SPACELIKE TWO-SURFACES IN STATIC MULTIPLY WARPED PRODUCT SPACETIMES

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After a brief summary of the geometry of static multiply warped product spacetimes, inequalities are given relating the Gaussian curvature of spacelike two-surfaces in these spacetimes with some principal sectional curvatures of the embedding space. Extremal and geodesic surfaces are respectively characterized if equality holds in an appropriate inequality.

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1. Introduction

Warped product manifolds appear frequently in general relativity as the model space for spacetime [1–3]. From the cosmological viewpoint, the Robertson–Walker spaces and their generalizations are well-known examples. Properties of geodesics and surfaces in these spacetimes were studied in *e.g.* [4, 5].

On the other hand it was shown that every static spacetime is isometric to a warped product manifold $M \times_f \mathbb{R}$, with warping function $f : M \to \mathbb{R}$ and were $\mathbb{R}$ has the negative definite metric $-dt^2$ (see *e.g.* [6]). The classical examples are the Schwarzschild and Reissner–Nordström spacetimes. However, these particular spaces not only are a warped product, but can be seen as a multiply warped product of two intervals with a two-dimensional Riemannian manifold. The geometry of general multiply warped product manifolds was studied in [7–9]. There, attention was mainly focused on the non-static multiply warped product spacetimes which are considered to be extensions of Kasner spaces or the interior Schwarzschild metric.

In the following we will focus on the static multiply warped product spacetimes which can be viewed as generalizations of the Schwarzschild space (see Remark 2.1). In Section 2 we recall basic notions on static multiply warped product spacetimes and give some examples. In Section 3 spacelike two-surfaces in these spacetimes are studied and we obtain several inequalities relating the Gaussian curvature of the surface to some principal sectional curvatures of the static multiply warped product spacetime. Equality in these inequalities characterizes either geodesic or extremal surfaces. We recall that a spacelike two-surface which is contained in a time-symmetric Cauchy hypersurface is extremal if and only if the outgoing light rays are marginally converging, *i.e.*, the trace of the shape operator corresponding with the future normal null direction vanishes. These extremal surfaces are called marginally outer trapped and play an important role in general relativity, where they are considered to be apparent horizons. The Penrose inequality, which was proved in the Riemannian case in [10] and [11], can then be seen as an inequality between the area of the extremal surface and the ADM mass.

2. Static multiply warped product spacetimes

We start with a smooth manifold $M = B_1 \times B_2 \times F$, where B_1 and B_2 are (possibly infinite) open intervals of $\mathbb{R}$ and F is a connected two-dimensional manifold. Let π_1, π_2 and σ be the projections onto B_1, B_2 and F, respectively. The vector field $X = \partial_r$ is taken to be the lift to M of the standard vector field d/dr on $B_1 \subset \mathbb{R}$ and $U = \partial_t$ is analogously taken to be the lift of the standard vector field d/dt on $B_2 \subset \mathbb{R}$. While X is spacelike, *i.e.*, $g(X,X) = 1$, the vector field U is timelike, *i.e.*, $g(U,U) < 0$. Holding r and t constant gives the slice

$$F(r,t) = \{r\} \times \{t\} \times F = \{(r,t,p) \mid p \in F\} .$$

We further denote the fibers by $F(r) = \{r\} \times B_2 \times F$ and $F(t) = B_1 \times \{t\} \times F$. As usual, the lift $\phi \circ \pi_1$ of a function $\phi \in \mathcal{F}(B_1)$ is again denoted by ϕ and we write ϕ' for $X(\phi)$.

Definition 2.1 *Let $f, h > 0$ be smooth functions on an open interval $B_1 \subset \mathbb{R}$, B_2 an open interval of $\mathbb{R}$ with negative definite metric and let F be a connected two-dimensional Riemannian manifold. Then, the multiply warped product*

$$M = B_1 \times_f B_2 \times_h F ,$$

is called a static multiply warped product spacetime.

Explicitly, the manifold $M = B_1 \times B_2 \times F$ has a line element

$$ds^2 = dr^2 - f^2(r)dt^2 + h^2(r)ds_F^2 , \tag{1}$$

with ds_F^2 the line element of F lifted to M. The metric tensor of the two-dimensional slice F will be denoted by g_F, its Levi–Civita connection by ${}^F\nabla$ and Gaussian curvature by $K^F = g_F({}^F R(A_1, A_2)A_2, A_1)$, with $\{A_1, A_2\}$ a g_F-orthonormal basis of the tangent space to the two-dimensional manifold F. The Levi–Civita connection on the static multiply warped product spacetime will be denoted by D and its corresponding curvature tensor by R.

Remark 2.1 The metric of the Reissner–Nordström spacetime can be written as

$$ds^2 = \frac{dr^2}{1 - \frac{2m}{r} + \frac{e^2}{r^2}} - \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right) dt^2 + r^2\Big(d\theta^2 + \sin(\theta)^2 d\phi^2\Big)\,,$$

with mass m and electric charge e. Using a transformation similar as in [12] this metric can be written as,

$$ds^2 = d\mu^2 - \left(1 - \frac{2m}{P^{-1}(\mu)} + \frac{e^2}{P^{-1}(\mu)^2}\right) dt^2 + P^{-1}(\mu)^2\Big(d\theta^2 + \sin(\theta)^2 d\phi^2\Big)\ ,$$

with $\mu = P(r) = \sqrt{r^2 - 2mr + e^2} + m \ln\Big(r - m + \sqrt{r^2 - 2mr + e^2}\Big)$. Hence, the Reissner–Nordström and Schwarzschild metric ($e = 0$) are both what we call static multiply warped product spacetimes. Perhaps generalized Schwarzschild or generalized Reissner–Nordström spacetime would be more appropriate, but these names refer already to other types of spacetimes (see *e.g.* [13–15]).

After a conformal transformation of a static multiply warped product metric g, with conformal function $\phi \in \mathcal{F}(B_1)$, *i.e.*, $\widetilde{g} = e^{\phi} g$, the metric $\widetilde{g}$ remains a static multiply warped product metric. Moreover, every static multiply warped product metric (1) can be written as

$$ds^2 = f(r)^2\Big\{ - dt^2 + f(r)^{-2} dr^2 + h(r)^2 f(r)^{-2} ds_F^2 \Big\}\,.$$

Hence, the metric ds^2 is conformal to the product metric

$$ds_p^2 = -dt^2 + du^2 + \frac{h(u)^2}{f(u)^2} ds_F^2\,,$$

with $u = \int f(r)^{-1} dr$. By an analogous reasoning as in [5], we find that t is a time function such that (M, ds_p^2) is stably causal. Further, (M, ds_p^2) is globally hyperbolic if and only if ds_F^2 is complete. Because the causal structure is invariant under conformal transformations, these properties also hold for (M, ds^2).

We can express the connection and curvature of a static multiply warped product spacetime using the results on multiply warped products [7]. As usual, we denote by $\mathcal{L}(F)$ the set of all lifts to M of vector fields on F. Thus, $A \in \mathcal{L}(F)$ if and only if $d\sigma(A) \in \mathfrak{X}(F)$ and $d\pi_i(A) = 0$, $i = 1, 2$.

Proposition 2.1 *Let $A, B \in \mathcal{L}(F)$ on M. Then,*

(a) $D_X X = 0$,

(b) $D_X U = D_U X = (f'/f)U$,

(c) $D_X A = D_A X = (h'/h)A$,

(d) $D_U U = f f' X$,

(e) $D_U A = D_A U = 0$,

(f) $D_A B = \sigma^\star({}^F\nabla_A B) - g(A, B)(h'/h)X$,

with $\sigma^\star({}^F\nabla_A B)$ the lift of ${}^F\nabla_A B$ on F to M.

If we denote by II^F the second fundamental form of the slice $F(r, t)$, it follows that

$$\mathrm{II}^F(A, B) = -\frac{h'}{h}\, g(A, B)\, X\,. \tag{2}$$

Corollary 2.1 *The slices $F(r, t)$ are non-trapped extrinsic spheres in M, i.e., totally umbilical spacelike surfaces with constant mean curvature and a spacelike mean curvature vector.*

Using the expressions for the covariant derivatives, the following result is straightforward.

Proposition 2.2 *Let (M, g) be a static multiply warped product spacetime. If $\xi \in \mathcal{L}(F)$ is a Killing vector of g_F, then ξ is also a Killing vector of g.*

Because U is a timelike Killing vector, there holds the following result.

Corollary 2.2 *Let (M, g) be a static multiply warped product spacetime with F a two-dimensional manifold of constant Gaussian curvature. Then, (M, g) admits a group G_4 of motions.*

The components of the Riemann–Christoffel curvature tensor, given by $R(V_1, V_2)V_3 = \Big(D_{V_1}D_{V_2} - D_{V_2}D_{V_1} - D_{[V_1,V_2]}\Big)V_3$, are as follows.

Proposition 2.3 *Let $A, B, C \in \mathcal{L}(F)$ on M. Then,*

(a) $R(U, X)X = -(f''/f)U$,

(b) $R(A, X)X = -(h''/h)A$,

(c) $R(A, U)U = (ff'h'/h)A$,

(d) $R(A, B)C = \sigma^\star({}^F R(A, B)C) + (h'/h)^2\{g(A, C)B - g(B, C)A\}$,

(e) $R(A, B)U = R(A, B)X = R(X, U)A = R(U, A)X = 0$.

It follows that the plane spanned by X and U has sectional curvature $-f''/f$ and a plane tangent to a slice $F(r, t)$ has sectional curvature $(K^F - h'^2)/h^2$. A plane spanned by X and a vector tangent to a slice $F(r, t)$ has curvature $-h''/h$, while a plane spanned by U and a vector tangent to $F(r, t)$ has curvature $-(f'h')/(fh)$.

The only non-vanishing components of the Ricci tensor of a static multiply warped product spacetime are

$$\mathrm{Ric}(X, X) \; = \; -\frac{f''}{f} - \frac{2h''}{h}\,, \quad \mathrm{Ric}(U, U) = ff'' + 2ff'\frac{h'}{h}\,,$$

and

$$\mathrm{Ric}(A, B) \; = \; -\left\{\frac{h''}{h} + \left(\frac{h'}{h}\right)^2 + \frac{(f'h')}{(fh)} - \frac{K^F}{h^2}\right\} g(A, B)\,.$$

Hence, if the static multiply warped product spacetime is non-vacuum, one can always find a real orthonormal basis with respect to which the Ricci tensor can be diagonalized.

Corollary 2.3 *Let (M, g) be a static multiply warped product spacetime. Then, (M, g) cannot be a solution of the Einstein field equations with source a null electromagnetic field.*

By the Einstein field equations we understand the equations

$$\mathrm{Ric}(X, Y) - \tfrac{1}{2}R\, g(X, Y) = \kappa_0 T(X, Y)\,,$$

with κ_0 Einstein's gravitational constant and T the (0,2) energy-momentum tensor.

Instead of an orthonormal basis, we now consider a null tetrad adapted to the warped product structure, *i.e.*, we take as null vectors

$$K = \frac{1}{\sqrt{2}}\left(\frac{1}{f}U + X\right), \quad L = \frac{1}{\sqrt{2}}\left(\frac{1}{f}U - X\right) \quad \text{and} \quad M = \frac{1}{\sqrt{2}h}(A_1 + iA_2)\,, \tag{3}$$

with $\{A_1, A_2\}$ a g_F-orthonormal basis of F such that $g(K,L) = -1$ and $g(M,\overline{M}) = 1$. Using the expressions for the covariant derivatives, the spin coefficients of the null tetrad are

$$\kappa = \sigma = \tau = \nu = \lambda = \pi = 0\,, \qquad \rho = \mu = -\frac{h'}{\sqrt{2}h}\,,$$

$$\varepsilon = \gamma = \frac{f'}{2\sqrt{2}f}\,,$$

$$\alpha = \frac{1}{2\sqrt{2}h}\left\{g_F\left({}^F\nabla_{A_2}A_2, A_1\right) - i\,g_F\left({}^F\nabla_{A_1}A_1, A_2\right)\right\}\,,$$

and

$$\beta = -\frac{1}{2\sqrt{2}h}\left\{g_F\left({}^F\nabla_{A_2}A_2, A_1\right) + i\,g_F\left({}^F\nabla_{A_1}A_1, A_2\right)\right\}\,.$$

Hence, every static multiply warped product spacetime contains two geodesic, shearfree and non-diverging null congruences. The Ricci scalars are

$$\Phi_{01} = \Phi_{02} = \Phi_{12} = 0\,, \qquad \Phi_{00} = \Phi_{22} = \frac{f'h'}{2fh} - \frac{h''}{2h}\,,$$

$$4\,\Phi_{11} = \frac{f''}{f} - \left(\frac{h'}{h}\right)^2 + \frac{K^F}{h^2}\,,$$

and the scalar curvature is

$$R = -\frac{2f''}{f} - \frac{4h''}{h} - 2\left(\frac{h'}{h}\right)^2 - \frac{4f'h'}{fh} + \frac{2K^F}{h^2}\,.$$

The Weyl scalars are $\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0$ and

$$6\,\Psi_2 = \frac{f''}{f} - \frac{h''}{h} + \left(\frac{h'}{h}\right)^2 - \frac{f'}{f}\frac{h'}{h} - \frac{K^F}{h^2}\,.$$

Theorem 2.1 *Every static multiply warped product spacetime is either of Petrov type D or O. Moreover, a static multiply warped product spacetime is conformally flat if and only if the warping functions satisfy the condition*

$$-\frac{h''}{h} - \frac{f'h'}{fh} = -\frac{f''}{f} + \frac{K^F}{h^2} - \left(\frac{h'}{h}\right)^2\,.$$

If follows that a conformally flat static multiply warped product spacetime has $K^F = \text{constant}$.

Theorem 2.2 *A static multiply warped product spacetime is a solution of Einstein's field equations with as source a cosmological constant, i.e.,* $\mathrm{Ric}(X, Y) = \frac{R}{4} g(X, Y)$, *if and only if* $f = c\,h'$, *with* $c \in \mathbb{R}_0$, *and*

$$\frac{h'''}{h'} - \left(\frac{h'}{h}\right)^2 + \frac{K^F}{h^2} = 0\,.$$

Note that this last condition can be geometrically interpreted as saying that the sectional curvature of a plane tangent to the slice $F(r, t)$ must be equal to the sectional curvature of a plane spanned by the vectors X and U.

Example 2.1 Let $K^F = 1$ and $h(r) = \cos(r)$. The spacetime (M, ds^2), with

$$ds^2 = dr^2 - \sin^2(r)dt^2 + \cos^2(r)\Big(d\theta^2 + \cos^2(\theta)d\phi^2\Big)\,,$$

is a conformally flat, static multiply warped product spacetime with source a cosmological constant $\Lambda = 3$.

Corollary 2.4 *A static multiply warped product spacetime is vacuum if and only if* $f = c\,h'$, *with* $c \in \mathbb{R}_0$, *and*

$$2\frac{h''}{h} + \left(\frac{h'}{h}\right)^2 - \frac{K^F}{h^2} = 0\,.$$

Example 2.2 Let $K^F = 0$ and $h(r) = r^{2/3}$. The spacetime (M, ds^2), with

$$ds^2 = dr^2 - r^{-2/3}dt^2 + r^{4/3}\Big(dx^2 + dy^2\Big)\,,$$

is a vacuum, Petrov type D static multiply warped product spacetime and belongs to the class of special Kasner metrics [16].

Theorem 2.3 *A static multiply warped product spacetime is a solution of Einstein's field equations with source a perfect fluid with 4-velocity* U *if and only if*

$$\frac{f''}{f} + \frac{h''}{h} - \frac{f'h'}{fh} - \left(\frac{h'}{h}\right)^2 + \frac{K^F}{h^2} = 0\,.$$

Example 2.3 Let $K^F = 0$, $f(r) = r^a$ and $h(r) = r^{a(a-1)/(a+1)}$, with $a \in \mathbb{R} - \{\pm 1, -1/3\}$. The spacetime (M, ds^2), with

$$ds^2 = dr^2 - f^2(r)dt^2 + h^2(r)\Big(dx^2 + dy^2\Big)\,,$$

is a Petrov type D, perfect fluid, static multiply warped product spacetime.

Example 2.4 Let $K^F = -1$, $f(r) = r^{1+\sqrt{3}}$ and $h(r) = r$. The spacetime (M, ds^2), with

$$ds^2 = dr^2 - r^{2+2\sqrt{3}}dt^2 + r^2\Big(du^2 + e^{2u}d\phi^2\Big)\,,$$

is a Petrov type D, perfect fluid, static multiply warped product spacetime.

The previous examples are contained in the complete list of Petrov type D and O static perfect fluid solutions obtained in [17].

A spacetime (M, g) is called *pseudo-symmetric* if there exists a function $L : M \to \mathbb{R}$, called the *double sectional curvature*, such that the condition

$$R \cdot R = L\, Q(g, R)\,,$$

holds for all points $p \in M$ with $Q(g, R)_p \neq 0$, where the $(0, 6)$ Tachibana tensor $Q(g, R)$ is defined as

$$\begin{aligned} Q(g,R)(X_1,X_2,X_3,X_4;X,Y) \;&:=\; -\Big((X\wedge Y)\cdot R\Big)(X_1,X_2,X_3,X_4) \\ &=\; R\Big((X\wedge Y)X_1,X_2,X_3,X_4\Big) \\ &\quad +R\Big(X_1,(X\wedge Y)X_2,X_3,X_4\Big) \\ &\quad +R\Big(X_1,X_2,(X\wedge Y)X_3,X_4\Big) \\ &\quad +R\Big(X_1,X_2,X_3(X\wedge Y)X_4\Big)\,, \end{aligned}$$

and the $(0, 6)$-tensor $R \cdot R$ is defined similarly by letting the curvature operator $R(X, Y)$ act as a derivation on the curvature tensor R. The canonical metrical endomorphism $X \wedge Y$ is defined by

$$(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y\,.$$

For further details on pseudo-symmetric manifolds and spacetimes, see *e.g.* [18–21].

Theorem 2.4 *A static multiply warped product spacetime is pseudo-symmetric if and only if $h' = c\,f$, with $c \in \mathbb{R}_0$. The double sectional curvature function then is $L = h''/h$. If $h' = 0$, the static multiply warped product spacetime is semi-symmetric.*

Proof: From the classification in [20] it follows that a Petrov type D spacetime is pseudo-symmetric if and only if the Ricci scalars $\Phi_{0i} = \Phi_{2i} = 0$, $i = 0, 1, 2$. The double sectional curvature function L is then given by $L = -(\Psi_2 + R/12)$. ◇

3. Characterizations of extremal and geodesic two-surfaces

We first recall some of the basic formulas of surface theory we will use in this section. Let S be a spacelike two-surface isometrically embedded in a static multiply warped product spacetime. Denote by ξ a normal vector field to the surface S and let $\{e_1, e_2\}$ be an orthonormal tangent basis to S. The Gauss and Weingarten formulas are

$$D_{e_i} e_j \; = \; \nabla_{e_i} e_j + \mathrm{II}(e_i, e_j)\,, \tag{4}$$

and

$$D_{e_i} \xi \; = \; -A_\xi(e_i) + \nabla^{\perp}_{e_i} \xi\,, \tag{5}$$

with ∇ the induced connection on the surface, II the second fundamental form, A_ξ the shape operator with respect to the normal direction ξ and $\nabla^{\perp}$ the normal connection in the normal bundle. The second fundamental form and the shape operator are related by

$$g(\mathrm{II}(e_i, e_j), \xi) = g(A_\xi(e_i), e_j)\,.$$

A surface is called *totally geodesic* if $\mathrm{II} = 0$. Congruences of these kind of surfaces in a spacetime were studied in [22]. Let $\{\xi_1, \xi_2\}$ be a basis of the normal bundle, with $g(\xi_1, \xi_1) = -g(\xi_2, \xi_2) = 1$. The mean curvature vector of the surface is defined as

$$2\boldsymbol{H} = \mathrm{Tr}(A_{\xi_1})\xi_1 - \mathrm{Tr}(A_{\xi_2})\xi_2\,.$$

If $\boldsymbol{H} = 0$ the surface is called *extremal*. Further, if $\boldsymbol{H}$ is a timelike vector everywhere on the surface, the surface is called *trapped*, if it is a null vector everywhere and non-vanishing in at least one point, the surface is called *marginally trapped*. If the mean curvature vector is spacelike in every point, the surface is called *non-trapped*.

The Gaussian curvature $\overline{K}$ of the surface is related to the curvature of the static multiply warped product spacetime through the Gauss equation,

$$\overline{K} \; = \; R(e_1, e_2, e_2, e_1) + g(\mathrm{II}(e_1, e_1), \mathrm{II}(e_2, e_2)) - g(\mathrm{II}(e_1, e_2), \mathrm{II}(e_1, e_2))\,,$$

or

$$\overline{K} \; = \; R(e_1, e_2, e_2, e_1) + \det(A_{\xi_1}) - \det(A_{\xi_2})\,.$$

3.1. Two-surfaces in fibers $F(t)$

Let S be a spacelike two-surface which is isometrically embedded in a static multiply warped product spacetime such that it lies entirely in a fiber $F(t_0) = B_1 \times \{t_0\} \times F$. Hence, U is a timelike normal to S. Denote by N

a spacelike unit normal to the surface. Then, the vector field $X \in \mathfrak{X}(M)$ can be decomposed as

$$X = X^T + \cos(\theta) N \,,$$

with X^T the tangent part of X to S and $\cos(\theta) = g(X,N)$. Let $\{e_1, e_2\}$ be an orthonormal basis of the tangent space to S at (r, t_0, p). Each basis vector can be written as

$$e_i = g(e_i, X)X + e_i^F \,, \qquad i = 1, 2 \,, \tag{6}$$

with e_i^F the projection onto the fiber $F(r, t_0)$.

From the Weingarten formula (5) and Proposition 2.1 it follows that $A_U = 0$. Hence, every spacelike surface lying entirely in the fiber $F(t_0)$ is non-trapped. Further, from (4) it readily follows that

$$\operatorname{div}(X^T) = \frac{h'}{h}\Big(1 + \cos(\theta)^2\Big) + 2\cos(\theta) H \,,$$

with H the length of the mean curvature vector $\boldsymbol{H}$.

Proposition 3.1 *Let S be a compact, spacelike two-surface in a static multiply warped product spacetime, which lies entirely in a fiber $F(t_0)$. Then,*

$$\int\limits_S \left\{ \frac{h'}{h}\Big(1 + \cos(\theta)^2\Big) + 2\cos(\theta) H \right\} dV = 0 \,. \tag{7}$$

Corollary 3.1 *Let M be a static multiply warped product spacetime. Then, M admits a compact extremal spacelike two-surface lying entirely in $F(t_0)$ if and only if it admits a totally geodesic spacelike slice $F(r, t_0)$.*

Proof: From (7) it follows that

$$\int\limits_S \frac{h'}{h}\Big(1 + \cos(\theta)^2\Big) dV = 0 \,.$$

Thus, there must exist a $r_0 \in I$ such that $h'(r_0) = 0$. Then, from (2) we see that $F(r_0, t_0)$ defines a totally geodesic spacelike slice. ◇

Using the expressions of the curvature tensor R of the static multiply warped product spacetime from Proposition 2.3 and the decomposition (6), it is a straightforward computation to show that

$$\begin{aligned} R(e_1, e_2, e_2, e_1) \;=\;& g\Big({}^F R\left(e_1^F, e_2^F\right) e_2^F, e_1^F \Big) \\ &+ \left(\frac{h'}{h}\right)^2 \Big(g\left(e_1^F, e_2^F\right)^2 \\ &- g\left(e_1^F, e_1^F\right) g\left(e_2^F, e_2^F\right) \Big) \\ &+ \frac{h''}{h} \Big(2g\left(e_1, X\right) g(e_2, X) g\left(e_1^F, e_2^F\right) \\ &- g\left(e_1, X\right)^2 g\left(e_2^F, e_2^F\right) \\ &- g\left(e_2, X\right)^2 g\left(e_1^F, e_1^F\right) \Big) . \end{aligned}$$

Because $g(e_i^F, e_j^F) = \delta_{ij} - g(e_i, X)g(e_j, X)$ and using that $g(X^T, X^T) = \sin(\theta)^2$, we obtain

$$R(e_1, e_2, e_2, e_1) = h^2 g_F\left({}^F R(e_1^F, e_2^F) e_2^F, e_1^F \right) - \left(\frac{h'}{h}\right)^2 \cos(\theta)^2 - \frac{h''}{h} \sin(\theta)^2 .$$

There holds that

$$\begin{aligned} K^F \;&=\; \frac{g_F\left({}^F R(e_1^F, e_2^F) e_2^F, e_1^F \right)}{g_F(e_1^F, e_1^F) g_F\left(e_2^F, e_2^F\right) - g_F\left(e_1^F, e_2^F\right)^2} \\ &=\; \frac{h^4}{\cos(\theta)^2} g_F\left({}^F R(e_1^F, e_2^F) e_2^F, e_1^F \right) . \end{aligned}$$

Using the characteristic equation for 2×2 matrices, $A_N^2 - \mathrm{Tr}(A_N) A_N + \det(A_N) I_2 = 0$, there holds the following.

Proposition 3.2 *Let S be a spacelike two-surface in a static multiply warped product spacetime, which lies entirely in a fiber $F(t_0)$. Then,*

$$\overline{K} + \frac{h''}{h} \sin(\theta)^2 + \left[\left(\frac{h'}{h}\right)^2 - \frac{K^F}{h^2} \right] \cos(\theta)^2 - 2H^2 \leq 0 .$$

Equality holds in every point of the surface if and only if the surface is totally geodesic.

Let $\chi(S)$ denote the Euler–Poincaré characteristic of the surface. Using the theorem of Gauss–Bonnet we obtain the following integral inequality.

Corollary 3.2 *Let S be a compact spacelike two-surface in a static multiply warped product spacetime, lying entirely in a fiber $F(t_0)$. Then,*

$$\chi(S) \leq \frac{1}{2\pi} \int\limits_S \left\{ \left[\frac{K^F}{h^2} - \left(\frac{h'}{h} \right)^2 \right] \cos(\theta)^2 + 2H^2 - \frac{h''}{h} \sin(\theta)^2 \right\} dV \,,$$

and equality holds if and only if the surface is totally geodesic.

3.2. Two-surfaces in fibers $F(r)$

Let now S be a spacelike two-surface which lies entirely in a fiber $F(r_0)$. Hence, X is a unit spacelike normal to S. Denote by ξ a timelike unit normal. Then,

$$U = U^T - f(r_0) \sinh(\phi)\, \xi \,,$$

with $\sinh(\phi) = g(U, \xi)/f$ and U^T is the tangent part of U to S. A tangent basis $\{e_1, e_2\}$ to S at (r_0, t, p) can be decomposed as

$$e_i = -\frac{1}{f^2} g(e_i, U) U + e_i^F \,.$$

From the Weingarten formula if follows that

$$\det(A_X) = \frac{f'h}{fh} \cosh(\phi)^2 - \left(\frac{h'}{h} \right)^2 \sinh(\phi)^2 \,.$$

Using this in the Gauss equation gives

$$\overline{K} = R(e_1, e_2, e_2, e_1) + \frac{f'h'}{fh} \cosh(\phi)^2 - \left(\frac{h'}{h} \right)^2 \sinh(\phi)^2 - \det(A_\xi) \,.$$

By a calculation, similar as in the previous section, we obtain

$$\overline{K} = -\frac{K^F}{h^2} \sinh(\phi)^2 - \det(A_\xi) \,.$$

From the definition of the mean curvature vector, there holds that

$$\mathrm{Tr}(A_\xi)^2 = \left[\frac{f'}{f} \cosh(\phi)^2 + \frac{h'}{h} \left(2 - \cosh(\phi)^2 \right) \right]^2 - 4H^2 \,.$$

Collecting these results gives the following.

Proposition 3.3 *Let S be a spacelike two-surface in a static multiply warped product spacetime, which lies entirely in a fiber $F(r_0)$. If S is assumed to be non-trapped, then*

$$\overline{K} + \frac{K^F}{h^2}\sinh(\phi)^2 + \frac{1}{2}\left[\frac{f'}{f}\cosh(\phi)^2 + \frac{h'}{h}\left(2-\cosh(\phi)^2\right)\right]^2 \geq 0\,.$$

Equality holds in every point of the surface if and only if the surface is extremal in M and geodesically in the fiber $F(r_0)$.

From [23] it follows that assuming the surface is non-trapped is natural since there do not exist compact trapped or marginally trapped surfaces in a static multiply warped product spacetime. Using the Gauss–Bonnet theorem we obtain an integral inequality on a compact non-trapped surface.

Corollary 3.3 *Let S be a compact, non-trapped surface in a static multiply warped product spacetime, which lies entirely in a fiber $F(r_0)$. Then,*

$$\chi(S) \geq \frac{1}{2\pi}\int_S \left\{-\frac{K^F}{h^2}\sinh(\phi)^2 - \frac{1}{2}\left[\frac{f'}{f}\cosh(\phi)^2 + \frac{h'}{h}\left(2-\cosh(\phi)^2\right)\right]^2\right\} dV\,,$$

and equality holds if and only if the surface is extremal.

3.3. Spacelike two-surfaces in a general position

Let S be a general spacelike two-surface in a static multiply warped product spacetime M. At a point $q \in S$ we can consider an orthonormal frame of T_qM, $\{\xi_1, \xi_2, e_1, e_2\}$, such that e_1 and e_2 are tangent to S at q and ξ_1 and ξ_2 are a spacelike and timelike normal respectively to T_qS. A null tetrad associated with this basis is

$$\widehat{K} = \tfrac{1}{\sqrt{2}}(\xi_1 + \xi_2)\,, \quad \widehat{L} = \tfrac{1}{\sqrt{2}}(\xi_2 - \xi_1)\,, \quad \widehat{M} = \tfrac{1}{\sqrt{2}}(e_1 + ie_2)\,.$$

This null tetrad is related to the null tetrad (3) in q through successive Lorentz transformations, *i.e.*, up to rescaling, the null vectors $\{K, L\}$ can be brought in the direction of $\{\widehat{K}, \widehat{L}\}$ as follows:

$$\begin{aligned}
\widehat{K} &= K + E\overline{M} + \overline{E}M + E\overline{E}L\,,\\
\widehat{M} &= M + EL + B(K + E\overline{M} + \overline{E}M + E\overline{E}L)\,,\\
\widehat{L} &= (1 + B\overline{E} + +\overline{B}E + B\overline{B}E\overline{E})L + B\overline{B}K + \overline{B}(1 + B\overline{E})M\\
&\quad +B(1 + \overline{B}E)\overline{M}\,,
\end{aligned} \tag{8}$$

with E and B complex functions.

The second fundamental form II of the surface can be written as

$$\mathrm{II}(e_i, e_j) = -g(\mathrm{II}(e_i, e_j), \widehat{L})\widehat{K} - g(\mathrm{II}(e_i, e_j), \widehat{K})\widehat{L}\,.$$

In particular, in terms of the Newman–Penrose spin coefficients, we find

$$\begin{aligned}
\mathrm{II}(e_1, e_1) &= \tfrac{1}{2}(\widehat{\lambda} + \overline{\widehat{\lambda}} + \widehat{\mu} + \overline{\widehat{\mu}})\widehat{K} - \tfrac{1}{2}(\widehat{\sigma} + \overline{\widehat{\sigma}} + \widehat{\rho} + \overline{\widehat{\rho}})\widehat{L}\,,\\
\mathrm{II}(e_2, e_2) &= -\tfrac{1}{2}(\widehat{\lambda} + \overline{\widehat{\lambda}} - \widehat{\mu} - \overline{\widehat{\mu}})\widehat{K} + \tfrac{1}{2}(\widehat{\sigma} + \overline{\widehat{\sigma}} - \widehat{\rho} - \overline{\widehat{\rho}})\widehat{L}\,,\\
\mathrm{II}(e_1, e_2) &= \tfrac{i}{2}(\widehat{\lambda} - \overline{\widehat{\lambda}} + \widehat{\mu} - \overline{\widehat{\mu}})\widehat{K} + \tfrac{i}{2}(\widehat{\sigma} - \overline{\widehat{\sigma}} + \widehat{\rho} - \overline{\widehat{\rho}})\widehat{L}\,.
\end{aligned}$$

The mean curvature vector then reads

$$\boldsymbol{H} = \tfrac{1}{2}(\widehat{\mu} + \overline{\widehat{\mu}})\widehat{K} - \tfrac{1}{2}(\widehat{\rho} + \overline{\widehat{\rho}})\widehat{L}\,.$$

Note that the second fundamental form of a spacelike surface in a spacetime is completely determined by the shear, expansion and vorticity of the two null congruences orthogonal to the surface.

The spin coefficients $\widehat{\rho}$, $\widehat{\mu}$, $\widehat{\sigma}$ and $\widehat{\lambda}$ can be related to the corresponding spin coefficients of the null tetrad $\{K, L, M, \overline{M}\}$ by means of the transformation (8),

$$\begin{aligned}
\widehat{\rho} &= \rho + 2E\alpha + E^2\sigma + 2E\overline{E}\varepsilon - \overline{E}L(E) - M(E)\\
&\quad + \overline{B}\Big(E(1 + E\overline{E})(2\varepsilon + \rho) + \overline{E}\sigma + 2E^2\alpha - 2E\overline{E}\overline{\alpha} + E^3\sigma\\
&\quad - K(E) + EM(E) - \overline{E}\,\overline{M}(E) - E\overline{E}L(E)\Big)\,,\\
\widehat{\sigma} &= \sigma - 2E\overline{\alpha} + E^2(\rho + 2\varepsilon) - EL(E) - \overline{M}(E)\\
&\quad + B\Big(E(1 + E\overline{E})(\rho + 2\varepsilon) + \overline{E}\sigma + 2E^2\alpha - 2E\overline{E}\overline{\alpha} + E^3\sigma\\
&\quad - K(E) - EM(E) - \overline{E}\,\overline{M}(E) - E\overline{E}L(E)\Big)\,,\\
\widehat{\mu} &= \mu + 2\overline{B}\Big(E(\rho + \varepsilon) - \overline{\alpha}\Big) + B\Big(E\sigma + \overline{E}\rho\Big)\\
&\quad + \overline{B}^2\Big(\sigma - 2E\overline{\alpha} + E^2(\rho + 2\varepsilon) - EL(E) - \overline{M}(E)\Big)\\
&\quad + 2B\overline{B}\Big(\varepsilon - \overline{E}\overline{\alpha} + E\alpha + E\overline{E}(\rho + \varepsilon) + E^2\sigma\Big)\\
&\quad + B\overline{B}^2\Big(E(1 + E\overline{E})(\rho + 2\varepsilon) + \overline{E}\sigma + 2E^2\alpha - 2E\overline{E}\overline{\alpha} + E^3\sigma\\
&\quad - K(E) - EM(E) - \overline{E}\,\overline{M}(E) - E\overline{E}L(E)\Big)\\
&\quad + B\Big(K(\overline{B}) + E\overline{M}(\overline{B}) + \overline{E}M(\overline{B}) + E\overline{E}L(\overline{B})\Big) + M(\overline{B}) + EL(\overline{B})\,,
\end{aligned}$$

$$\begin{aligned}\widehat{\lambda} \;=\; & \sigma + \overline{B}\Big(3E\sigma + \overline{E}(\rho + 2\varepsilon) + 2\alpha\Big) \\ & + \overline{B}^2\Big(\rho + 2\varepsilon + 4E\alpha + 3E^2\sigma + 2E\overline{E}(\rho + 2\varepsilon) - 2\overline{E}\overline{\alpha} - \overline{E}L(E) - M(E)\Big) \\ & + \overline{B}^3\Big(E(1 + E\overline{E})(\rho + 2\varepsilon) + \overline{E}\sigma + 2E^2\alpha - 2E\overline{E}\overline{\alpha} + E^3\sigma \\ & - K(E) - EM(E) - \overline{E}\,\overline{M}(E) - E\overline{E}L(E)\Big) \\ & + \overline{B}\Big(K(\overline{B}) + E\overline{M}(\overline{B}) + \overline{E}M(\overline{B}) + E\overline{E}L(\overline{B})\Big) + \overline{M}(\overline{B}) + \overline{E}L(\overline{B}) \ .\end{aligned}$$

Proposition 3.4 *Let S be a spacelike two-surface in a static multiply warped product spacetime.*

1. *The surface S is totally geodesic if and only if $\widehat{\lambda} = -\widehat{\overline{\lambda}} = -\widehat{\mu}$ and $\widehat{\sigma} = -\widehat{\overline{\sigma}} = -\widehat{\rho}$.*
2. *The surface S is extremal if and only if $\widehat{\mu} + \widehat{\overline{\mu}} = \widehat{\rho} + \widehat{\overline{\rho}} = 0$.*
3. *The surface S is trapped if and only if $\widehat{\mu} + \widehat{\overline{\mu}} = -(\widehat{\rho} + \widehat{\overline{\rho}}) \neq 0$.*
4. *The surface S is marginally trapped if and only if $\widehat{\mu} + \widehat{\overline{\mu}} = 0$ and $\widehat{\rho} + \widehat{\overline{\rho}} \neq 0$, or $\widehat{\rho} + \widehat{\overline{\rho}} = 0$ and $\widehat{\mu} + \widehat{\overline{\mu}} \neq 0$.*
5. *The surface is totally umbilical if and only if $\widehat{\sigma} = -\widehat{\overline{\sigma}} = \frac{1}{2}(\widehat{\overline{\rho}} - \widehat{\rho})$ and $\widehat{\lambda} = -\widehat{\overline{\lambda}} = \frac{1}{2}(\widehat{\overline{\mu}} - \widehat{\mu})$.*

We finally give some necessary and sufficient conditions for a compact spacelike surface to be part of a fiber $F(r)$ or $F(t)$. The vectors X and U can be decomposed into parts tangent and normal to the surface S as

$$X = X^T + X^N \ \text{ and } \ U = U^T + U^N \ .$$

Denote by $\{e_1, e_2\}$ an orthonormal tangent basis to the surface as above. These vectors can be written as

$$e_i = g(e_i, X)X - \frac{1}{f^2}g(e_i, U)U + e_i^F \ ,$$

for $i = 1, 2$ and where e_i^F is the part tangent to the slice $F(r, t)$. Using the Gauss and Weingarten formulas we find

$$D_{e_i}X^T = -\frac{f'}{f^3}g(e_i, U)U + \frac{h'}{h}e_i^F + A_{X^N}(e_i) - \nabla^{\perp}_{e_i}X^N \ ,$$

such that

$$\begin{aligned}\operatorname{div}(X^T) &= -\frac{f'}{f^3} g\left(U^T, U^T\right) \\ &\quad + \frac{h'}{h}\left(2 - g\left(X^T, X^T\right) + \frac{1}{f^2} g\left(U^T, U^T\right)\right) + \operatorname{Tr}\left(A_{X^N}\right).\end{aligned}$$

Because $g(U^N, U^N) = -f^2 - g(U^T, U^T) < 0$, the vector U^N is a timelike, normal vector to the surface. Denote by ξ the unitary, spacelike normal to the surface and to U^N. Then,

$$\begin{aligned}\operatorname{Tr}\left(A_{X^N}\right) &= g\left(X^N, \xi\right) \operatorname{Tr}(A_\xi) + \frac{g\left(X^N, U^N\right)}{g\left(U^N, U^N\right)} \operatorname{Tr}(A_{U^N}) \\ &= 2g(X, \boldsymbol{H}).\end{aligned}$$

Proposition 3.5 *Let S be a compact spacelike two-surface in a static multiply warped product spacetime. Then,*

$$\begin{aligned}&\int_S \left\{-\frac{f'}{f^3} g(U^T, U^T)\right. \\ &\quad \left. + \frac{h'}{h}\left(2 - g(X^T, X^T) + \frac{1}{f^2} g(U^T, U^T)\right) + 2g(X, \boldsymbol{H})\right\} dV = 0.\end{aligned}$$

If the static multiply warped product spacetime is pseudo-symmetric, then h' is signed.

Corollary 3.4 *Let S be a compact spacelike two-surface in a pseudo-symmetric, static multiply warped product spacetime with $h' > 0$. Then,*

$$\int_S \left\{-\frac{f'}{f^3} g(U^T, U^T) + \frac{h'}{h}\left(2 - g(X^T, X^T)\right) + 2g(X, \boldsymbol{H})\right\} dV \leq 0,$$

and equality holds if and only if the surface S lies in a fiber $F(t)$.

Corollary 3.5 *Let S be a compact spacelike surface in a pseudo-symmetric, static multiply warped product spacetime with $h' > 0$. Then,*

$$\int_S \left\{-\frac{f'}{f^3} g(U^T, U^T) + \frac{h'}{h}\left(2 + \frac{1}{f^2} g(U^T, U^T)\right) + 2g(X, \boldsymbol{H})\right\} dV \geq 0,$$

and equality holds if and only if the surface S lies in a fiber $F(r)$.

Analogous results hold if we assume that $h' < 0$, now with opposite inequality signs.

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