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A COMPARISON OF THE CUT-OFF EFFECTS FOR TWISTED MASS, OVERLAP AND CREUTZ FERMIONS AT TREE-LEVEL OF PERTURBATION THEORY*

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In this paper we investigate the cut-off effects at tree-level of perturbation theory for three different lattice regularizations of fermions — maximally twisted mass Wilson, overlap and Creutz fermions. We show that all three kinds of fermions exhibit the expected $\mathcal{O}(a^2)$ scaling behaviour in the lattice spacing. Moreover, the size of these cut-off effects for the considered quantities *i.e.* the pseudoscalar correlation function $C_{\rm PS}$, the mass $m_{\rm PS}$ and the decay constant $f_{\rm PS}$ is comparable for all of them.

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1. Introduction

The main goal of Lattice Field Theory is to study the non-perturbative aspects of quantum field theories, in particular Quantum Chromodynamics (QCD). For example, Lattice QCD is a regularization of QCD which consists in putting the theory on a four dimensional lattice (discretization) with lattice spacing a, whose inverse is the ultraviolet cutoff of the theory. The discretization of bosons is relatively straightforward, but when one tries to

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discretize fermions in the naive way, the notorious fermion doubling problem emerges — instead of one fermion in the continuum limit, one has as many as 2^d fermions, where d is the space-time dimensionality. As it was originally proposed by Wilson [1], the doubling problem can be solved if a different fermionic discretization is chosen, the so called Wilson fermions. After Wilson's proposal, many alternative fermion discretizations which remove the doubling problem have been suggested and are still appearing. However, as stated by the Nielsen–Ninomiya theorem [2], new problems will always appear when removing the doublers; in order to eliminate the fermion doubling problem one has to pay the price of either explicitly breaking chiral symmetry (even in the massless limit), or giving up locality or translational invariance.

Much of the effort of lattice QCD goes into finding a lattice theory without doublers which keeps the largest possible number of symmetries, and at the same time reaches the continuum limit as fast as possible (the dependence on the inverse cutoff, a, is as small as possible *e.g.* $\mathcal{O}(a^2)$ leading cut-off dependence is better than $\mathcal{O}(a)$).

In this paper, we investigate the cut-off effects at tree-level of perturbation theory of three different discretizations of fermions — twisted mass Wilson fermions at maximal twist (MTM), overlap fermions and Creutz fermions, at a fixed value of the physical quark mass. We have presented a similar analysis for a different value of the quark mass in [3]. Here we compare the results.

The MTM fermions [4, 5] are relatively cheap to simulate and they are by now a widely used fermion discretization. Although similar to Wilson fermions, they retain a subgroup of chiral symmetry which guarantees automatic $\mathcal{O}(a)$ improvement, *i.e.* $\mathcal{O}(a^2)$ leading cut-off effects. The price to pay to have a residual chiral symmetry, is to break a subgroup of the isospin symmetry transformation[1].

It has been shown by Ginsparg and Wilson [7] that there is a way to preserve chiral symmetry on the lattice, even without the doubler modes, if the corresponding Dirac operator obeys a relation now called the Ginsparg–Wilson relation. It is a non-standard realisation of chiral symmetry [8], because the Dirac operator no longer anticommutes with γ_5 at non-zero lattice spacing, but it only anticommutes with a lattice modified version of γ_5. A particularly simple form of a Dirac operator that obeys the Ginsparg–Wilson relation has been found by Neuberger [9]. The main disadvantage of overlap fermions is that they are much more costly to simulate — by a factor of 30–120 in comparison with MTM fermions[2].

[1] For a recent review of twisted mass fermions see [6].

[2] For a review of overlap fermions see *e.g.* [10]. For a comparison with twisted mass fermions see [11].

The recently proposed Creutz fermions [12] represent the class of minimally-doubled fermions[3]. They describe two flavours of quarks and preserve exact chiral symmetry. However, they break a number of discrete symmetries and isospin symmetry and this would make their simulation very difficult[4].

2. Setup

2.1. Correlation functions

In order to investigate the cut-off dependence of the pseudoscalar meson mass and decay constant we have to first calculate the correlation function corresponding to this meson[5]. Despite the fact that we work at tree-level of perturbation theory, we will refer to the pseudoscalar meson as the "pion".

The charged pions are described by the following interpolating operator:

$$\mathcal{P}^{\pm}(x) \equiv \mathcal{P}^1(x) \mp i\mathcal{P}^2(x)\,, \tag{1}$$

where the pseudoscalar density $\mathcal{P}^a(x) = \bar{\psi}(x)\gamma_5(\tau^a/2)\psi(x)$ (for $a = 1, 2, 3$) and τ^a are the Pauli matrices.

The time dependence of the correlation function $C_{\mathrm{PS}}(t)$ is thus given by:

$$C_{\mathrm{PS}}(t) = -\sum_{\vec{x}} \langle 0|\mathcal{P}^+(x)\mathcal{P}^-(0)|0\rangle\,. \tag{2}$$

Performing all the possible Wick's contractions, one obtains the dependence of the pion correlation function on the quark propagator, $S_\mu(p)$, given by

$$C_{\mathrm{PS}}(t) = \frac{N_{\mathrm{c}}N_{\mathrm{D}}}{L^3T^2} \sum_{p_4} \sum_{p_4'} \sum_{\vec{p}} \sum_{\mu} e^{i(p_4-p_4')t}\, S_\mu(\vec{p}, p_4) S^*_\mu(\vec{p}, p_4')\,. \tag{3}$$

N_{c} denotes the number of colours and N_{D} the number of Dirac components. $L = aN$ is the physical extent of the lattice in the spatial directions (N is the number of lattice sites in all spatial directions) and $T = aN_4$ the physical extent in the temporal direction (N_4 the number of lattice points in the time direction). The possible choices of the index μ will be explained below (see Eq. (10)). The numerical computation of correlation functions consists in directly evaluating the expression (3).

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[3] Other examples of minimally-doubled fermions were given by Karsten [13] and Wilczek [14].

[4] For a discussion of this aspect of Creutz fermions see [15].

[5] For a pedagogical introduction to the methods we have used in this work see [16].

2.2. Quark propagators

We present in this section the analytical expressions of the quark propagators, in momentum space and at tree-level of perturbation theory, for the three kinds of lattice fermions considered in our analysis.

Wilson twisted mass fermions

$$S_{\rm tm}(p) = \frac{-i\mathring{p}_\mu\gamma_\mu\mathbb{1}_f + M(p)\mathbb{1}\mathbb{1}_f - i\mu_q\gamma_5\tau_3}{\sum_\mu \mathring{p}_\mu^2 + M(p)^2 + \mu_q^2}\,, \tag{4}$$

where

$$\mathring{p}_\mu = \frac{1}{a}\sin(ap_\mu)\,, \qquad \hat{p}_\mu = \frac{2}{a}\sin\left(\frac{ap_\mu}{2}\right)\,, \qquad M(p) = m_0 + \frac{a}{2}\sum_\mu \hat{p}_\mu^2\,, \tag{5}$$

$\mathbb{1}$ and $\mathbb{1}_{\rm f}$ are the identity matrices in Dirac and flavour space, respectively. μ_q is the twisted quark mass and m_0 the untwisted quark mass. The maximal twist setup consists in setting the untwisted mass to zero[6] such that the quark mass is only given by the twisted mass.

Overlap fermions

$$S_{\rm ov}(p) = \frac{-i\left(1-\frac{ma}{2}\right)F(p)^{-1/2}\mathring{p}_\mu\gamma_\mu + \mathcal{M}(p)\mathbb{1}}{\left(1-\frac{ma}{2}\right)^2 F(p)^{-1}\sum_\mu \mathring{p}_\mu^2 + \mathcal{M}(p)^2}\,, \tag{6}$$

where

$$\begin{aligned} F(p) &= 1 + \frac{a^4}{2}\sum_{\mu<\nu}\hat{p}_\mu^2\hat{p}_\nu^2\,, \\ \mathcal{M}(p) &= \frac{1}{a}\left(1+\frac{ma}{2} - \left(1-\frac{ma}{2}\right)F(p)^{-1/2}\left(1-\frac{a^2}{2}\sum_\mu \hat{p}_\mu^2\right)\right), \end{aligned} \tag{7}$$

m is the bare overlap quark mass.

Creutz fermions

$$S_{\rm C}(p) = \frac{-i\sum_\mu p_\mu\bar{\gamma}_\mu + m_0\mathbf{1}}{\sum_\mu\sum_\rho p_\mu p_\mu \bar{a}_{\rho\mu}\bar{a}_{\rho\mu} + \sum_{\mu\neq\nu}\sum_\rho p_\mu p_\nu \bar{a}_{\rho\mu}\bar{a}_{\rho\nu} + m_0^2}\,, \tag{8}$$

[6] This can be done exactly only at tree-level. A fine tuning is required in the interacting theory.

where

$$\bar{a} = \frac{1}{R}\begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -\frac{3\sqrt{1-C^2}}{C} & -\frac{3\sqrt{1-C^2}}{C} & -\frac{3\sqrt{1-C^2}}{C} & -\frac{3\sqrt{1-C^2}}{C} \end{pmatrix},$$

$\bar{\gamma} = \bar{a}^T\gamma$, m_0 — bare quark mass, C — lattice geometry parameter, R — C-dependent normalisation factor needed to obtain the correct continuum limit. We consider two values of C: $C = 3/\sqrt{10}$ ($R = 2$)[7] and $C = 3/\sqrt{14}$ ($R = 2\sqrt{2}$)[8].

We also consider a modification of Creutz's action suggested by Borici [17]. We call the corresponding fermions the "Borici fermions" and the quark propagator for them is

$$S_{\mathrm{B}}(p) = \frac{-i\sum_\mu G_\mu(ap)\gamma_\mu + m_0\,\mathbb{1}}{\sum_\mu G_\mu(ap)^2 + m_0^2}\,, \tag{9}$$

where the functions $G_\mu(ap)$ are

$$\begin{aligned} G_1(ap) &= \mathring{p}_1 - \frac{a}{4}\left[\hat{p}_1^2 + \hat{p}_2^2 - \hat{p}_3^2 - \hat{p}_4^2\right], \\ G_2(ap) &= \mathring{p}_2 - \frac{a}{4}\left[-\hat{p}_1^2 + \hat{p}_2^2 - \hat{p}_3^2 - \hat{p}_4^2\right], \\ G_3(ap) &= \mathring{p}_3 - \frac{a}{4}\left[-\hat{p}_1^2 - \hat{p}_2^2 + \hat{p}_3^2 - \hat{p}_4^2\right], \\ G_4(ap) &= \mathring{p}_4 - \frac{a}{4}\left[-\hat{p}_1^2 - \hat{p}_2^2 - \hat{p}_3^2 + \hat{p}_4^2\right]. \end{aligned}$$

Quark propagator decomposition. All of the quark propagators are matrix expressions that can be decomposed in terms of the gamma matrices and the identity matrix:

$$S(p) = S_U(p)\mathbb{1} + \sum_\mu S_\mu(p)\gamma_\mu\,, \tag{10}$$

where $\mu = 1,2,3,4$ for overlap and Creutz fermions and $\mu = 1,2,3,4,5$ for twisted mass fermions.

[7] This value corresponds to the hypercubic lattice.

[8] This value corresponds to a highly symmetric lattice geometry, which is the 4-dimensional analogue of graphene structure.

3. Scaling tests

In this section we present the scaling tests performed on the pseudoscalar correlation functions, masses and decay constants. We employ the following strategy; we fix $Nm = 0.8$ (where m is the bare quark mass in lattice units) and calculate the correlators towards the continuum limit ($a \to 0$). At tree level of perturbation theory this is equivalent to the limit $N \to \infty$. The time extent is always set to be larger than and an integer multiple of the spatial extent.

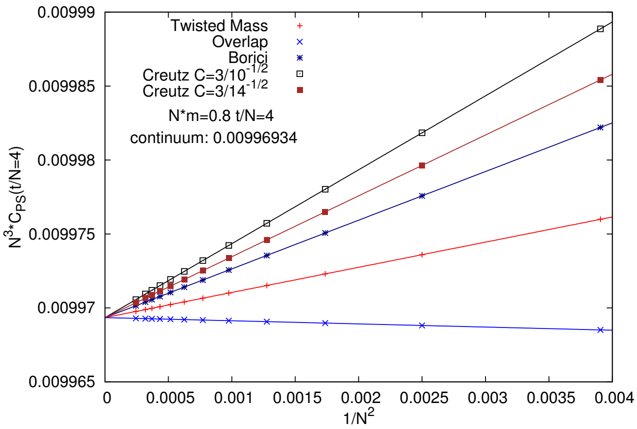

Fig. 1. Cut-off effects and continuum limit of the pseudoscalar correlation function.

In Fig. 1 we show the correlation function at a fixed physical time $t/N = 4$, which is large enough to allow for a reliable extraction of the ground state contribution. To compare different fermion discretizations, we extract the coefficients, Table I, of the fitting curves shown in Fig. 1. We use the following form of the fitting function:

$$N^3 C_{\mathrm{PS}} = a + b\frac{1}{N^2} + c\frac{1}{N^4}\,. \tag{11}$$

TABLE I

Fit coefficients for the pseudoscalar correlation function.

$N^3 C_{\mathrm{PS}}(t/N = 4)$	a	b	c
MTM	0.00996934	0.00170143	0.00002268
OVERLAP	0.00996934	−0.00021268	−0.00006924
BORICI	0.00996934	0.00329653	−0.00116956
CREUTZ — $C = 3/\sqrt{10}$	0.00996934	0.00499799	0.000201048
CREUTZ — $C = 3/\sqrt{14}$	0.00996934	0.00412066	−0.00143986

Fig. 2 shows the pseudoscalar mass and in Table II we have gathered the coefficients of the following fit:

$$Nm_{\rm PS} = a + b\frac{1}{N^2} + c\frac{1}{N^4}\,. \tag{12}$$

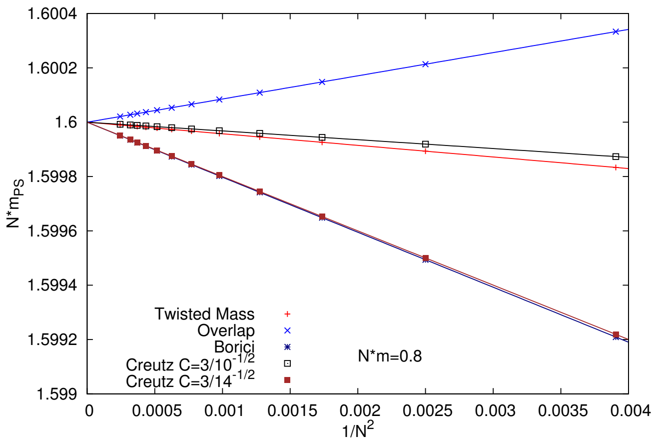

Fig. 2. Cut-off effects and continuum limit of the pseudoscalar mass.

Fig. 3 presents the pseudoscalar decay constant and Table III the coefficients of the following fit:

$$Nf_{\rm PS} = a + b\frac{1}{N^2} + c\frac{1}{N^4}\,. \tag{13}$$

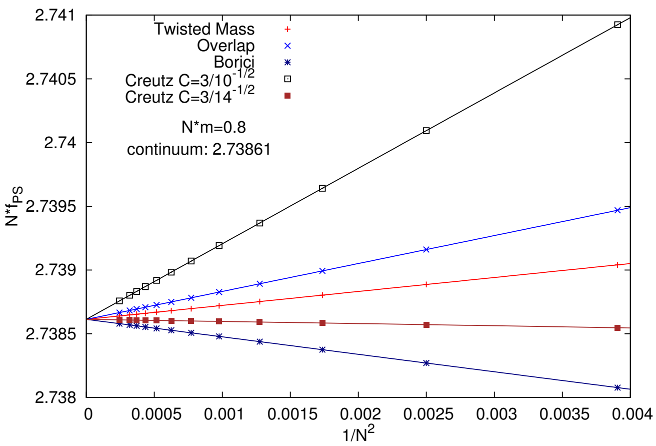

Fig. 3. Cut-off effects and continuum limit of the pion decay constant.

TABLE II

Fit coefficients for the pseudoscalar mass.

$Nm_{\rm PS}$	a	b	c
MTM	1.6	−0.042667	0.003045
OVERLAP	1.6	0.085333	0.008182
BORICI	1.6	−0.202667	0.058999
CREUTZ — $C=3/\sqrt{10}$	1.6	−0.032000	−0.106063
CREUTZ — $C=3/\sqrt{14}$	1.6	−0.200000	0.029668

TABLE III

Fit coefficients for the pseudoscalar decay constant.

$Nf_{\rm PS}$	a	b	c
MTM	2.73861	0.109545	−0.004307
OVERLAP	2.73861	0.219089	0.028606
BORICI	2.73861	−0.136931	−0.027123
CREUTZ — $C=3/\sqrt{10}$	2.73861	0.593370	−0.388244
CREUTZ — $C=3/\sqrt{14}$	2.73861	−0.015977	−0.195636

All types of fermions show the expected behaviour in the lattice spacing — $\mathcal{O}(a^2)$ scaling violations. This is due to the exact chiral symmetry for overlap and Creutz fermions, and to the residual chiral symmetry for MTM.

The continuum limit for each observable is always the same for every discretization (and the expected one for the mass at tree-level of perturbation theory), thus providing a first check of consistency of the corresponding lattice regularizations here analyzed. The magnitude of the $\mathcal{O}(a^2)$ effects is, however, very different for different discretizations and depends on the observable under consideration. For example, for the correlation function at a fixed physical time, the smallest effects are exhibited by overlap fermions and the largest by Creutz fermions with $C=3/\sqrt{10}$. For the pion mass, however, the $\mathcal{O}(a^2)$ scaling violations are the smallest for Creutz fermions with $C=3/\sqrt{10}$ and the largest for Borici fermions. Therefore, there are no definite conclusions, from this scaling test at tree-level, of which fermions exhibit the smallest $\mathcal{O}(a^2)$ effects. The only clear regularity that we observe is that the discretization errors for twisted mass fermions at maximal twist are rather small for all observables that we have considered. Even the recently proposed Creutz fermions and their modification by Borici, which break a number of important discrete symmetries, do not suffer from very large $\mathcal{O}(a^2)$ scaling violations at tree-level of perturbation theory and thus cannot be excluded from this point of view.

We have compared the results for the scaling behaviour here presented with the ones discussed in [3], where the same study for a different value of the quark mass ($Nm=0.5$) was performed. We observe that, as expected, for all the quantities which have a well-defined continuum limit, the relative discretization errors (ratio of the coefficient of the $\mathcal{O}(a^2)$ effects with respect to the continuum value) decrease when decreasing the quark mass independently of the action considered, while the difference of the relative discretization errors can vary between the actions considered here when changing the quark mass.

4. Conclusion

We have performed a scaling test of three different lattice fermion regularisations at tree-level of perturbation theory; the widely used twisted mass and overlap fermions and also the recently proposed minimally-doubled Creutz fermions. All these discretizations lead to the same continuum limit and are $\mathcal{O}(a)$-improved, but the relative sizes of the $\mathcal{O}(a^2)$ effects depend strongly on the observable we choose for the analysis. Therefore, we cannot exclude or put a preference, from the tree-level study of the lattice artifacts, on any particular fermion discretization of the three here considered. It will, therefore, be interesting to test these discretizations in the interacting theory in practical simulations.

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