STOCHASTIC MULTIRESONANCE IN THE ISING MODEL ON SCALE-FREE NETWORKS*

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Stochastic resonance is investigated in the Ising model with ferromagnetic coupling on scale-free networks with various scaling exponents $\gamma > 2$ of the degree distributions $p(k) \propto k^{-\gamma}$, subjected to a weak oscillating magnetic field. In the case $2 < \gamma < 3$ and for slow to moderate frequencies of the input signal the linear response theory and numerical simulations in the mean-field approximation predict the occurrence of stochastic multiresonance, with the spectral power amplification as a function of temperature exhibiting double maxima in the vicinity of and below the crossover temperature for the ferromagnetic transition. In the case $\gamma > 3$ the spectral power amplification is expected to exhibit single maximum close to the critical temperature. These predictions are qualitatively confirmed by Monte Carlo simulations of the Ising model on scale-free networks obtained from a preferential attachment growing procedure.

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1. Introduction

Stochastic resonance (SR) [1] is a phenomenon where noise plays a constructive role by enhancing response of a nonlinear system to a periodic signal (for review see Ref. [2–4]) This response can be characterized, *e.g.*, by the spectral power amplification (SPA), defined as the strength of the Fourier component of the output signal at the frequency of the input signal divided by the strength of the input signal, which exhibits a maximum at non-zero noise intensity. An interesting extension of SR is stochastic multiresonance, where the response to the periodic signal is enhanced for many different values of the noise intensity, which results in multiple maxima of the SPA [5–7]. SR was first demonstrated in low-dimensional potential bistable

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systems [1, 8, 9], then also in networks of bistable systems with mean-field (MF) coupling [10], regular arrays with nearest-neighbor coupling [11–13], small-world networks [14] (with partial rewiring of regular connections [15]) and scale-free networks [16]. As an example of a coupled system exhibiting SR the Ising model with ferromagnetic coupling was also studied, with a weak periodic magnetic field as the input signal, time-dependent magnetization as the output signal, and thermal fluctuations playing the role of noise [17–25]. In the case of regular arrays of spins in more than one dimension [17, 20], networks of spins with global MF coupling [22], small-world [23] and scale-free networks [25] the response of the Ising model to the oscillating magnetic field was maximum in the vicinity of the critical temperature for the ferromagnetic transition due to the divergence of the magnetic susceptibility.

It is known that many weblike structures as the Internet, world-wide web, power supply networks, *etc.*, which are of high importance for the modern society, have scale-free (SF) topology, *i.e.*, their degree distribution (distribution of the number of edges, or connections, per node) obeys a power scaling law $p(k) \propto k^{-\gamma}$, usually with $\gamma < 2$. SF networks belong to a general class of complex networks whose study is a rapidly developing area in statistical physics (for review see Ref. [26, 27]). Investigation of the critical properties of the Ising model on SF networks (with spins located in the network nodes and the edges corresponding to non-zero ferromagnetic exchange interactions between them) revealed that the temperature of the ferromagnetic transition can exhibit strong size dependence, in particular in networks with $2 < \gamma \leq 3$ [28–34]. SR in the Ising model with ferromagnetic coupling on the SF network was investigated for a particular case of a Barabási–Albert network with $\gamma = 3$ [35] and also showed strong dependence on the number of nodes [25]. In this contribution certain aspects of SR in the Ising model with ferromagnetic coupling on SF networks with arbitrary $\gamma < 2$ are analyzed both in the MF approximation and by means of Monte Carlo (MC) simulations. It is shown that in the case of networks with $2 < \gamma < 3$ stochastic multiresonance can occur, *i.e.*, the output signal can exhibit maximum periodicity for two different resonance temperatures, below and in the vicinity of the ferromagnetic transition point; for $\gamma \geq 3$ multiresonance is not observed. This effect can be qualitatively explained using linear response theory (LRT) in the MF approximation.

2. The model and methods of analysis

Let us consider a complex network with N nodes and with the degree distribution $p(k)$. The Ising model with ferromagnetic coupling on such network consists of $i = 1, 2, \ldots N$ spins with two possible orientations $\sigma_i = \pm 1$ located in the nodes and subjected to thermal noise. The exchange integral between the spins σ_i, σ_j is $J_{ij} = J > 0$ if there is an edge between nodes i, j, and $J_{ij} = 0$ otherwise. In order to observe SR the input periodic signal in the form of the external oscillating magnetic field $h(t) = h_0 \sin \omega_0 t$ is applied to all spins. The Hamiltonian for the model is

$$H = -\frac{1}{\langle k \rangle} \sum_{i,j=1}^{N} J_{ij} \sigma_i \sigma_j - h_0 \sin \omega_0 t \sum_{i=1}^{N} \sigma_i \,, \tag{1}$$

where $\langle k \rangle$ is the average degree of nodes. The model obeys the Glauber thermal-bath dynamics, with the transition rates between two spin configurations which differ by a single flip of one spin, *e.g.*, that in the node i, in the form

$$w_i(\sigma_i) = \frac{1}{2} \left[1 - \sigma_i \tanh \left(\frac{I_i(t)}{T} \right) \right] , \tag{2}$$

where

$$I_i(t) = \frac{J}{\langle k \rangle} \sum_{j=1}^{k_i} \sigma_j(t) + h_0 \sin \omega_0 t \tag{3}$$

is a local field acting on the spin i (with degree k_i) at time t, T is the temperature, and the sum in Eq. (3) runs over all k_i neighbours of the node i. The output signal is the time-dependent order parameter $S(t)$, which in the case of the Ising model on complex networks is defined as "weighted" magnetization (with spin states multiplied by the corresponding node degrees) [30],

$$S(t) = (N \langle k \rangle)^{-1} \sum_{i=1}^{N} k_i \sigma_i(t) \,. \tag{4}$$

In order to observe SR the SPA is evaluated from the output signal,

$$\begin{aligned} \mathrm{SPA} &= |P_1|^2 / h_0^2 \,, \\ P_1 &= \lim_{\tau \to \infty} \frac{1}{\tau} \sum_{t=0}^{\tau-1} S(t) e^{-i\omega_0 t} \,, \end{aligned} \tag{5}$$

and the dependence of the SPA on the temperature T is analyzed.

SR in the above-mentioned model can be investigated by means of MC simulations. Besides, the system dynamics can be studied in the MF approximation. A continuous-time equation for the MF value of the order parameter (4), denoted as $\langle S(t) \rangle$, can be derived similarly as in the case of the Ising model on regular arrays [20]. The Master equation for the probability that at time t the system is in the spin configuration $(\sigma_1, \sigma_2, \ldots \sigma_N)$ is

$$\begin{aligned}\frac{d}{dt}P(\sigma_1,\sigma_2,\ldots,\sigma_j,\ldots\sigma_N;t) =& -\sum_{j=1}^{N} w_j\left(\sigma_j\right)P\left(\sigma_1,\sigma_2,\ldots,\sigma_j,\ldots\sigma_N;t\right)\\ &+\sum_{j=1}^{N} w_j(-\sigma_j)P\left(\sigma_1,\sigma_2,\ldots,-\sigma_j,\ldots\sigma_N;t\right).\end{aligned} \tag{6}$$

Multiplying both sides of Eq. (6) by σ_i and performing an ensemble average, denoted by $\langle\ \rangle$, yields

$$\frac{d\langle\sigma_i\rangle}{dt} = -\langle\sigma_i\rangle + \left\langle \tanh\left(\frac{I_i(t)}{T}\right)\right\rangle . \tag{7}$$

In the MF approximation the spins σ_j in Eq. (3) are replaced by $\langle\sigma_j\rangle$, and the resulting MF value of the local field, denoted as $\langle I_i(t)\rangle$, is inserted in Eq. (7). Moreover, following the argument in Ref. [31], the network nodes can be divided into classes according to their degrees k. Then the average values of all spins located in the nodes belonging to the class with degree k are equal and denoted as $\langle\sigma_k\rangle$. Under such assumptions the sum over the network nodes can be replaced by a sum over the classes of nodes with different degrees. For example, the MF value of the order parameter becomes

$$\langle S(t)\rangle = \sum_{k=m}^{k_{\max}} \frac{kp(k)}{\langle k\rangle}\langle\sigma_k(t)\rangle , \tag{8}$$

where m and $k_{\max}$ are the minimum and maximum degrees of nodes, respectively. Similarly, taking into account that the probability that a link from a node i points to a node with degree k is $kp(k)/\sum_l lp(l) = kp(k)/\langle k\rangle$, the MF value of the local field in Eq. (7) is

$$\langle I_i(t)\rangle = \frac{1}{\langle k\rangle}Jk_i \sum_{k=m}^{k_{\max}} \frac{kp(k)}{\langle k\rangle}\langle\sigma_k\rangle + h_i(t) . \tag{9}$$

Multiplying both sides of Eq. (7) by k_i, performing the sum over all nodes i and replacing it by the sum over all classes of nodes as in Eq. (8), the equation for the continuous-time dynamics of $\langle S(t)\rangle$ is finally obtained,

$$\frac{d\langle S\rangle}{dt} = -\langle S\rangle + \sum_{k=m}^{k_{\max}} \frac{p(k)k}{\langle k\rangle}\tanh\left(\frac{Jk\langle S\rangle}{\langle k\rangle T} + \frac{h_0}{T}\sin\omega_0 t\right) . \tag{10}$$

For a given degree distribution $p(k)$, Eq. (10) can be solved numerically. For $h_0 \to 0$ analytic solution is also possible in the framework of the LRT.

3. Mean-field approximation

3.1. Linear response theory

Let us constraint our attention to SF networks with the degree distribution $p(k) = Ak^{-\gamma}$, $\gamma > 2$, where A is a normalization constant, and with the minimum degree of nodes m. For $N \to \infty$ nodes with arbitrarily large k are present in the network, and $A = (\gamma - 1)\, m^{\gamma-1}$. However, in networks with finite N the distribution $p(k)$ has a cutoff at a maximum value $k = k_{\max}$, which for $2 < \gamma \leq 3$ can be estimated from the condition $\int_{k_{\max}}^{\infty} p(k)dk < N^{-1}$ (hence, it is practically impossible to find a node with degree $k > k_{\max}$), which yields $k_{\max} = mN^{\frac{1}{\gamma-1}}$, and for $\gamma > 3$ in practice scales as $k_{\max} \propto N^{1/2}$ [36].

In the absence of the magnetic field the system evolves towards a stable equilibrium with the corresponding value of the order parameter $\langle S \rangle_0$ which can be obtained as a stable fixed point of Eq. (10) with $h_0 = 0$,

$$\langle S \rangle_0 = \int\limits_{m}^{k_{\max}} \frac{Ak^{-\gamma+1}}{\langle k \rangle} \tanh\left(\frac{Jk\langle S \rangle_0}{\langle k \rangle T} \right) dk\,. \tag{11}$$

The corresponding stationary value of the magnetization $\langle M \rangle_0$ is then

$$\langle M \rangle_0 = \sum_{m}^{k_{\max}} p(k) \tanh\left(\frac{Jk\langle S \rangle_0}{\langle k \rangle T} \right) = \int\limits_{m}^{k_{\max}} Ak^{-\gamma} \tanh\left(\frac{Jk\langle S \rangle_0}{\langle k \rangle T} \right) dk \tag{12}$$

(in Eqs. (11), (12) summation was replaced by integration). Eq. (11) has one stable fixed point $\langle S \rangle_0 = 0$ for $T > T_{\mathrm{c}}$ corresponding to the paramagnetic phase, and two stable symmetric fixed points $\pm\langle S \rangle_0$ with $\langle S \rangle_0 > 0$ for $T > T_{\mathrm{c}}$ corresponding to the ferromagnetic phase. The temperature $T_{\mathrm{c}} = J\langle k^2 \rangle / \langle k \rangle^2$, where $\langle k^2 \rangle$ is the second moment of the distribution $p(k)$, depends on the scaling exponent γ and, possibly, on the number of nodes N (note that due to normalization of the exchange energy in Eq. (1) to $\langle k \rangle$ the formula for T_{c} differs from that in Ref. [27, 31]). For $\gamma > 3$ there system undergoes a ferromagnetic phase transition at the critical temperature $T_{\mathrm{c}} = J\frac{\gamma-2}{(\gamma-1)(\gamma-3)}$. For $\gamma \leq 3$ the critical temperature diverges in the thermodynamic limit, however, for finite N there is a crossover temperature: $T_{\mathrm{c}} \propto \ln N$ for $\gamma = 3$ and $T_{\mathrm{c}} \propto N^{\frac{3-\gamma}{\gamma-1}}$ for $\gamma < 3$, separating the ordered and disordered phases [28–33]. The above-mentioned predictions were qualitatively confirmed by MC simulations [28, 34], but for $\gamma < 3$ the crossover temperature T_{c} turned out to be very sensitive to possible correlations between the degrees of the connected nodes and the dependence of T_{c} on N

was characterized by a scaling exponent different from that derived in the MF approximation [34].

In the framework of the LRT it is assumed that under the influence of the periodic field with $h_0 \to 0$ for given T the MF order parameter $\langle S(t)\rangle$ oscillates around the stable stationary state, *i.e.*, $\langle S(t)\rangle = \langle S\rangle_0 + \xi(t)$, where $\xi(t) \to 0$. Inserting this into Eq. (10), expanding the tanh function in the Taylor series up to linear terms with respect to $Jk\xi(t)\left(\langle k\rangle T\right)^{-1} + h_0 T^{-1}\sin\omega_0 t$, and replacing the summation with integration yields

$$\begin{aligned}
\frac{d\xi}{dt} &= -\frac{\xi}{\tau_{\mathrm{MF}}} + \frac{h_0 Q}{T}\sin\omega_0 t\,, \qquad (13)\\
\tau_{\mathrm{MF}} &= \left[1 - \frac{J}{T\langle k\rangle^2}\sum_{k=m}^{k_{\max}} p_k k^2 \cosh^{-2}\left(\frac{Jk\langle S\rangle_0}{\langle k\rangle T}\right)\right]^{-1}\\
&= \begin{cases} \left\{\frac{A}{\langle k\rangle\langle S\rangle_0}\left[m^{-\gamma+2}\tanh\left(\frac{Jm\langle S\rangle_0}{\langle k\rangle T}\right) - k_{\max}^{-\gamma+2}\tanh\left(\frac{Jk_{\max}\langle S\rangle_0}{\langle k\rangle T}\right)\right] - 3 + \gamma\right\}^{-1}, & \text{for } T \le T_{\mathrm{c}},\\ \left(1 - \frac{T_{\mathrm{c}}}{T}\right)^{-1}, & \text{for } T > T_{\mathrm{c}}, \end{cases}\\
Q &= \frac{1}{\langle k\rangle}\sum_{k=m}^{k_{\max}} p_k k \cosh^{-2}\left(\frac{Jk\langle S\rangle_0}{\langle k\rangle T}\right)\\
&= \begin{cases} \frac{A}{J\langle S\rangle_0}\left[k_{\max}^{-\gamma+1}\tanh\left(\frac{Jk_{\max}\langle S\rangle_0}{\langle k\rangle T}\right) - m^{-\gamma+1}\tanh\left(\frac{Jm\langle S\rangle_0}{\langle k\rangle T}\right) - (-\gamma+1)\frac{\langle M\rangle_0}{A}\right], & \text{for } T \le T_{\mathrm{c}},\\ 1 & \text{for } T > T_{\mathrm{c}}, \end{cases}
\end{aligned}$$

where τ_{MF} is the MF relaxation time (to evaluate τ_{MF} and Q for $T \le T_{\mathrm{c}}$ the integration by parts was performed, and Eqs. (11), (12) were taken into account). It should be noted that for large N the parameter $Jk\xi(t)\left(\langle k\rangle T\right)^{-1}$ need not be small because the maximum degree $k_{\max}$ can be large. Since in the linear approximation $\xi \propto h_0$ (see below), for networks with large N the signal amplitude must be vanishingly small for the LRT to hold.

The asymptotic solution of Eq. (13) is

$$\begin{aligned}
\xi(t) &= \xi_0 \sin\left(\omega_0 t - \theta\right)\,,\\
\xi_0 &= \frac{h_0 Q}{T}\left(\frac{1}{\tau_{\mathrm{MF}}^2} + \omega_0^2\right)^{-1/2}\\
\theta &= \arctan\left(\omega_0 \tau_{\mathrm{MF}}\right)\,. \qquad (14)
\end{aligned}$$

Thus the SPA is

$$\mathrm{SPA} = \frac{\xi_0^2}{4h_0^2}\,. \qquad (15)$$

Hence, the SPA is proportional to $\left|\frac{\partial S}{\partial h}(\omega_0)\right|^2$, where $\frac{\partial S}{\partial h}(\omega)$ is a dynamical susceptibility of the order parameter $S(t)$.

In the paramagnetic phase with $T > T_\mathrm{c}$ there is $\langle S \rangle_0 = 0$, $Q = 1$ and

$$\mathrm{SPA} = \frac{1}{4T^2}\left[\left(1 - \frac{T_\mathrm{c}}{T}\right)^2 + \omega_0^2\right]^{-1/2}, \tag{16}$$

which is a monotonically decreasing function of T. This result is the same as in the MF approximation for the Ising model on regular lattices [18–21].

Exemplary curves SPA *versus* T resulting from Eq. (15) are shown in Fig. 1 for a relatively large network with $N = 10000$ and different frequencies of the magnetic field ω_0. It can be seen that for $\gamma = 2.5$ and for

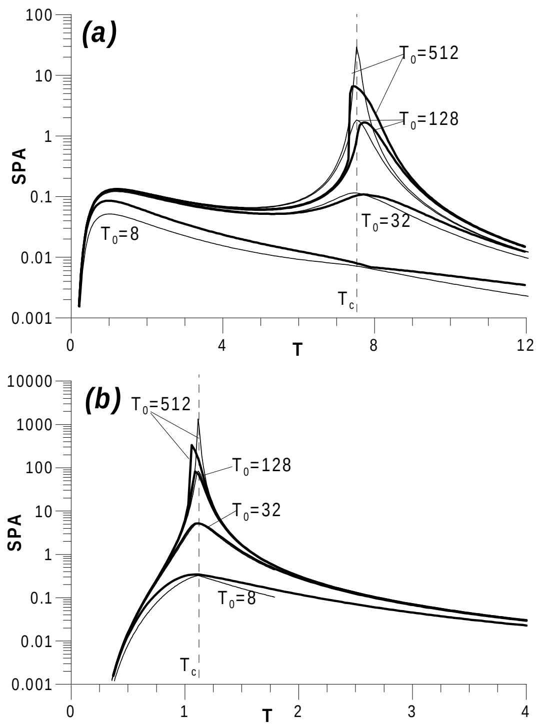

Fig. 1. SPA *versus* T for the Ising model on a SF network with the degree distribution $p(k) = (\gamma - 1)\, m^{\gamma-1} k^{-\gamma}$ predicted by the LRT in the MF approximation, Eq. (15) (thin solid lines), and obtained from numerical simulations of Eq. (10) (thick solid lines). The results are shown for $N = 10000$, $m = 5$, $J = 1$, $h_0 = 0.01$, different frequencies $\omega_0 = 2\pi/T_0$ of the oscillating magnetic field and (a) $\gamma = 2.5$, (b) $\gamma = 5$. T_c is the critical (crossover) temperature for the ferromagnetic transition evaluated in the MF approximation as explained in Sec. 3.1.

moderate and small frequencies ω_0 the LRT predicts double maxima of the SPA corresponding to two different resonance temperatures $T_{\rm r}$ (Fig. 1(a)). One maximum occurs at $T_{\rm r} = T_{\rm c}$ due to the divergence of the magnetic susceptibility in the vicinity of the critical temperature for the ferromagnetic transition, while the other one occurs in the ferromagnetic phase, at $T_{\rm r} < T_{\rm c}$. The heights of both maxima are comparable for moderate ω_0. For $\omega_0 \to 0$ the height of the maximum at $T_{\rm r} = T_{\rm c}$ significantly increases while that of the maximum at $T_{\rm r} < T_{\rm c}$ saturates so that in the adiabatic limit the former maximum becomes dominant. In contrast, for high frequencies ω_0 the maximum at $T_{\rm r} = T_{\rm c}$ disappears and only that at $T_{\rm r} < T_{\rm c}$ remains. It turns out that qualitatively similar dependence of the SPA on T and ω_0 is predicted for networks with large N and γ in the range $2 < \gamma < 3$. Thus, for moderate and small frequencies of the magnetic field the LRT in the MF approximation predicts the occurrence of stochastic multiresonance in the Ising model with ferromagnetic coupling on SF networks with $2 < \gamma < 3$, while for high ω_0 SR with a single maximum of the curve SPA *versus* T is expected.

The case of SF network with $\gamma = 3$ is a limiting one: stochastic multiresonance is neither predicted by the LRT in the MF approximation nor observed in MC simulations of the Ising model on the Barabási–Albert network [25]. Similarly, for SF networks with $\gamma > 3$ the curves SPA *versus* T resulting from Eq. (15) exhibit only a single maximum at $T_{\rm r} = T_{\rm c}$ which takes a form of a sharp peak in the adiabatic limit (Fig. 1(b)).

3.2. Numerical simulations

Numerical simulations of the MF equation (10) were performed for the same parameters as used to obtain the curves SPA *versus* T from the LRT in Sec. 3.1, and for $h_0 = 0.01$ (the same as in the MC simulations, cf. Sec. 4). In the case of SF networks with $\gamma > 3$ the resulting curves SPA *versus* T coincide with those predicted by the LRT, especially for high and moderate frequencies of the oscillating magnetic field (Fig. 1(b)). For $2 < \gamma < 3$ the agreement is worse (Fig. 1(a)). As mentioned in Sec. 3.1 for $2 < \gamma \leq 3$ the maximum degree of nodes diverges with N as $k_{\rm max} \propto N^{\frac{1}{\gamma-1}}$, thus the assumption of the linear response is limited to vanishingly small amplitudes of the input signal h_0. Nevertheless, simulations of the MF equation (10) confirm the possibility of the occurrence of stochastic multiresonance in the Ising model with ferromagnetic coupling on SF networks with $2 < \gamma < 3$ for moderate and small frequencies of the magnetic field.

4. Monte Carlo simulations

Since the pioneering work of Barabási and Albert [35] it has been known that SF networks can be constructed using the preferential attachment growing procedure. In this paper SR was investigated by means of MC simulations in the Ising model on SF networks with different scaling exponents γ obtained in the following way [27]. First, a small number $m + 1$ of fully connected nodes is fixed. Then, step by step, new nodes are added, and each new node is connected to existing nodes with m edges according to the following probabilistic rule: Probability of linking to a node i is $p_i = (k_i + B) / \sum_i (k_i + B)$, where k_i is the actual degree of the node i, $\sum_i k_i$ is the actual number of edges in the whole network, and B is a tunable parameter representing the initial attractiveness of each node. The growth process is continued until the total number of nodes N is reached, when the network structure is frozen. For large N, this preferential attachment rule results in the network with the mean node degree $\langle k \rangle = 2m$ and the degree distribution $p(k) \propto k^{-\gamma(B,m)}$ for $k \gg B$, with $\gamma\,(B, m) = 3 + B/m$. In particular, for $B = 0$ the original Barabási–Albert network with $\gamma = 3$ is recovered. However, it should be noted that in networks obtained in this way the distribution $p(k)$ for small k deviates from the power scaling law.

It was observed that in the absence of the magnetic field the Ising model on the above-mentioned network shows ferromagnetic transition at the temperature T_c which for $B > 0$ is weakly, and for $B \leq 0$ strongly dependent on N. T_c was estimated from MC simulations as the temperature where the fluctuations of the order parameter, $\delta S^2 = \langle S^2 \rangle - \langle |S| \rangle^2$, were maximum (the brackets denote averaging over many MC simulation steps and many random realizations of the network; besides, the absolute value of S instead of S appears in the second bracket since in the ferromagnetic phase the order parameter performs jumps between the two equivalent orientations, and such large fluctuations can lead to underestimation of T_c). The resulting values of T_c differ significantly from those predicted from the MF approximation, in particular for $2 < \gamma < 3$; this is mainly due to the fact that the distribution $p(k)$ for small k deviates from the power scaling law which affects the value of the maximum degree of nodes $k_{\max}$.

In order to observe SR in the Ising model on the above-mentioned SF network MC simulations were performed of the system with $N = 10000$, $m = 5$, different frequencies of the input signal ω_0 and parameters B which yield different scaling exponents γ. The amplitude of the magnetic field was $h_0 = 0.01$ since for smaller values prohibitively long simulation times were necessary to obtain reliable curves SPA *versus* T due to the intrinsic thermal fluctuations in the system. Typically, the simulation time was 2^{15} steps of the MC algorithm (with one step corresponding to updating N spins), and the results were averaged over 10 random realizations of the network.

Exemplary curves SPA *versus* T from the MC simulations are shown in Fig. 2. They qualitatively resemble those evaluated using the LRT in the MF approximation for similar values of γ, ω_0 (*cf.* Fig. 1). In particular, for the SF network with $\gamma = 2.2$ ($B = -4$, Fig. 2(a)) and for small to moderate frequencies ω_0 double maxima of the SPA can be seen, one at $T_{\rm r} \geq T_{\rm c}$ and the other one at $T_{\rm r} < T_{\rm c}$, in the ferromagnetic phase. In the adiabatic limit $\omega_0 \to 0$ the former maximum is dominant, while for high frequencies of the magnetic field that corresponding to the ferromagnetic phase is more pronounced. Thus, stochastic multiresonance is observed in the MC simulations in the region of the parameter space γ, ω_0 where its occurrence is predicted by the LRT in the MF approximation. However, it should be pointed out that, for the particular model of SF network under study, the scaling exponent γ for the tails of the degree distribution $p(k)$

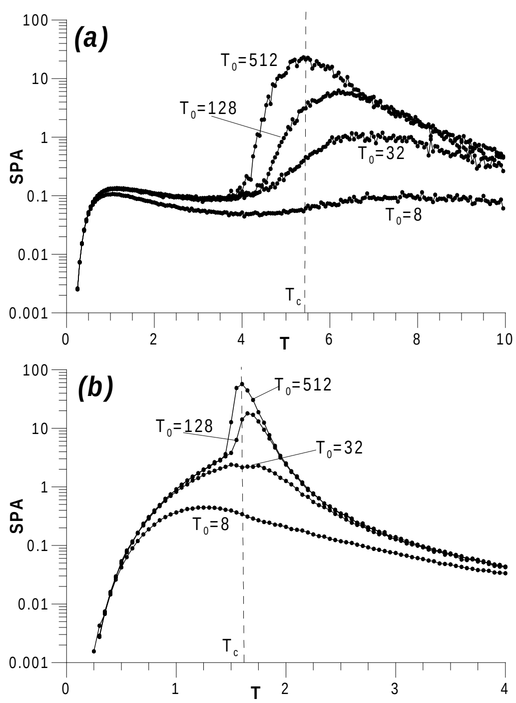

Fig. 2. SPA *versus* T from the MC simulations of the Ising model on a SF network constructed using the preferential attachment growing procedure described in Sec. 4. The results are shown for $N = 10000$, $m = 5$, $J = 1$, $h_0 = 0.01$, different frequencies $\omega_0 = 2\pi/T_0$ of the oscillating magnetic field and (a) $B = -4$ ($\gamma = 2.2$), (b) $B = 10$ ($\gamma = 5$). $T_{\rm c}$ is the critical (crossover) temperature for the ferromagnetic transition estimated from MC simulations with $h_0 = 0$ as a point where the fluctuations of the order parameter S are maximum, as explained in Sec. 4.

must be close to 2 for the multiresonance to occur. In contrast with the predictions of the LRT, for $\gamma > 2.4$ the curves SPA *versus* T obtained from MC simulations exhibit only one maximum. For example, for $\gamma = 5$ ($B = 10$, Fig. 2(b)) the curves SPA *versus* T exhibit one maximum which for small ω_0 is located slightly above the critical temperature T_c.

5. Summary and conclusions

SR in the Ising model with ferromagnetic coupling on SF networks with various scaling exponents $\gamma > 2$ of their degree distributions $p(k) \propto k^{-\gamma}$ was investigated using the LRT in the MF approximation and MC simulations. In the latter case, the network under study was obtained from the preferential attachment growing procedure, with the initial attractiveness ascribed to each node. The input signal had a form of the oscillating magnetic field and the output one was the time-dependent order parameter $S(t)$ (sum of magnetic moments at each node weighted by node degrees). For networks with $2 < \gamma < 3$ and for slow to moderate frequencies of the input signal stochastic multiresonance was observed, with the curves SPA *versus* T exhibiting double maxima at the resonance temperatures in the vicinity of and below the crossover temperature for the ferromagnetic transition. These double maxima should not be confused with the double maxima of various input–output correlation functions observed in the vicinity of T_c in the studies of SR in the Ising model on regular arrays and with MF coupling [20, 22], which appear due to the divergence of the relaxation time at the critical point. For networks with $\gamma \geq 3$ a typical picture of SR occurs, with the curves SPA *versus* T exhibiting a single maximum at the resonance temperature close to the critical one.

In this contribution attention was focused on the search for multiple maxima of the curves SPA *versus* T evaluated from the time series of $S(t)$; however, if the magnetization $M(t)$ is assumed as the output signal, multiresonance can appear for a wider range of γ, *e.g.*, in the Ising model on the Barabási–Albert network [25]. It should be also mentioned that SR on SF networks with $\gamma \leq 3$ can exhibit strong size effects. Detailed discussion of these problems is beyond the scope of this contribution.

REFERENCES

[1] R. Benzi, A. Sutera, A. Vulpiani, *J. Phys.* **A14**, L453 (1981).

[2] L. Gammaitoni, P. Hänggi, P. Jung, F. Marchesoni, *Rev. Mod. Phys.* **70**, 223 (1998).

[3] V.S. Anischenko, A.B. Neiman, F. Moss, L. Schimansky-Geier, *Phys. Usp.* **42**, 7 (1999) [*Usp. Fiz. Nauk* **169**, 7 (1999)].

[4] F. Moss, *Stochastic Resonance: Looking Forward*, in: J. Walleczek (Ed.), *Self-Organized Biological Dynamics & Nonlinear Control*, p. 236, Cambridge University Press, Cambridge 2000.

[5] J.M.G. Vilar, J.M. Rubí, *Phys. Rev. Lett.* **78**, 2882 (1997).

[6] J.M.G. Vilar, J.M. Rubí, *Physica A* **264**, 1 (1999).

[7] S. Matyjaśkiewicz, A. Krawiecki, J.A. Hołyst, L. Schimansky-Geier, *Phys. Rev.* **E68**, 016216 (2003).

[8] B. McNamara, K. Wiesenfeld, *Phys. Rev.* **A39**, 4854 (1989).

[9] P. Jung, P. Hänggi, *Phys. Rev.* **A44**, 8032 (1991).

[10] P. Jung, U. Behn, E. Pantazelou, F. Moss, *Phys. Rev.* **A46**, R1709 (1992).

[11] J.F. Lindner, B.K. Meadows, W.L. Ditto, M.E. Inchiosa, A.R. Bulsara, *Phys. Rev. Lett.* **75**, 3 (1995); *Phys. Rev.* **E53**, 2081 (1996).

[12] M. Löcher, D. Cigna, E.R. Hunt, G.A. Johnson, F. Marchesoni, L. Gammaitoni, M.E. Inchiosa, A.R. Bulsara, *Chaos* **8**, 604 (1998).

[13] N. Sungar, J.P. Sharpe, S. Weber, *Phys. Rev.* **E62**, 1413 (2000).

[14] Zhuo Gao, Bambi Hu, Gang Hu, *Phys. Rev.* **E65**, 016209 (2001).

[15] D.J. Watts, S.H. Strogatz, *Nature* **393**, 440 (1998).

[16] A. Krawiecki, *Physica A* **333**, 505 (2004).

[17] Z. Néda, *Phys. Rev.* **E51**, 5315 (1995).

[18] J. Javier Brey, A. Prados, *Phys. Lett.* **A216**, 240 (1996).

[19] U. Siewert, L. Schimansky-Geier, *Phys. Rev.* **E58**, 2843 (1998).

[20] Kwan-tai Leung, Z. Néda, *Phys. Lett.* **A246**, 505 (1998).

[21] Kwan-tai Leung, Z. Néda, *Phys. Rev.* **E59**, 2730 (1999).

[22] B.J. Kim, P. Minnhagen, H.J. Kim, M.Y. Choi, G.S. Jeon, *Europhys. Lett.* **56**, 333 (2001).

[23] H. Hong, Beom Jun Kim, M.Y. Choi, *Phys. Rev.* **E66**, 011107 (2002).

[24] M. Acharyya, *Phys. Rev.* **E59**, 218 (1999).

[25] A. Krawiecki, *Int. J. Modern Phys.* **B18**, 1759 (2004).

[26] R. Albert, A.-L. Barabási, *Rev. Mod. Phys.* **74**, 47 (2002).

[27] S.N. Dorogovtsev, J.F.F. Mendes, *Adv. Phys.* **51**, 1079 (2002).

[28] A. Aleksiejuk, J.A. Hołyst, D. Stauffer, *Physica A* **310**, 260 (2002).

[29] A. Aleksiejuk, *Int. J. Modern Phys.* **C13**, 1415 (2002).

[30] G. Bianconi, *Phys. Lett.* **A303**, 166 (2002).

[31] M. Leone, A. Vázquez, A. Vespignani, R. Zecchina, *Eur. Phys. J.* **B28**, 191 (2002).

[32] S.N. Drogovtsev, A.V. Goltsev, J.F.F. Mendes, *Phys. Rev.* **E66**, 016104 (2002).

[33] F. Iglói, L. Turban, *Phys. Rev.* **E66**, 036140 (2002).

[34] C.P. Herrero, *Phys. Rev.* **E69**, 067109 (2004).

[35] A.-L. Barabási, R. Albert, *Science* **286**, 509 (1999).

[36] Z. Burda, A. Krzywicki, *Phys. Rev.* **E67**, 046118 (2003).

[37] M. Catanzaro, M. Boguna, R. Pastor-Satorras, *Phys. Rev.* **E71**, 027103 (2005).