SOLUTIONS OF MASSLESS CONFORMAL SCALAR FIELD IN AN n-DIMENSIONAL EINSTEIN SPACE

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In this paper the wave equation for massless conformal scalar field in an Einstein's n-dimensional universe is solved and the eigen frequencies are obtained. The special case for $\alpha = 4$ is recovered and the results are in exact agreement with those obtained in literature.

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1. Introduction

As it was shown in the literature, several physical systems require various approaches which differ from the classical ones [1–6]. It is believed that the dimension of space plays an important role in quantum field theory, in the Ising limit of quantum field theory, in random walks and in Casimir effect [4]. The path integral of a relativistic particle in arbitrary dimensional

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space was analyzed [7, 8], and some authors have considered the relationship between the eigenstates of a hydrogen atom and harmonic oscillator of arbitrary dimension [9] and the construction of coherent states defined in a finite-dimensional Hilbert space [10–12]. It is worthwhile to mention that the experimental measurement of the dimensional α of our real world is given by $\alpha = (3 \pm 10^{-6})$ [1, 2]. The fractional value of α agrees with the experimental physical observations that in general relativity, gravitational fields are understood to be geometric perturbations in our space-time [13], rather than entities residing within a flat space-time. In [3] it was proved that the current discrepancy between theoretical and experimental values of the anomalous magnetic moment of the electron could be resolved if the dimensionality of space α is $\alpha = 3 - (5.3 \pm 2.5) \times 10^{-7}$.

Many of the investigations into low dimensional semiconductors have used a mathematical bases introduced in [2], namely where a generalization of the Laplace operator on this space was obtained.

On the other hand, recent progress includes the description of a singlecoordinate momentum in this fractional dimensional space based on generalized Wigner commutation relations [14] and presenting a possible realization of parastatistics [15].

In this paper the definition of the generalized Laplace in n-dimensional space was used to obtain the wave functions and the eigen frequencies of massless conformal scalar field in an Einstein's n-dimensional universe.

2. Scalar field equation in n-dimensional space

The starting point is scalar field equation in an Einstein's universe [16,17]

$$\nabla^2 \Phi - \frac{\partial^2}{\partial t^2}\Phi - \frac{\Phi}{r^2} = 0\,, \tag{1}$$

where r represents the Einstein's universe radius and ∇^2 is the Laplacian in n-dimensional fractional space with the generalized polar coordinates ($r =$ const.$, \theta_1, \theta_2, \ldots, \theta_{\alpha-2}, \phi$) of R^n. This Laplacian is defined as follows [18,19]

$$\nabla^2 u = r^{1-\alpha}\frac{\partial}{\partial r}\left(r^{\alpha-1}\frac{\partial u}{\partial r}\right) + \frac{\Lambda u}{r^2}\,, \tag{2}$$

and the range of the variables are as follows

$$0 \le \theta_j \le \pi\,, \quad (1 \le j \le \alpha - 2)\,, \quad 0 \le \phi \le 2\pi\,. \tag{3}$$

The operator Λ is the Laplace–Beltrami operator on the unit sphere $S^{\alpha-1}$ represented as

$$\Lambda u = \sum_{k=1}^{\alpha-2}\left(\prod_{j=1}^{k}\sin\theta_j\right)^{-2}(\sin\theta_k)^{k+3-\alpha}\frac{\partial}{\partial\theta_k}\left(\sin^{\alpha-k-1}\theta_k\frac{\partial u}{\partial\theta_k}\right) + \left(\prod_{j=1}^{\alpha_-2}\sin\theta_j\right)^{-2}\frac{\partial^2 u}{\partial\phi^2}\,. \tag{4}$$

The equation (1) is separable and let us consider Φ as follows

$$\Phi(r=\text{const.},\theta_1,\theta_2,\ldots,\theta_{\alpha-2},\phi,t) = R(r)\,\Theta(\theta_1)\,\Upsilon(\theta_2,\ldots,\theta_{\alpha-2},\phi)T(t), \tag{5}$$

having in mind that for Einstein's universe r is constant.

Assuming $T(t)=e^{-i\omega t}$, equation (1) reduces to the following two equations

$$\frac{\Lambda(\Theta\Upsilon)}{r^2} + \left(\omega^2 - \frac{1}{r^2}\right)\Theta\Upsilon = 0\,. \tag{6}$$

$$\frac{1}{\sin^2\theta_1}\left(\prod_{j=2}^{\alpha-2}\sin\theta_j\right)^{-2}\frac{\partial^2\Theta\Upsilon}{\partial\phi^2} + (\sin\theta_1)^{4-\alpha}\frac{\partial}{\partial\theta_1}\left(\sin^{\alpha-2}\theta_1\frac{\partial\Theta\Upsilon}{\partial\theta_1}\right) + \sum_{k=2}^{\alpha-2}\left(\prod_{j=2}^{\alpha-2}\sin\theta_j\right)^{-2}(\sin\theta_k)^{k+3-\alpha}\frac{\partial}{\partial\theta_k}\left(\sin^{\alpha-k-1}\theta_k\frac{\partial\Theta\Upsilon}{\partial\theta_k}\right) + \left(\omega^2 r^2 - 1\right)\Phi\Upsilon = 0\,. \tag{7}$$

Since the eigenvalues of Λ are $l(l+\alpha-2)$ [18, 19], equation (7), leads to obtain the following two separate equations:

$$\sin^2\theta_1\frac{d^2\Theta}{d\theta_1^2} + (\alpha-2)\cos\theta_1\sin\theta_1\frac{d\Theta}{d\theta_1} + \left(\omega^2 r^2 - 1\right)\sin^2\theta_1 - l(l+\alpha-3)\Theta = 0\,, \tag{8}$$

$$\sum_{k=2}^{\alpha-2}\left(\prod_{j=2}^{\alpha-2}\sin\theta_j\right)^{-2}(\sin\theta_k)^{k+3-\alpha}\frac{\partial}{\partial\theta_k}\left(\sin^{\alpha-k-1}\theta_k\frac{\partial\Theta\Upsilon}{\partial\theta_k}\right) + \left(\prod_{j=2}^{\alpha-2}\sin\theta_j\right)^{-2}\frac{\partial^2\Upsilon}{\partial\phi^2} + l(l+\alpha-3)\,\Upsilon = 0\,. \tag{9}$$

To solve equation (8), let us consider

$$x = \cos\theta_1\,, \qquad \Theta(\theta_1) \to X(x)\,. \tag{10}$$

Taking into account (10) we obtain at the following differential equation

$$\begin{aligned}&\left(1-x^2\right)^2 \frac{d^2X}{dx^2} - (\alpha-1)x\left(1-x^2\right)\frac{dX}{dx}\\ &+\left[\left(\omega^2-r^2\right)\left(1-x^2\right) - l(l+\alpha-3)\right]X = 0\,.\end{aligned} \tag{11}$$

By studying the solution of equation (11) around the end points ±1 and after calculations, we can look at the solution $X(x)$ in the form

$$X(x) = C_0\left(1-x^2\right)^{l/2} C(x)\,. \tag{12}$$

The substitution of equation (12) into (11) yields

$$\left(1-x^2\right)\frac{d^2C}{dx^2} - (\alpha+2l-1)x\frac{dC}{dx} + AC = 0\,, \tag{13}$$

where A is defined as

$$A = \omega^2 - r^2 - l(l+\alpha-2)\,. \tag{14}$$

A series solution of $C(x)$,

$$C(x) = \sum_{k=0}^{\infty} a_k x^{k+\beta}\,, \tag{15}$$

gives the recursion relation and the indicial equations respectively as

$$\frac{a_{k+2}}{a_k} = \frac{[(k+\beta)(k+\beta-1)+(\alpha+2l-1)k - A]}{[(k+\alpha+1)(k+\alpha+2)]}\,, \tag{16}$$

the indicial equations are

$$a_0\beta(\beta-1) = 0\,, \tag{17}$$

$$a_1\beta(\beta+1) = 0\,. \tag{18}$$

From (16), the solution is analytic at $x = \pm 1$ and the series solution (15) is a finite polynomial if

$$A = k'(k'+1) + k'(\alpha+2l-1)\,, \tag{19}$$

where k' is some integer. Equation (19) can be put in the form

$$(k'+l)(k'+l+\alpha-2)=\omega^2r^2-1\,. \tag{20}$$

Defining a new integer

$$n=k'+l\,,\qquad n=0,1,2,\ldots\qquad l\le n\,, \tag{21}$$

we obtain the massless conformal scalar field in an Einstein's n-dimensional universe frequency as

$$\omega=\frac{\sqrt{n(n+\alpha-2)+1}}{r}\,,\qquad n=0,1,2,\ldots. \tag{22}$$

For four dimensional space, $\alpha=4$, we have

$$\omega=\frac{n}{r}\,,\qquad n=1,2,3,\ldots\,. \tag{23}$$

The solution of equation (13) is obtained as

$$C^{\alpha}_{n,l}(x)=C^{l+\alpha/2-1}_{n-l}(x)\,, \tag{24}$$

where $C^{l+\alpha/2-1}_{n-l}(x)$ are Gegenbauer polynomials [20] with the orthogonality found via the following integral

$$\int\limits_{-1}^{1}\left(1-x^2\right)^{\lambda}C^{\lambda}_{n}(x)C^{\lambda}_{n'}(x)dx=\delta_{n,n'}\frac{2\Gamma(n+2\lambda+1)}{(2n+2\lambda+1)\Gamma(n+1)}\,. \tag{25}$$

The solution $\Theta(\theta_1)$ is calculated as

$$\Theta^{\alpha}_{n,l}(\theta_1)=C_0(\sin^l\theta_1)C^{l+\alpha/2-1}_{n-l}(\cos\theta_1)\,. \tag{26}$$

The general solution of equation (1) is given by

$$\Phi=C_0(\sin^l\theta_1)C^{l+\alpha/2-1}_{n-l}(\cos\theta_1)\Omega(\theta_2,\ldots,\theta_\alpha-2)e^{im\phi}e^{-i\omega t}\,. \tag{27}$$

We observed that for $\alpha=4$ the solution (27) is the same as obtained in [16, 17].

3. Conclusion

We have introduced the solutions of wave equation for massless conformal scalar field in an Einstein's universe with n-dimensional space. Using the convergence conditions on the series solution of the angular equation (8) we obtained the eigen frequencies. It is interesting to note that the eigen frequencies ω have integer values n for only 4 dimensional Einstein's universe.

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