimate if there is a systematic way of improving its accuracy. The semiclassical approach belongs to this category. One chooses a saddle-point classical field and then has to take into account quantum fluctuations about it. Part of the fluctuations are ultra-violet and are thus the same as in empty space. Therefore, their role is to

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renormalize the bare coupling constant; at this point the famous dimensional transmutation occurs, when the ultraviolet cutoff in a proper combination with the bare coupling constant forms the QCD scale parameter Λ , the only dimensional scale that henceforth will be in the theory. What is left, is a series in the 't Hooft *running* coupling $\lambda \equiv N\alpha_s/2\pi$ coming from loop expansion in the background of classical configurations.

The argument of the running coupling λ is determined by the largest scale in the vacuum, $\max(T, n^{\frac{1}{4}})$, where T is temperature, and n is the mean 4D density of the (large) classical field configurations. For example, near the deconfinement temperature $T \approx T_c \approx \Lambda$ the running coupling is approximately $\lambda = [(11/3) \ln(4\pi T/\Lambda e^{\gamma_E})]^{-1}|_{T=\Lambda} \approx 1/7$ [1]. The numerically large factor 2π in the argument of the logarithm is not accidental but related to the fact that it is actually not the temperature itself but rather the Matsubara frequency $2\pi T$ that defines the scale. At zero temperature many QCD specialists believe that α_s does not grow above the value of 0.5, which gives $\lambda \approx 1/4$. Therefore, in the whole range of temperatures within the confining phase the semiclassical approximation is expected to yield the accuracy of 15–25%, already in the 1-loop approximation (provided the saddle point is chosen correctly!) with a possibility for rapid improvement when higher loops are taken into account. We shall see, however, that the actual accuracy can be much better than this estimate. It is not a too big price to pay if confinement, the most challenging riddle in 35 years, is explained in simple terms.

We shall be considering the pure Yang–Mills theory based on the $SU(N)$ gauge group in a broad range of temperatures between 0 and T_c , the deconfinement phase transition temperature. Although the formalism we use is designed for nonzero T , we shall see that the physical observables we find (such as the string tension) have a finite limit when $T \rightarrow 0$. In this limit the nonzero temperature can be thought of as an infrared regulator. After all, our world's temperature is 2.7 K $\neq 0$.

Confinement, as we understand it today and learn from lattice experiments with a pure glue theory, has in fact many facets, and all have to be explained. Let us enumerate the main:

- the average Polyakov line in any N -ality nonzero representation of the $SU(N)$ group is zero below T_c and nonzero above it,
- the potential energy of two static colour sources (defined through the correlation function of two Polyakov lines) asymptotically rises linearly with the separation; the slope called the string tension depends only on the N -ality of the sources,

- the average of the spatial Wilson loop decays exponentially with the area spanning the contour; at vanishing temperatures the spatial (“magnetic”) string tension has to coincide with the “electric” one, for all representations,
- the mass gap: no massless gluons left in the spectrum.

Remarkably, all these requirements are satisfied already in a semiclassical approximation if one uses an ensemble of dyons as a saddle point in the Yang–Mills partition function [2]. That paper with Victor Petrov is the base for this presentation.

2. Yang–Mills theory at nonzero temperatures

The Yang–Mills (YM) partition function can be written as a path or functional integral over the spatial components of the connection $A_i(t, \mathbf{x})$ satisfying the periodic boundary conditions up to a gauge transformation $\Omega(\mathbf{x})$ over which one has to integrate separately [3]:

$$\mathcal{Z} = \sum_{\text{gauge invariant states}} \langle n | e^{-\beta\mathcal{H}} | n \rangle = \int D\Omega(\mathbf{x}) DA_i(\mathbf{x}) \left[\beta = \frac{1}{T} \right] \times \int_{A_i(\mathbf{x})}^{A_i(\mathbf{x})\Omega(\mathbf{x})} DA_i(t, \mathbf{x}) \exp \left[-\frac{1}{g^2} \int_0^\beta dt \int d^3\mathbf{x} \text{Tr} \left(\dot{A}_i \dot{A}_i + B_i B_i \right) \right], \quad (1)$$

where $B_i = \epsilon_{ijk} (\partial_j A_k - \frac{i}{2} [A_j, A_k])$ is the magnetic field strength and $A_i^\Omega \equiv \Omega^\dagger A_i \Omega + i\Omega^\dagger \partial_i \Omega$ is the gauge-transformed potential. A_i, B_i are $N \times N$ matrices belonging to the $\mathfrak{su}(N)$ algebra while Ω is an element of the $SU(N)$ group.

One can rewrite the partition function in a more customary form by introducing gauge-transformed integration variables A_i that are strictly periodic in time, and trading Ω for the time component of the YM potential A_4 that is also periodic:

$$\mathcal{Z} = \int DA_\mu(t, \mathbf{x}) \exp \left(-\frac{1}{2g^2} \int_0^{\frac{1}{T}} dt \int d^3\mathbf{x} \text{Tr} F_{\mu\nu} F_{\mu\nu} \right), \quad (2)$$

$$A_\mu \left(t + \frac{1}{T}, \mathbf{x} \right) = A_\mu(t, \mathbf{x}),$$

where $F_{\mu\nu}$ is the usual field strength. This form stresses the fact that Euclidean $O(4)$ symmetry is restored as $T \rightarrow 0$.

An important variable is the Polyakov loop: in the formulation (2) it is the path-ordered exponent

$$L(\mathbf{x}) = \mathcal{P} \exp \left(i \int_0^{\frac{1}{T}} dt A_4(t, \mathbf{x}) \right) \quad (= \Omega(\mathbf{x})). \quad (3)$$

In the formulation (1) it is nothing but the $SU(N)$ matrix $\Omega(\mathbf{x})$ over which there is a final integration in Eq. (1). The eigenvalues of $L(\mathbf{x})$ are gauge invariant; we parameterize them as

$$L = \text{diag} (e^{2\pi i \mu_1}, e^{2\pi i \mu_2}, \dots, e^{2\pi i \mu_N}), \quad \mu_1 + \dots + \mu_N = 0, \quad (4)$$

and assume that the phases of these eigenvalues are ordered: $\mu_1 \leq \mu_2 \leq \dots \leq \mu_N \leq \mu_{N+1} \equiv \mu_1 + 1$. We shall call the set of N phases $\{\mu_m\}$ the “holonomy” for short. Apparently, shifting μ 's by integers does not change the eigenvalues, hence all quantities have to be periodic in all μ 's.

The holonomy is said to be “trivial” if L belongs to one of the N elements of the group center Z_N . For example, in $SU(3)$ the three trivial holonomies are

$$\begin{aligned} 1. \quad \mu_1 = \mu_2 = \mu_3 = 0 & \implies L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ 2. \quad \mu_1 = -\frac{2}{3}, \mu_2 = \frac{1}{3}, \mu_3 = \frac{1}{3} & \implies L = e^{\frac{2\pi i}{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ 3. \quad \mu_1 = -\frac{1}{3}, \mu_2 = -\frac{1}{3}, \mu_3 = \frac{2}{3} & \implies L = e^{-\frac{2\pi i}{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Trivial holonomy corresponds to equal μ 's, *modulo* unity. Out of all possible combinations of μ 's a distinguished role is played by equidistant μ 's:

$$\mu_m^{\text{conf}} = -\frac{1}{2} - \frac{1}{2N} + \frac{m}{N}, \quad \text{Tr } L = 0. \quad (5)$$

For example, in $SU(3)$ it is

$$\mu_1 = -\frac{1}{3}, \mu_2 = 0, \mu_3 = \frac{1}{3} \implies L = \begin{pmatrix} e^{-\frac{2\pi i}{3}} & 0 & 0 \\ 0 & e^{\frac{0\pi i}{3}} & 0 \\ 0 & 0 & e^{\frac{2\pi i}{3}} \end{pmatrix}, \quad \text{Tr } L = 0. \quad (6)$$

We shall call it “most non-trivial” or “confining” holonomy as it corresponds to $\text{Tr } L = 0$ which is the 1st confinement requirement.

Immediately, an interesting question arises: Imagine we take the YM partition function, be it in form (1) or (2), and integrate out all degrees of freedom except the eigenvalues $\{\mu_m\}$ of the Polyakov loop $L(\mathbf{x})$ (or $\Omega(\mathbf{x})$), which, in addition, we take slowly varying in space. What is the effective action for μ 's? What set of μ 's is preferred dynamically by the YM system of fields?

In general, it is a difficult calculational problem that can be addressed using various approximations but in one case the result is known exactly. It is the case of the supersymmetric $\mathcal{N}=1$ version of the YM theory (SYM) where in addition to gluons there are gluinos in the adjoint representation. In order not to spoil supersymmetry one takes not the real temperature but rather a 4D space compactified in the time direction, $R^3 \times S^1$. The difference is that in the “real temperature” case one uses periodic conditions in the Euclidean time direction for boson fields (gluons) and *antiperiodic* conditions for fermion fields (gluinos) — that spoils supersymmetry; in the “compactification” case one implies periodic conditions for both kinds of fields, what supports supersymmetry. However, we shall anyway call the inverse circumference of the compactified time direction “temperature” for short.

There is no perturbative contribution to the potential energy in question as function of μ 's (directly related in this case to the holomorphic superpotential) because of the supersymmetric cancellation between boson and fermion loops, and the only contribution is nonperturbative coming from dyons. It can be reliably computed in the limit of high “temperatures” and then claimed to be actually independent of temperature owing to the holomorphy typical in supersymmetry. The result [4] is that the potential energy of the system has the minimum at precisely the “most non-trivial” or “confining” holonomy (5).

In the non-supersymmetric pure YM theory, there is a perturbative effective action for slowly varying μ 's. It can be understood as gluon loop(s) in the background of a slowly varying field $A_4(\mathbf{x})$. The effective action can be expanded in the number of gradients of μ 's. The zero-order term, the potential energy with no derivatives, has been computed long ago in Refs. [3,5]:

$$P^{\text{pert}} = V \frac{(2\pi)^2 T^3}{3} \sum_{m>n}^N (\mu_m - \mu_n)^2 [1 - (\mu_m - \mu_n)]^2 \Big|_{\text{mod } 1}. \quad (7)$$

Since the piece with no derivatives implies constant μ 's, it has to be proportional to the 3-volume V , and hence to T^3 by dimensions. P^{pert} has exactly N zero minima when all μ 's are equal *modulo* unity. Hence, P^{pert} says that at high temperatures the system prefers one of the N trivial holonomies corresponding to the Polyakov loop being one of the N elements of the center Z_N , see Fig. 1, top.

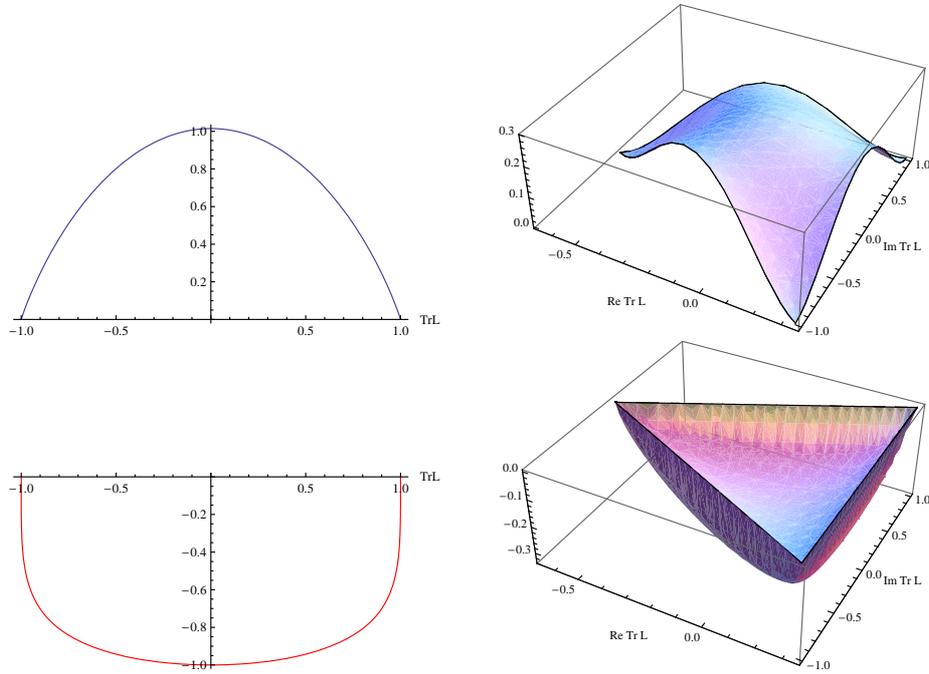


Fig. 1. The perturbative (top) *versus* nonperturbative (bottom) potential energy as a function of the Polyakov line for the SU(2) (left) and SU(3) (right) groups. The perturbative potential energy has minima where the Polyakov loop is one of the N elements of the center Z_N and is maximal at the “confining” holonomy. The nonperturbative potential energy, on the contrary, has a single and non-degenerate minimum at the confining holonomy corresponding to $\text{Tr } L = 0$.

However, gradient terms in the effective action indicate that there is a problem with the trivial-holonomy points, already at the perturbative level. Indeed, the two-derivative term is [1]

$$\begin{aligned}
 S_{2\text{-der}}^{\text{pert}} &= \sum_{m>n}^N \int d^3x [\partial_i(\mu_m - \mu_n)]^2 \frac{11}{12} T \left[H(\mu_m - \mu_n) + 2 \log \left(\frac{4\pi T}{\Lambda e^{\gamma_E}} \right) \right], \\
 H(\nu) &= [\psi(\nu) + \psi(1 - \nu) + 2\gamma_E]_{\text{mod } 1}, \\
 \psi(\nu) &= \frac{d}{d\nu} \ln \Gamma(\nu).
 \end{aligned} \tag{8}$$

Since $\psi(\nu) \approx -1/\nu$ at small ν , the gradient term becomes negative near “trivial” holonomy, which signals its instability even in perturbation theory.

We shall show below that a semiclassical configuration — an ensemble of dyons with quantum fluctuations about it — generates a nonperturbative free energy shown in Fig. 1, bottom. It has the opposite behavior of

the perturbative potential energy, having the minimum at the equidistant (confining) values of the μ 's. There is a fight between the perturbative and nonperturbative contributions to the free energy [6]. Since the perturbative contribution is $\sim T^4$ with respect to the nonperturbative one, it certainly wins when temperatures are high enough, and the system is then forced into one of the N vacua thus breaking spontaneously the Z_N symmetry. At low temperatures the nonperturbative contribution prevails forcing the system into the confining vacuum. At a critical T_c there is a confinement–deconfinement phase transition. It turns out to be of the second order for $N = 2$ but the first order for $N = 3$ and higher, in agreement with lattice findings.

3. Dyon saddle points

Dyons or Bogomolny–Prasad–Sommerfield (BPS) monopoles [7] are (anti) self-dual solutions of the nonlinear Maxwell equations, $D_\mu^{ab} F_{\mu\nu}^b = 0$. In $SU(N)$ there are exactly N kinds of “fundamental” dyons with Coulomb asymptotics for both electric and magnetic fields (hence the term “dyon”):

$$\pm \mathbf{E} = \mathbf{B} \Big|_{|\mathbf{x}| \rightarrow \infty} \frac{1}{2} \frac{\mathbf{x}}{|\mathbf{x}|^3} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (9)$$

Dyon solutions are labeled by the holonomy or the set of μ 's at spatial infinity:

$$A_4(|\mathbf{x}| \rightarrow \infty) \rightarrow 2\pi T \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix} \quad (10)$$

(we illustrate it for the case of $SU(3)$). The explicit expressions for the solutions in various gauges can be found *e.g.* in the Appendix of Ref. [8]. Inside the cores which are of the size $\sim 1/(T(\mu_{m+1} - \mu_m))$, the fields are large, nonlinearity is essential. The action density is time-independent everywhere and is proportional to the temperature. Isolated dyons are thus 3D objects but with finite action $S_{\text{dyon}} = (8\pi^2/g^2)(\mu_{m+1} - \mu_m)$ independent of temperature (here $\mu_{N+1} \equiv \mu_1 + 1$). The full action of all N kinds of well-separated dyons together is that of one standard instanton: $S_{\text{inst}} = 8\pi^2/g^2$.

In the semiclassical approach, one has first of all to find the statistical weight with which a given classical configuration enters the partition function. It is given by $\exp(-\text{Action})$, times the determinant $^{-1/2}$ from small quantum oscillations about the saddle point. For an isolated dyon as a saddle-point configuration, this factor diverges linearly in the infrared region owing to the slow Coulomb decrease of the dyon field (9). It means that

isolated dyons are not acceptable as saddle points: they have zero weight, despite finite classical action. However, one may look for classical solutions that are superpositions of N fundamental dyons, with zero a net magnetic charge. The small-oscillation determinant must be infrared-finite for such classical solutions, if they exist.

4. Instantons with non-trivial holonomy

Remarkably, the needed classical solution has been found a decade ago by Kraan and van Baal [9] and independently and simultaneously by Lee and Lu [10], see also [11]. We shall call them for short the “KvBLL instantons”; an alternative name is “calorons with non-trivial holonomy”. The solution was first found for the $SU(2)$ group but soon generalized to the arbitrary $SU(N)$ [12]. A nice overview of the solutions has been presented by Pierre van Baal at the 2003 School in Zakopane [13]. We shall mention only the essentials here.

The general solution depends on Euclidean time t and space \mathbf{x} and is parameterized by $3N$ positions of N kinds of “constituent” dyons in space $\mathbf{x}_1, \dots, \mathbf{x}_N$ and their $U(1)$ phases ψ_1, \dots, ψ_N . All in all, there are $4N$ collective coordinates characterizing the solution (called the moduli space), of which the action $S_{\text{inst}} = 8\pi^2/g^2$ is in fact independent, as it should be for a general solution with a unity topological charge. The solution also depends explicitly on temperature T and on the holonomy μ_1, \dots, μ_N :

$$A_{\mu}^{\text{KvBLL}} = \bar{A}_{\mu}^a(t, \mathbf{x}; \mathbf{x}_1, \dots, \mathbf{x}_N, \psi_1, \dots, \psi_N; T, \mu_1, \dots, \mu_N). \quad (11)$$

The solution is a relatively simple expression given by elementary functions. If the holonomy is trivial (all μ 's are equal *modulo* unity) the expression takes the form of the strictly periodic $O(3)$ symmetric caloron [14] reducing further to the standard $O(4)$ symmetric BPST instanton [15] in the $T \rightarrow 0$ limit. At small temperatures but arbitrary holonomy, the KvBLL instanton also has only a small $\mathcal{O}(T)$ difference with the standard instanton.

One can plot the action density of the KvBLL instanton in various corners of the parameter (moduli) space, see Fig. 2.

When all dyons are far apart one observes N static (*i.e.* time-independent) objects, the isolated dyons. As they merge, the configuration is not static anymore, it becomes a *process* in time. In the limiting case of a complete merger, the configuration becomes a 4D lump resembling the standard instanton. The full (integrated) action is exactly the same $S_{\text{inst}} = 8\pi^2/g^2$ for any choice of the dyon separations. It means that classically dyons do not interact. However, they do experience a peculiar interaction at the quantum level to which we proceed.

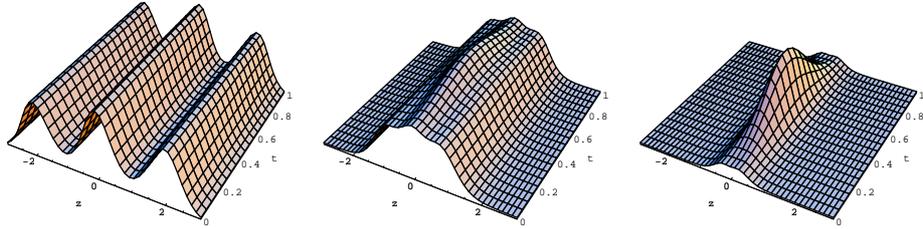


Fig. 2. Action density inside the SU(3) KvBLL instanton as a function of time and one space coordinate, for large (left), intermediate (middle) and small (right) separations between the three constituent dyons.

5. Quantum weight of a neutral cluster of N dyons

Remarkably, the small-oscillation determinant about the KvBLL instanton can be computed exactly; this has been first done for the SU(2) group in Ref. [16] and later generalized to SU(N) in Ref. [17]. The quantum weight of the KvBLL instanton can be schematically written as an integral over $3N$ coordinates of dyons (the weight does not depend on the U(1) angles ψ_m , hence they can be integrated out):

$$W_1 = \int d\mathbf{x}_1 \dots d\mathbf{x}_N \sqrt{\det g} \left(\frac{4\pi \mu^4}{g^4 T} \right)^N \exp \left(-\frac{8\pi^2}{g^2} \right) (\text{Det}(-\Delta))_{\text{reg, norm}}^{-1}, \quad (12)$$

where g is the full $4N \times 4N$ metric tensor of the moduli space, defined as the zero modes overlap matrix, and $\text{Det}(-\Delta)$ is the functional determinant over nonzero modes, normalized to the free one and regularized by the background Pauli–Villars method; μ is the Pauli–Villars ultra-violet cutoff and g^2 is the bare coupling constant defined at that cutoff. The Jacobian $\det g$ turns out to be a square of the determinant of an $N \times N$ matrix $G^{(1)}$ such that $\sqrt{\det g} = \det G^{(1)}$ where

$$G_{mn}^{(1)} = \delta_{mn} \left(4\pi\nu_m + \frac{1}{T|\mathbf{x}_m - \mathbf{x}_{m-1}|} + \frac{1}{T|\mathbf{x}_m - \mathbf{x}_{m+1}|} \right) - \frac{\delta_{m,n-1}}{T|\mathbf{x}_m - \mathbf{x}_{m+1}|} - \frac{\delta_{m,n+1}}{T|\mathbf{x}_m - \mathbf{x}_{m-1}|}, \quad (13)$$

$$\nu_m = \mu_{m+1} - \mu_m, \quad \sum_{m=1}^N \nu_m = 1,$$

is a matrix whose entries are Coulomb interactions between dyons that are nearest neighbors in kind. The Coulomb interactions in the zero mode overlap matrix arise naturally from the Coulomb asymptotics of the dyon field (9), so it is quite simple to check that Eq. (13) is correct at large separations.

A nontrivial fact is that Eq. (13) is actually *exact* for all separations between dyons, including the case when they strongly overlap like in Fig. 2, right. This has been first conjectured by Lee, Weinberg and Yi [18] and then proved to be indeed exact at all separations by a direct calculation by Kraan [19] and later checked in Ref. [20]. In the last paper it has been also shown that in the limit of trivial holonomy ($\mu_m = 0$) or vanishing temperature the measure given by Eqs. (12)–(13) reduces to the standard instanton measure written in terms of the conventional “center-size-orientation”, which is a rather nontrivial but gratifying statement.

The functional determinant over nonzero modes $\text{Det}^{-1}(-\Delta)$ together with the classical action and the Pauli–Villars cutoff combine into the renormalized scale parameter Λ_{PV} , times a function of dyon separations, Λ_{PV} and T [16,17]. It is a complicated function which, for the time being, we approximate by its most essential part: a constant equal to $\exp(-P^{\text{pert}})$, where P^{pert} is the perturbative gluon loop (7) in the background of a constant field A_4 (10). This part is necessarily present in $\text{Det}^{-1}(-\Delta)$ as most of the 3D space outside the instanton’s core is just a constant A_4 background, and indeed the calculation [16,17] exhibits this piece which is the only one proportional to the 3-volume.

Therefore, we write the weight of the KvBLL instanton *i.e.* a neutral cluster of N different-kind dyons as

$$W_1 \approx \int d\mathbf{x}_1 \dots d\mathbf{x}_N \det G_1 f^N \exp(-P^{\text{pert}}(\mu_1, \dots, \mu_N)), \quad (14)$$

where the fugacity f is

$$f = \frac{4\pi}{g^4} \frac{\Lambda^4}{T} = \mathcal{O}(N^2). \quad (15)$$

The bare coupling constant g^2 in the pre-exponent is renormalized and starts to “run” only at the 2-loop level not considered here. Eventually, its argument will be the largest scale in the vacuum, be it the temperature or the equilibrium density of dyons.

6. Quantum weight of many dyons

In the vacuum problem, one needs to use not one but $\mathcal{O}(V)$ number of KvBLL instantons as the saddle point. Solutions with the topological charge greater than 2 will be hardly ever known explicitly as their construction runs into the problem of resolving the nonlinear Atiyah–Drinfeld–Hitchin–Manin–Nahm constraints. At present 2-instanton solutions characterized by a nontrivial holonomy have been found [21] but it is insufficient. Nevertheless, the moduli space *measure* of an arbitrary number K of KvBLL instantons can

be constructed despite the lack of explicit solutions, at least in the approximation which seems to be relevant for the large-volume thermodynamics, if not exactly.

When one takes a configuration of K instantons each made of N different-kind dyons one encounters also same-kind dyons for which the metric (13) is inapplicable. However, the case of identical dyons has been considered separately by Gibbons and Manton [22]. The integration measure for K identical dyons following from that work is, in our notations,

$$\begin{aligned}
 W^{\text{ident}} &= \frac{1}{K!} \int d\mathbf{x}_1 \dots d\mathbf{x}_K \det G_{K \times K}^{\text{ident}}, \\
 G_{ij}^{\text{ident}} &= \begin{cases} 4\pi\nu_m - \sum_{k \neq i} \frac{2}{T|\mathbf{x}_i - \mathbf{x}_k|}, & i = j \\ \frac{2}{T|\mathbf{x}_i - \mathbf{x}_j|}, & i \neq j \end{cases}, \quad (16)
 \end{aligned}$$

where the identity factorial is inserted to avoid counting same configurations more than once.

As in the case of different-kind dyons, this result for the metric can be easily obtained at large separations from considering the asymptotics of the zero modes' overlap. However, in contrast to the different-kind dyons, it is not possible to prove that this expression is correct at all separations. Moreover, such an extension of Eq. (16) is probably wrong. The metric for two same-kind dyons has been found exactly at all separations by Atiyah and Hitchin [23]: it is more complicated than what follows from Eq. (16) at $K = 2$ but differs from it by terms that are exponentially small at large separations [24]. We shall neglect the difference and use the Gibbons–Manton metric at face value. The point is, Eq. (16) imposes very strong repulsion between same-kind dyons (as does the exact Atiyah–Hitchin metric), hence the range of the moduli space where the two metrics differ is, statistically, not frequently visited by dyons. We do not have a proof that all thermodynamic quantities will be computed correctly with this simplification: proving or disproving it is an interesting and important problem. To remain on the safe side, one has to admit today that the metric (16) is applicable if the dyon ensemble is sufficiently dilute, that is at high temperatures. Nevertheless, physical observables we compute have a smooth limit even at $T \rightarrow 0$. Therefore, it may well prove to be a correct computation at any temperatures, but this remains to be seen.

It is possible to combine the metric tensors for different-kind (13) and same-kind (16) dyons into one metric appropriate for the moduli space of K_1 dyons of kind 1, K_2 dyons of kind 2, ..., K_N dyons of kind N . It is a matrix whose dimension is the total number of dyons, that is a $(K_1 + \dots + K_N) \times (K_1 + \dots + K_N)$ matrix:

$$\begin{aligned}
G_{mi,nj} = & \delta_{mn}\delta_{ij} \left(4\pi\nu_m + \sum_k \frac{1}{T|\mathbf{x}_{mi}-\mathbf{x}_{m-1,k}|} + \sum_k \frac{1}{T|\mathbf{x}_{mi}-\mathbf{x}_{m+1,k}|} \right. \\
& \left. - 2 \sum_{k \neq i} \frac{1}{T|\mathbf{x}_{mi}-\mathbf{x}_{mk}|} \right) \\
& - \frac{\delta_{m,n-1}}{T|\mathbf{x}_{mi}-\mathbf{x}_{m+1,j}|} - \frac{\delta_{m,n+1}}{T|\mathbf{x}_{mi}-\mathbf{x}_{m-1,j}|} + 2 \frac{\delta_{mn}}{T|\mathbf{x}_{mi}-\mathbf{x}_{mj}|} \Big|_{i \neq j}, \quad (17)
\end{aligned}$$

where \mathbf{x}_{mi} is the coordinate of the i^{th} dyon of kind m . Since the statistical weight of a configuration of dyons is large when $\det G$ is large and small when it is small, $\det G$ imposes an attraction between dyons that are nearest neighbors in kind, and a repulsion between same-kind dyons. The coefficients $-1, 2, -1$ in front of the Coulomb interactions are actually the scalar products of the Cartan generators that determine the asymptotics of the dyons' field, see Eq. (9).

The matrix G has the following nice properties:

- symmetry: $G_{mi,nj} = G_{nj,mi}$;
- overall “neutrality”: the sum of Coulomb interactions in non-diagonal entries cancel those on the diagonal: $\sum_{nj} G_{mi,nj} = 4\pi\nu_m$;
- identity loss: dyons of the same kind are indistinguishable, meaning mathematically that $\det G$ is symmetric under permutation of any pair of dyons ($i \leftrightarrow j$) of the same kind m . Dyons do not “know” to which instanton they belong to;
- factorization: in the geometry when dyons fall into K well separated neutral clusters of N dyons of different kinds in each, $\det G$ factorizes into a product of exact integration measures for K KvBLL instantons, $\det G = (\det G^{(1)})^K$ where $G^{(1)}$ is given by Eq. (13);
- last but not least, the metric corresponding to G is hyper-Kähler, as it should be for the moduli space of a self-dual classical field [23]. In fact, it is a severe restriction on the metric.

7. Ensemble of dyons

In the semiclassical approximation we thus replace the YM partition function (2) by the partition function of an interacting ensemble of an arbitrary number of dyons of N kinds:

$$Z = \sum_{K_1 \dots K_N} \frac{1}{K_1! \dots K_N!} \prod_{m=1}^N \prod_{i=1}^{K_m} \int (d\mathbf{x}_{mi} f) \det G(\mathbf{x}), \quad (18)$$

where \mathbf{x}_{mi} is the coordinate of the i^{th} dyon of kind m , the matrix G is given by Eq. (17) and the fugacity f is given by Eq. (15). The overall exponent of the perturbative potential energy as function of the holonomy $\{\mu\}$ is understood, as in Eq. (14).

The ensemble defined by a determinant of a matrix whose dimension is the number of particles, is not a usual one. More customary, the interaction is given by the Boltzmann factor $\exp(-U_{\text{int}}(\mathbf{x}_1, \dots))$. Of course, one can always present the determinant in that way using the identity $\det G = \exp(\text{Tr} \log G) \equiv \exp(-U_{\text{int}})$ but the interactions will then include three-, four-, five- ... body forces. At the same time, it is precisely the determinant form of the interaction that makes the statistical physics of dyons an exactly solvable problem.

8. Dyons' free energy: confining holonomy preferred

The partition function (18) can be computed directly and exactly, just by writing the determinant of G by definition as a sum of permutations of products of the matrix entries. The result is astonishingly simple: all Coulomb interactions cancel exactly after integration over dyons' positions, provided the overall neutrality condition is satisfied, *viz.* $K_1 = K_2 = \dots = K_N = K$; otherwise the partition function is divergent. Therefore, the recipe for computing the partition function is just to impose the neutrality condition and then to throw out all Coulomb interactions! We have thus to take the product of $(4\pi\nu_m)$'s from the diagonal of G :

$$\mathcal{Z} = \sum_K \frac{(4\pi fV)^{KN}}{(K!)^N} \prod_{m=1}^N \nu_m^K.$$

The quantity $4\pi fV$ is dimensionless and large for large volumes V . The sum can be, therefore, computed from the saddle point in K using the Stirling asymptotics for large factorials, and we obtain

$$\mathcal{Z} = \exp\left(4\pi fVN (\nu_1\nu_2 \dots \nu_N)^{\frac{1}{N}}\right), \quad \nu_1 + \nu_2 + \dots + \nu_N = 1. \quad (19)$$

By definition, $F = -T \log \mathcal{Z}$ is the nonperturbative dyon-induced free energy as function of the holonomy; for $N = 2, 3$ it is plotted in Fig. 1, bottom. Evidently, it has the minimum at

$$\nu_1 = \nu_2 = \dots = \nu_N = \frac{1}{N} \quad (20)$$

corresponding to equidistant, that is confining values of μ 's (5)! At the minimum, the free energy is

$$F_{\text{min}} = -T \log \mathcal{Z}_{\text{min}} = -4\pi fVT = -\frac{16\pi^2}{g^4} \Lambda^4 V = \frac{N^2}{4\pi^2} \frac{\Lambda^4}{\lambda^2} V \quad (21)$$

and there are no Coulomb corrections to this result. In the last equation we have introduced the N -independent 't Hooft coupling $\lambda \equiv \alpha_s N/2\pi$.

We note that the free energy is $\mathcal{O}(N^2)$ as expected on general N -counting grounds and that it is temperature-independent. It corresponds to $\log \mathcal{Z}$ being proportional to the 4-volume $V^{(4)} = V/T$, demonstrating the expected extensive behavior at low temperature.

9. Statistical physics of dyons as a Quantum Field Theory

Although the Coulomb interactions of dyons cancel exactly in the free energy, the dyon ensemble defined by Eq. (18) is not a free gas but a highly correlated system. To facilitate computing observables through correlation functions, we rewrite Eq. (18) as an equivalent quantum field theory. As a byproduct, we shall also check that the result for the free energy (19) is correct.

To proceed to the quantum field theory description we use two mathematical tricks.

1. “Fermionization” (Berezin [25]). It is helpful to exponentiate the Coulomb interactions rather than keeping them in $\det G$. To that end one presents the determinant of a matrix as an integral over a finite number of anticommuting Grassmann variables:

$$\det(G_{AB}) = \int \prod_A d\psi_A^\dagger d\psi_A \exp\left(\psi_A^\dagger G_{AB} \psi_B\right).$$

Now we have the two-body Coulomb interactions in the exponent and it is possible to use the second trick presenting Coulomb interactions with the help of a functional integral over an auxiliary boson field.

2. “Bosonization” (Polyakov [26]). One can write

$$\begin{aligned} \exp\left(\sum_{m,n} \frac{Q_m Q_n}{|\mathbf{x}_m - \mathbf{x}_n|}\right) &= \int D\phi \exp\left[-\int d\mathbf{x} \left(\frac{1}{16\pi} \partial_i \phi \partial_i \phi + \rho \phi\right)\right] \\ &= \exp\left(\int \rho \frac{4\pi}{\Delta} \rho\right), \\ \rho &= \sum Q_m \delta(\mathbf{x} - \mathbf{x}_m). \end{aligned}$$

After applying the first trick the “charges” Q_m become Grassmann variables but after applying the second one, it becomes easy to integrate them out since the square of a Grassmann variable is zero. In fact one needs $2N$ boson fields v_m, w_m to reproduce diagonal elements of G and $2N$ anticommuting (“ghost”) fields χ_m^\dagger, χ_m to present the non-diagonal elements. The

chain of identities is accomplished in Ref. [2] and the result for the partition function (18) is, identically, a path integral defining a quantum field theory in 3 dimensions

$$\mathcal{Z} = \int D\chi^\dagger D\chi Dv Dw \exp \int d^3x \left\{ \frac{T}{4\pi} \left(\partial_i \chi_m^\dagger \partial_i \chi_m + \partial_i v_m \partial_i w_m \right) + f \left[(-4\pi\mu_m + v_m) \frac{\partial \mathcal{F}}{\partial w_m} + \chi_m^\dagger \frac{\partial^2 \mathcal{F}}{\partial w_m \partial w_n} \chi_n \right] \right\}, \tag{22}$$

$$\mathcal{F} = \sum_{m=1}^N e^{w_m - w_{m+1}}. \tag{23}$$

The fields v_m have the meaning of the asymptotic Abelian electric potentials of dyons

$$\begin{aligned} (A_4)_{mn} &= \delta_{mn} A_{m4}, \\ \frac{A_{m4}(\mathbf{x})}{T} &= 2\pi\mu_m - \frac{1}{2}v_m(\mathbf{x}), \\ \mathbf{E}_m &= \nabla A_{m4}, \end{aligned} \tag{24}$$

while w_m have the meaning of the dual (or magnetic) Abelian potentials. Note that the kinetic energy for the v_m, w_m fields has only the mixing term $\partial_i v_m \partial_i w_m$ which is nothing but the Abelian duality transformation $\mathbf{E} \cdot \mathbf{B}$. The function $\mathcal{F}(w)$ (23) where one assumes a cyclic summation over m , is known as the periodic (or affine) Toda lattice.

Although the Lagrangian in Eq. (22) describes a highly nonlinear interacting quantum field theory, it is in fact exactly solvable! To prove it, one observes that the fields v_m enter the Lagrangian only linearly, therefore one can integrate them out. It leads to a functional δ -function:

$$\int Dv_m \longrightarrow \delta \left(-\frac{T}{4\pi} \partial^2 w_m + f \frac{\partial \mathcal{F}}{\partial w_m} \right). \tag{25}$$

This δ -function restricts possible fields w_m over which one still has to integrate in Eq. (22). Let \bar{w}_m be a solution to the argument of the δ -function. Integrating over small fluctuations about \bar{w} gives the Jacobian

$$\text{Jac} = \det^{-1} \left(-\frac{T}{4\pi} \partial^2 \delta_{mn} + f \frac{\partial^2 \mathcal{F}}{\partial w_m \partial w_n} \Big|_{w=\bar{w}} \right). \tag{26}$$

Remarkably, exactly the same functional determinant (but in the numerator) arises from integrating over the ghost fields, for any background \bar{w} .

Therefore, all quantum corrections cancel *exactly* between the boson and ghost fields (a characteristic feature of supersymmetry), and the ensemble of dyons is basically governed by a classical field theory.

To find the ground state we examine the fields' potential energy being $-4\pi f \mu_m \partial \mathcal{F} / \partial w_m$ which we prefer to write restoring $\nu_m = \mu_{m+1} - \mu_m$ and \mathcal{F} as

$$\mathcal{P} = -4\pi f V \sum_m \nu_m e^{w_m - w_{m+1}} \quad (27)$$

(the volume factor arises for constant fields w_m). One has first to find the stationary point in w_m for a given set of ν_m 's. It leads to the equations

$$\frac{\partial \mathcal{P}}{\partial w_m} = 0,$$

whose solution is

$$e^{w_1 - w_2} = \frac{(\nu_1 \nu_2 \nu_3 \dots \nu_N)^{\frac{1}{N}}}{\nu_1}, \quad e^{w_2 - w_3} = \frac{(\nu_1 \nu_2 \nu_3 \dots \nu_N)^{\frac{1}{N}}}{\nu_2}, \quad \text{etc.} \quad (28)$$

Putting it back into Eq. (27) we obtain

$$\mathcal{P} = -4\pi f V N (\nu_1 \nu_2 \dots \nu_N)^{\frac{1}{N}}, \quad \nu_1 + \nu_2 + \dots + \nu_N = 1, \quad (29)$$

which is exactly what one gets from a direct calculation of the partition function, outlined in the previous section, see Eq. (19). The minimum is achieved at the equidistant, confining value of the holonomy, see Eqs. (5), (20). Using field-theoretic methods, we have also proven that the result is exact, as all potential quantum corrections cancel. It is in line with the exact cancelation of the Coulomb interactions in the determinant.

Given this cancelation, the key finding — that the dyon-induced free energy has the minimum at the confining value of holonomy — is trivial. If all Coulomb interactions cancel after integration over dyons' positions, the weight of a many-dyon configuration is the same as if they were infinitely dilute (although they are not). Then the weight, what concerns the holonomy, is proportional to the product of diagonal matrix elements of G in the dilute limit, that is to the normalization integrals for dyon zero modes. These are nothing but the field strengths $F_{\mu\nu}$ of individual dyons, hence the normalization is proportional to the product of the dyon actions $\sim \nu_m$ where $\nu_m = \mu_{m+1} - \mu_m$ and $\nu_N = \mu_1 + 1 - \mu_N$ such that $\nu_1 + \nu_2 + \dots + \nu_N = 1$. The sum of all N kinds of dyons' actions is fixed and equal to the instanton action, however, it is the *product* of actions that defines the weight. The product is maximal when all actions are equal, hence the equidistant or confining μ 's are statistically preferred. Thus, the average Polyakov line is zero, $\langle \text{Tr } L \rangle = 0$.

10. Heavy quark potential

The field-theoretic representation of the dyon ensemble enables one to compute various Yang–Mills correlation functions in the semiclassical approximation. The key observables relevant to confinement are the correlation function of two Polyakov lines (defining the heavy quark potential), and the average of large Wilson loops. A detailed calculation of these quantities is performed in Ref. [2]; here we only present the results and discuss the meaning.

10.1. N -ality and k -strings

From the viewpoint of confinement, all irreducible representations of the $SU(N)$ group fall into N classes: those that appear in the direct product of any number of adjoint representations, and those that appear in the direct product of any number of adjoint representations with the irreducible representation being the rank- k antisymmetric tensor, $k = 1, \dots, N-1$. “ N -ality” is said to be zero in the first case and equal to k in the second. N -ality-zero representations transform trivially under the center of the group Z_N ; the rest acquire a phase $2\pi k/N$.

One expects that there is no asymptotic linear potential between static color sources in the adjoint representation as such sources are screened by gluons. If a representation is found in a direct product of some number of adjoint representations and a rank- k antisymmetric representation, the adjoint ones “cancel out” as they can be all screened by an appropriate number of gluons. Therefore, from the confinement viewpoint all N -ality = k representations are equivalent and there are only $N - 1$ string tensions $\sigma_{k,N}$ being the coefficients in the *asymptotic* linear potential for sources in the antisymmetric rank- k representation. They are called “ k -strings”. The representation dimension is $d_{k,N} = N!/(k!(N-k)!)$ and the eigenvalue of the quadratic Casimir operator is $C_{k,N} = (N+1)/(2N)k(N-k)$.

The value $k = 1$ corresponds to the fundamental representation whereas $k = N - 1$ corresponds to the representation conjugate to the fundamental [quarks and anti-quarks]. In general, the rank- $(N - k)$ antisymmetric representation is conjugate to the rank- k one; it has the same dimension and the same string tension, $\sigma_{k,N} = \sigma_{N-k,N}$. Therefore, for odd N all string tensions appear in equal pairs; for even N , apart from pairs, there is one privileged representation with $k = N/2$ which has no pair and is real. The total number of different string tensions is thus $[N/2]$.

The behavior of $\sigma_{k,N}$ as function of k and N is an important issue as it discriminates between various confinement mechanisms. On general N -counting grounds one can only infer that at large N and $k \ll N$, $\sigma_{k,N}/\sigma_{1,N} = (k/N)(1 + \mathcal{O}(1/N^2))$. Important, there should be no $\mathcal{O}(N^{-1})$ correc-

tion [27]. A popular version called ‘‘Casimir scaling’’, according to which the string tension is proportional to the Casimir operator for a given representation (it stems from an idea that confinement is somehow related to the modification of a one-gluon exchange at large distances), does not satisfy this restriction.

10.2. Correlation function of Polyakov lines

To find the potential energy $V_{k,N}$ of static ‘‘quark’’ and ‘‘antiquark’’ transforming according to the antisymmetric rank- k representation, one has to consider the correlation of Polyakov lines in the appropriate representation:

$$\left\langle \text{Tr } L_{k,N}(\mathbf{z}_1) \text{Tr } L_{k,N}^\dagger(\mathbf{z}_2) \right\rangle = \text{const.} \exp\left(-\frac{V_{k,N}(\mathbf{z}_1 - \mathbf{z}_2)}{T}\right). \quad (30)$$

Far away from dyons’ cores the field is Abelian and in the field-theoretic language of Eq. (22) is given by Eq. (24). Therefore, the Polyakov line in the fundamental representation is

$$\text{Tr } L(\mathbf{z}) = \sum_{m=1}^N Z_m, \quad Z_m = \exp\left(2\pi i \mu_m - \frac{i}{2} v_m(\mathbf{z})\right). \quad (31)$$

In the general antisymmetric rank- k representation

$$\text{Tr } L_{k,N}(\mathbf{z}) = \sum_{m_1 < m_2 < \dots < m_k}^N Z_{m_1} Z_{m_2} \dots Z_{m_k}, \quad (32)$$

where cyclic summation from 1 to N is assumed.

The average (30) can be computed from the quantum field theory (22). Inserting the two Polyakov lines (32) into Eq. (22) we observe that the Abelian electric potential v_m enters linearly in the exponent as before. Therefore, it can be integrated out, leading to a δ -function for the dual field w_m , which is now shifted by the source (*cf.* Eq. (25)):

$$\int Dv_m \longrightarrow \prod_m \delta\left(-\frac{T}{4\pi} \partial^2 w_m + f \frac{\partial \mathcal{F}}{\partial w_m} - \frac{i}{2} \delta(\mathbf{x} - \mathbf{z}_1)(\delta_{mm_1} + \dots + \delta_{mm_k}) + \frac{i}{2} \delta(\mathbf{x} - \mathbf{z}_2)(\delta_{mm_1} + \dots + \delta_{mm_k})\right). \quad (33)$$

One has to find the dual field $w_m(\mathbf{x})$ nullifying the argument of this δ -function, plug it into the action

$$\exp\left(\int d\mathbf{x} \frac{4\pi f}{N} \mathcal{F}(w)\right), \quad (34)$$

and sum over all sets $\{m_1 < m_2 < \dots < m_k\}$, $\{n_1 < n_2 < \dots < n_k\}$ with the weight $\exp(2\pi i(m_1 + \dots + m_k - n_1 - \dots - n_k)/N)$. The Jacobian from resolving the δ -function again cancels exactly with the determinant arising from ghosts. Therefore, the calculation of the correlator (30), sketched above, is exact.

At large separations between the sources $|z_1 - z_2|$, the fields w_m resolving the δ -function are small and one can expand the Toda chain:

$$\begin{aligned} \mathcal{F}(w) &= \sum_m e^{w_m - w_{m+1}} \approx N + \frac{1}{2} w_m \mathcal{M}_{mn} w_n, \\ \frac{\partial \mathcal{F}}{\partial w_m} &\approx \mathcal{M}_{mn} w_n, \end{aligned} \tag{35}$$

where

$$\mathcal{M} = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 & -1 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -1 & 0 & 0 & \dots & -1 & 2 \end{pmatrix}. \tag{36}$$

As apparent from Eq. (35), the eigenvalues of \mathcal{M} determine the spectrum of the dual fields w_m . There is one zero eigenvalue which decouples from everywhere, and $N - 1$ nonzero eigenvalues

$$\mathcal{M}^{(k)} = \left(2 \sin \frac{\pi k}{N} \right)^2, \quad k = 1, \dots, N - 1. \tag{37}$$

Certain orthogonality relation imposes the selection rule: the asymptotics of the correlation function of two Polyakov lines in the antisymmetric rank- k representation is determined by precisely the k^{th} eigenvalue. We obtain [2]

$$\left\langle \text{Tr } L_{k,N}(z_1) \text{Tr } L_{k,N}^\dagger(z_2) \right\rangle \stackrel{z_{12} \rightarrow \infty}{\sim} \text{const. exp} \left(-|z_1 - z_2| M \sqrt{\mathcal{M}^{(k)}} \right), \tag{38}$$

where M is the “dual photon” mass,

$$M^2 = \frac{4\pi f}{T} = \frac{16\pi^2 \Lambda^4}{g^4 T^2} = \mathcal{O}(N^2). \tag{39}$$

Comparing it with the definition of the heavy quark potential (30) we find that there is an asymptotically linear potential between static “quarks” in any N -ality nonzero representation, with the k -string tension

$$\sigma_{k,N} = MT \sqrt{\mathcal{M}^{(k)}} = 2MT \sin \frac{\pi k}{N} = \frac{\Lambda^2}{\lambda} \frac{N}{\pi} \sin \frac{\pi k}{N}. \tag{40}$$

This is the so-called “sine regime”: it has been found before in certain supersymmetric theories [28]. Lattice simulations [29] support this regime, whereas another lattice study [30] gives somewhat smaller values but within two standard deviations from the values following from Eq. (40). For a general discussion of the sine regime for k -strings, which is favored from many viewpoints, see [27].

We see that at large N and $k \ll N$, $\sigma_{k,N}/\sigma_{1,N} = (k/N)(1 + \mathcal{O}(1/N^2))$, as it should be on general grounds, and that all k -string tensions have a finite limit at zero temperature.

11. Area law for large Wilson loops

When dealing with the ensemble of dyons, it is convenient to use a gauge where A_4 is diagonal (*i.e.* Abelian). This necessarily implies Dirac string singularities sticking from dyons, which are however gauge artifacts as they do not carry any energy. Moreover, the Dirac strings’ directions are also subject to the freedom of the gauge choice. For example, one can choose the gauge in which N dyons belonging to a neutral cluster are connected by Dirac strings. This choice is, however, not convenient for the ensemble as dyons have to loose their “memory” to what particular instanton they belong to. The natural gauge is where all Dirac strings of all dyons are directed to infinity along some axis, *e.g.* along the z axis. The dyons’ field in this gauge is given explicitly in Ref. [8] (for the SU(2) group).

In this gauge, the magnetic field of dyons beyond their cores is Abelian and is a superposition of the Abelian fields of individual dyons. For large Wilson loops we are interested in, it is this superposition field of a large number of dyons that contributes most as they have a slowly decreasing $1/|\mathbf{x}-\mathbf{x}_i|$ asymptotics, hence the use of the field outside the cores is justified. Owing to self-duality,

$$[B_i(\mathbf{x})]_{mn} = [\partial_i A_4(\mathbf{x})]_{mn} = -\frac{T}{2} \delta_{mn} \partial_i v_m(\mathbf{x}), \quad (41)$$

cf. Eq. (24). Since A_i is Abelian beyond the cores, one can use the Stokes theorem for the spatial Wilson loop:

$$W \equiv \text{Tr } \mathcal{P} \exp i \oint A_i dx^i = \text{Tr} \exp i \int B_i d^2 \sigma^i = \sum_m \exp \left(-i \frac{T}{2} \int d^2 \sigma^i \partial_i v_m \right). \quad (42)$$

Eq. (42) may look contradictory as we first use $B_i = \text{curl} A_i$ and then $B_i = \partial_i A_4$. Actually there is no contradiction as the last equation is true up to Dirac string singularities which carry away the magnetic flux. If the Dirac string pierces the surface spanning the loop it gives a quantized contribution

$\exp(2\pi i \times \text{integer}) = 1$; one can also use the gauge freedom to direct Dirac strings parallel to the loop surface in which case there is no contribution from the Dirac strings at all.

Let us take a flat Wilson loop lying in the (xy) plane at $z = 0$. Then Eq. (42) is continued as

$$\begin{aligned} W &= \sum_m \exp \left(-i \frac{T}{2} \int_{x,y \in \text{Area}} d^3x \partial_z v_m \delta(z) \right) \\ &= \sum_m \exp \left(i \frac{T}{2} \int_{x,y \in \text{Area}} d^3x v_m \partial_z \delta(z) \right). \end{aligned} \tag{43}$$

It means that the average of the Wilson loop in the dyons ensemble is given by the partition function (22) with the source

$$\sum_m \exp \left(i \frac{T}{2} \int d^3x v_m \frac{d\delta(z)}{dz} \theta(x, y \in \text{Area}) \right),$$

where $\theta(x, y \in \text{Area})$ is a step function equal to unity if x, y belong to the area inside the loop and equal to zero otherwise. As in the case of the Polyakov lines the presence of the Wilson loop shifts the argument of the δ -function arising from the integration over the v_m variables, and the average Wilson loop in the fundamental representation is given by the equation

$$\begin{aligned} \langle W \rangle &= \sum_{m_1} \int Dw_m \exp \left(\int d\mathbf{x} \frac{4\pi f}{N} \mathcal{F}(w) \right) \det \left(-\frac{T}{4\pi} \partial^2 \delta_{mn} + f \frac{\partial^2 \mathcal{F}}{\partial w_m \partial w_n} \right) \\ &\times \prod_m \delta \left(-\frac{T}{4\pi} \partial^2 w_m + f \frac{\partial \mathcal{F}}{\partial w_m} + \frac{iT}{2} \delta_{mm_1} \frac{d\delta(z)}{dz} \theta(x, y \in \text{Area}) \right). \end{aligned} \tag{44}$$

Therefore, one has to solve the non-linear equations on w_m 's with a source along the surface of the loop

$$-\partial^2 w_m + M^2 (e^{w_m - w_{m+1}} - e^{w_{m-1} - w_m}) = -2\pi i \delta_{mm_1} \frac{d\delta(z)}{dz} \theta(x, y \in \text{Area}), \tag{45}$$

for all m_1 , plug it into the action $(4\pi f/N)\mathcal{F}(w)$, and sum over m_1 . In order to evaluate the average of the Wilson loop in a general antisymmetric rank- k representation, one has to take the source in Eq. (45) as $-2\pi i \delta'(z) (\delta_{mm_1} + \dots + \delta_{mm_k})$ and sum over $m_1 < \dots < m_k$ from 1 to N , see Eq. (32). Again, the ghost determinant cancels exactly the Jacobian from the fluctuations of w_m about the solution, therefore the classical-field calculation is exact.

Contrary to the case of the Polyakov lines, one cannot, generally speaking, linearize Eq. (45) in w_m but has to solve the non-linear equations as they are. The Toda equations (45) with a $\delta'(z)$ source in the r.h.s. define “pinned soliton” solutions $w_m(z)$ that are $1d$ functions in the direction transverse to the surface spanning the Wilson loop but do not depend on the coordinates x, y provided they are taken inside the loop. Beyond that surface $w_m = 0$. Along the perimeter of the loop, w_m interpolate between the soliton and zero. For large areas, the action (34) is, therefore, proportional to the area of the surface spanning the loop, which gives the famous area law for the average Wilson loop. The coefficient in the area law, the “magnetic” string tension, is found from integrating the action on the solution $w_m(z)$ in the z direction.

The exact solutions of Eq. (45) for any N and any representation k have been found in Ref. [2], and the resulting “magnetic” string tension turns out to be

$$\sigma_{k,N} = \frac{A^2}{\lambda} \frac{N}{\pi} \sin \frac{\pi k}{N}, \quad (46)$$

which coincides with the “electric” string tension (40) found from the correlators of the Polyakov lines, for all k -strings!

Several comments are in order here.

- The “electric” and “magnetic” string tensions should coincide only in the limit $T \rightarrow 0$ where the Euclidean $O(4)$ symmetry is restored. Both calculations have been in fact performed in that limit as we have ignored the temperature-dependent perturbative potential (7). If it is included, the “electric” and “magnetic” string tensions split.
- Despite that the theory (22) is 3-dimensional, with the temperature entering just as a parameter in the Lagrangian, it “knows” about the restoration of Euclidean $O(4)$ symmetry at $T \rightarrow 0$.
- The “electric” and “magnetic” string tensions are technically obtained in very different ways: the first is related to the mass of the elementary excitation of the dual fields w_m , whereas the latter is related to the mass of the dual field soliton.

12. Cancellation of gluons in the confinement phase

To prove confinement, it is insufficient to demonstrate the area law for large Wilson loops and the zero average for the Polyakov line: it must be shown that there are no massless gluons left in the spectrum. We give an argument that this indeed happens in the dyon vacuum.

A manifestation of massless gluons in perturbation theory is the Stefan–Boltzmann law for the free energy:

$$-\frac{T}{V} \log \mathcal{Z}_{\text{SB}} = \frac{F_{\text{SB}}}{V} = -\frac{\pi^2}{45} T^4 (N^2 - 1). \quad (47)$$

It is proportional to the number of gluons $N^2 - 1$ and has the T^4 behavior characteristic of massless particles. In the confinement phase, neither is permissible: If only glueballs are left in the spectrum the free energy must be $\mathcal{O}(N^0)$ and the temperature dependence must be very weak until $T \approx T_c$ where it abruptly rises owing to the excitation of many glueballs.

As explained in Section 8, the ensemble of dyons has a nonperturbative free energy

$$\frac{F_{\text{dyon}}}{V} = -\frac{N^2}{2\pi^2} \frac{\Lambda^4}{\lambda^2}. \quad (48)$$

It is $\mathcal{O}(N^2)$ but temperature-independent. We have doubled F_{min} from Eq. (21) keeping in mind that there are also anti-dyons and assuming that their interactions with dyons is not as strong as the interactions between dyons and anti-dyons separately, as induced by the determinant measure (17), therefore treating dyons and anti-dyons as two independent “liquids”. (By the same logic, the string tension (40) has to be multiplied by $\sqrt{2}$ as due to anti-dyons.)

Dyons force the system to have the “most nontrivial” holonomy (5). For that holonomy, the perturbative potential energy (7) is at its maximum equal to

$$\frac{F_{\text{pert, max}}}{V} = \frac{\pi^2}{45} T^4 \left(N^2 - \frac{1}{N^2} \right). \quad (49)$$

The full free energy is the sum of the three terms above.

We see that the leading $\mathcal{O}(N^2)$ term in the Stefan–Boltzmann law is *canceled* by the potential energy precisely at the confining holonomy point and nowhere else! In fact it seems to be the only way how $\mathcal{O}(N^2)$ massless gluons can be canceled out of the free energy, and the main question shifts to why does the system prefer the “most nontrivial” holonomy. Dyons answer that question.

13. Deconfinement phase transition

As the temperature rises, the perturbative free energy grows as T^4 and eventually it overcomes the negative nonperturbative free energy (48). At this point, the trivial holonomy for which both the perturbative and non-perturbative free energy are zero, becomes favorable. Therefore, an estimate

of the critical deconfinement temperature comes from equating the sum of Eq. (48) and Eq. (49) to zero, which gives

$$T_c^4 = \frac{45}{2\pi^4} \frac{N^4}{N^4 - 1} \frac{\Lambda^4}{\lambda^2}. \quad (50)$$

As expected, it is stable in N . A more robust quantity, both from the theoretical and lattice viewpoints, is the ratio $T_c/\sqrt{\sigma}$ since in this ratio the poorly known parameters Λ and λ cancel out:

$$\frac{T_c}{\sqrt{\sigma}} = \left(\frac{45}{4\pi^4} \frac{\pi^2 N^2}{(N^4 - 1) \sin^2 \frac{\pi}{N}} \right)^{\frac{1}{4}} \xrightarrow{N \rightarrow \infty} \frac{1}{\pi} \left(\frac{45}{4} \right)^{\frac{1}{4}} + \mathcal{O} \left(\frac{1}{N^2} \right). \quad (51)$$

In Table I, we compare the values from Eq. (51) to those measured in lattice simulations of the pure $SU(N)$ gauge theories [31]; there is a surprisingly good agreement. A detailed study of the thermodynamics of the phase transition will be published elsewhere.

TABLE I

	SU(3)	SU(4)	SU(6)	SU(8)
$T_c/\sqrt{\sigma}$, theory	0.6430	0.6150	0.5967	0.5906
$T_c/\sqrt{\sigma}$, lattice	0.6462(30)	0.6344(81)	0.6101(51)	0.5928(107)

14. Relation to other suggestions to explain confinement

Several mechanisms of confinement have been suggested in the past. The most popular are

- condensation of monopoles, or the dual Meissner effect [32, 33],
- proliferation of center vortices [32, 34], see a modern overview [35].

These two mechanisms and in particular lattice evidence supporting them have been reviewed by Greensite [35] and we are not going to repeat it here. What is important, both monopoles and vortices are identified on a lattice by fixing the gauge — choosing the “maximally Abelian” gauge in the first case and the “maximally center” gauge in the second. If this gauge-fixing procedure is applied to the dyon vacuum of the present paper, the maximally Abelian gauge would probably reveal lattice monopoles where dyons are placed, and a subsequent application of the maximally center gauge would probably reveal center vortices which would be nothing but the

phantom Dirac strings connecting dyons. Therefore, lattice findings that “there is no confinement without Abelian monopoles” and that “there is no confinement without center vortices” is presumably in no contradiction with the vacuum being formed by dyons. Moreover, recently there have been direct observations of dyons on the lattice by the Humboldt Universität — ITEP group, see [36] and further references therein.

Some time ago we have observed that standard instantons are also capable of yielding confinement, provided the instanton size distribution falls off as $1/\rho^3$ at large sizes ρ [37]. This regime implies, however, that large-size instantons inevitably overlap, since in 4D the packing fraction is proportional to the fourth moment of the size distribution $\overline{\rho^4}$ which is divergent. Therefore, the usual instantons’ “center-size-orientation” parameterization being all right for dilute systems is inapplicable for the confinement purposes. One needs a parametrization of the collective instantons’ coordinates that is as good for overlapping solutions as it is for dilute ones.

In an analogous 2D CP^{N-1} model also possessing instantons such a parameterization has long been known: instantons there are parameterized by the positions of N kinds of “instanton quarks”. The measure of the moduli space of multi-instantons is fortunately known exactly [38] and is given by a holomorphic function of the instanton quarks’ coordinates. The measure is invariant under permutation of the instanton quarks (they should not ‘know’ what instanton they belong to) and is perfectly valid for overlapping instantons, as well as for dilute ones. In the latter case the measure becomes the product of instanton “center-size-orientation” measures [39].

In the 4D YM theory a similar parameterization of multi-instantons has long been sought, starting from the pioneering work of Callan, Dashen and Gross who suggested “merons” as instanton constituents [40], but that did not work as merons had a divergent action. Zhitnitsky [41], Petrov and myself put much effort in identifying “instanton quarks” for the YM solutions but real progress has been achieved in constructing the KvBLL instantons [9, 10] whose constituents have been found to be the BPS monopoles, or dyons. The price is that one is obliged to take nonzero temperatures, however if one is interested in the zero-temperature case, T can be considered as an infrared regulator which is safe to put to zero at the end, if needed.

The measure of the multi-instanton space (17) is now written in terms of the coordinates of the constituent dyons. The metric is hyper-Kähler (which is the 4D analogue of holomorphy in 2D), the measure is invariant under permutation of dyons (they should not “know” what instanton they belong to) and is presumably valid for overlapping instantons, as well as for dilute ones. In the latter case the measure becomes the product of the instanton “center-size-orientation” measures [20]. Therefore, it seems to be the solution of a long-standing problem.

Two steps in modernizing the semi-classical “instanton liquid” model [6] are critical in getting confinement:

- generalizing instantons in such a way that they can have arbitrary holonomy, and allowing nontrivial holonomy, despite that in perturbation theory it is forbidden,
- writing the quantum weight of instantons with nontrivial holonomy through coordinates of constituent dyons, such that it is applicable for overlapping instantons.

What happens, can be summarized as follows:

- The ensemble of dyons favors dynamically the “most nontrivial” or confining value of the holonomy. This is almost clear, given that the weight is proportional to the product of individual actions of N kinds of dyons.
- Dyons form a sort of Coulomb plasma (but an exactly solvable variant of it) with an appearance of the Debye mass both for “electric” and “magnetic” (dual) photons. The first give rise to the exponential fall-off of the correlation of two Polyakov lines, *i.e.* to the linear heavy-quark potential, the second yield the area law for spatial Wilson loops.
- $N^2 - 1$ massless gluons cancel out from the free energy, and only massive (string?) excitations are left.

We do not see the quantum-mechanical condensation of monopoles; it is hence a new mechanism of confinement.

15. Why does it work and what should be done next?

The reason why a semiclassical approximation works well for strong interactions (where all dimensionless quantities are, generally speaking, of the order of unity) is not altogether clear. A possible justification has been outlined in Section 1: After UV renormalization is performed about the classical saddle points and the scale parameter Λ appears as the result of the dimensional transmutation, further quantum corrections to the saddle point is a series in the running 't Hooft coupling λ whose argument is typically the largest scale in the theory, in this case $\max(T, n^{1/4})$ where n is the 4D density of the dyons. An estimate shows that the running λ is between 1/4 at zero temperature and 1/7 or less at critical temperature. Therefore, although these numbers are “of the order of unity”, in practical terms they indicate that high order loop corrections are not too large. Let us recall that quite an accurate computation of anomalous dimensions in critical phenomena from the ϵ -expansion by Fisher and Wilson [42] is based on truncating the Taylor expansion in ϵ at the first couple of terms, where $\epsilon = 1$ or sometimes 2!¹

¹ I take the opportunity to thank Michael Fisher and Valery Pokrovsky for a discussion of this numerical miracle.

Unfortunately, approximations made in Ref. [2] and reproduced above are not limited to higher loop corrections. We have (i) ignored dyon interactions induced by the small oscillation determinant over nonzero modes, except the potential energy as function of the holonomy, (ii) ignored the interactions of dyons of different duality, treating them as two noninteracting “liquids”, (iii) conjectured a simple form of the dyon measure which may be incorrect when two *same-kind* dyons come close. Although certain justification for these approximations can be put forward (see above and Ref. [2]) it is desirable not to use them at all, and that may be possible.

These mathematical problems are of course in the line, as well as further physical problems, probably the most urgent being switching in light dynamical quarks into the dyon vacuum, that is moving into the realm of the real-world QCD. The main problem there is the spontaneous breaking of chiral symmetry. Although we do not think that its mechanism will differ dramatically from that found in Ref. [43], as due to the delocalization of the near-zero fermion modes, it would be very interesting to see how the ensuing effective chiral Lagrangian “knows” about the confinement.

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