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GEOMETRICAL BROWNIAN MOTION DRIVEN BY COLOR NOISE*

Ryszard Zygadło

Jagellonian University, Institute of Physics
ul. Reymonta 4, 30-059 Kraków, Poland

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The geometrical Brownian motion driven by Gaussian or dichotomous color noise is considered. The ordinary Malthusian evolution is observed for long times, however the initial values seem lowered and additionally, in the case of dichotomous noise, the rate of growth is decreased. In the latter case the possibility of arbitrage is shown explicitly.

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1. Introduction

The continuous-time description of a wide range of natural and social processes uses the stochastic differential equations (SDE) in order to regard the presence of fluctuation. The case of additive noise usually corresponds to an overdamped Brownian motion in some external potential and it is particularly important for nonequilibrium thermodynamics (see, *e.g.*, [1] and references therein). The case of multiplicative noise came both from population dynamics as the random growth rate or random carrying capacity models [2,3], from financial analysis (*e.g.*, as the Black and Scholes equation) [4–6], and from physical studies of a critical slowing down [7] and noise-induced transitions [1,8]. In most application the Gaussian white noise (GWN) is considered, not only because of a relatively simply description (*e.g.*, within Fokker–Planck theory), but also because of *nonanticipating* properties of the related Ito SDE. Let us remind [9,10] that the Ito equation

$$dx_t = f(x)dt + g(x)d \circ W_t \,, \tag{1}$$

where the $\circ$ sign is to indicate that the equation is *interpreted* according to the Ito definition that $\langle g(x)d \circ W_t \rangle = 0$ (nonanticipating property), results with the following *regression* equation for averages

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$$d\langle x_t\rangle = \langle f(x)\rangle dt\,, \tag{2}$$

and (consequently) with the conventional drift term $f(x)$ at the level of Fokker–Planck equation. W_t is the Wiener process normalized by the condition

$$\langle \exp(yW_t)\rangle = \exp\left(Dty^2\right)\,, \tag{3}$$

or, equivalently,

$$\langle \xi_t\xi_0\rangle = 2D\delta(t)\,, \tag{4}$$

where $\xi_t \equiv dW_t/dt$ is a GWN. The importance of Eq. (2) is particularly visible if a process of interest is considered to be *fair* according to the game theory (driftless Ito process $f(x) \equiv 0$, *martingale*) or if $f(x)$ is a linear (or affinic) function (*e.g.*, linear relaxation, Malthusian growth). Then $\langle f(x)\rangle = f(\langle x\rangle)$, so the properties of deterministic evolution are exactly reflected within stochastic generalization. On the other hand the well known consequence of the nonanticipating property is that the ordinary rules of differentiation and integration are no longer valid, being replaced by the specific Ito calculus. Particularly, it turns out that $x_t \equiv x(t, W_t)$, considered as a function of two variables, represents the solution to the Ito Eq. (1) only if the usual condition $\partial x/\partial W = g(x)$ and the *unusual* one $\partial x/\partial t = f(x) - Dg(x)g'(x)$ are satisfied. This means that using the ordinary calculus the same process $x(t, W_t)$ is considered to be the solution of the Stratonovich equation

$$dx_t = [f(x) - Dg(x)g'(x)]dt + g(x)dW_t \tag{5}$$

(in our notation without $\circ$ sign) and in such sense both Eqs. (1) and (5) are equivalent. The term Dgg' is called "spurious drift". The Stratonovich interpretation is more popular in a physical literature because the well recognized (ordinary) methods of transforming the variables and solving the differential equations can be used. Going beyond the white noise approximation (perfect randomness, ideal market, *etc.*) one should use appropriate correlated (color) noise instead and, due to its *nonsingular* character, consequently rather the Stratonovich form, Eq. (5), of the kinetic equation. In contrast to the GWN case the exact results are rarely known even in the asymptotic state. Because in the present paper we are particularly interested in nonstationary developing processes, so we restrict ourselves to the linear geometrical Brownian motion (GBM) model.

2. Geometrical Brownian motion

In his pioneering work [11] concerned on financial markets Bachelier adopts *arithmetical* Brownian motion

$$\dot{x}_t = rx + \xi_t \tag{6}$$

to describe the evolution of stock prices. Here $r > 0$ is an intrinsic growth rate often identified simply with the *interest* rate. Except for the sign $r = -\gamma < 0$, where γ is the *friction* coefficient, it is the famous Langevin equation, for the Brownian particle's velocity, of the Einstein–Smoluchowski theory of Brownian motion [12]. Because the solutions of Eq. (6) are Gaussian (Ornstein–Uhlenbeck processes [13]), they are in fact not well suited for modeling prices, which are the nonnegative quantities. Assuming an independent and Gaussian character of the *relative* changes Samuelson [4] and, independently, Black and Scholes (B&S) [5] have obtained

$$dx_t = rxdt + xd \circ W_t \tag{7}$$

or, equivalently,

$$\dot{x}_t = [r - D]x + x\xi_t \tag{8}$$

if the Stratonovich interpretation is used. The stochastic solution

$$x_t = x_0 e^{(r-D)t} \exp(W_t) \tag{9}$$

immediately follows from Eq. (8). Using Eq. (3) one verifies that

$$\langle x_t \rangle = x_0 e^{(r-D)t} \langle \exp(W_t) \rangle = x_0 e^{rt} \tag{10}$$

is in agreement with Eq. (7). Eq. (10) shows that the average return, related to the passive investment "buy and hold," is determined by the interest rate r. Because the discounted price $\tilde{x}_t = e^{-rt} x_t$ is a martingale the consequence of the games theory is that no other strategy can lead to a better result. The Eq. (7) is simultaneously the simplest random growth rate model associated to the pure Malthusian evolution, where r is a positive difference between birth and death rate. The expected in future value of x_t depend on the initial capital (or population) x_0 and interest (or growth) rate r according to Eq. (10). We are going to analyze to which extent this result is changed if the driving color noise is used. We will consider two cases:

The Gaussian color noise (GCN) (or the stationary Ornstein–Uhlenbeck process) is defined as the Gaussian process of a zero mean and an exponentially decaying autocorrelation function [9]

$$K(t) \equiv \langle \xi_t \xi_0 \rangle = D\tau^{-1} \exp(-t/\tau)\,, \tag{11}$$

where $\tau > 0$ is the correlation-time. Because (as a generalized function) $K(t) \to 2D\delta(t)$ for $\tau \to 0$ the GCN (11) approaches GWN (4) if the correlation-time goes to zero.

Another exponentially correlated process, of a different origin, is the dichotomous Markov process (DM) $\xi_t = \sigma(-1)^{N_t}$, where σ is a binary variable equal $\pm|\sigma|$ with probability $1/2$ and N_t is a Poisson counting process with parameter λ [8, 9, 14, 15]

$$\langle \xi_t \xi_0 \rangle = \sigma^2 \exp(-2\lambda t)\,. \tag{12}$$

It may be shown that at the limit $\sigma^2 \to \infty$, $\lambda \to \infty$, $\sigma^2/2\lambda = D = \text{const}$ the GWN (4) is also recovered [14]. The correlation-time of DM is $1/2\lambda$ $(= \tau)$.

Note that the above mentioned GWN-limit procedures are consistent with the Stratonovich interpretation. Thus we will study Eq. (8) with color noise (11) or (12).

3. GBM with GCN

Let

$$\dot{x}_t = (r - D)x + x\xi_t\,, \tag{13}$$

where ξ_t is GCN (11). Then

$$x_t = x_0 e^{(r-D)t} \exp\left[\int_0^t \xi_s ds\right]\,. \tag{14}$$

Using the general formula for stationary Gaussian processes and after that Eq. (11)

$$\left\langle \exp\left[\int_0^t \xi_s ds\right]\right\rangle = \exp\left[\int_0^t ds_1 \int_0^{s_1} ds_2 K(s_1 - s_2)\right] = \exp\left[Dt - D\tau(1 - e^{-t/\tau})\right]\,, \tag{15}$$

one obtains

$$\langle x_t \rangle = x_0 e^{rt} \exp\left[-D\tau(1 - e^{-t/\tau})\right] \approx x_0 e^{-D\tau} e^{rt} \approx x_0(1 - D\tau)e^{rt}. \tag{16}$$

Comparing to Eq. (10) the result (16) shows that the long-time growth rate r remains unchanged in the presence of GCN, however the considered process looks like beginning from the lowered value $x_0 e^{-D\tau} \approx x_0(1 - D\tau)$. Within economical language it means that the presence of a color noise in Eq. (13) introduces certain correlations between successive changes of prices. In contrast to the ideal market model the historical information about prices can be in principle useful to improve the strategy of investment. The *provision* to be payed is the ratio $D\tau$ of an initial investment.

4. GBM model with DM

Again, according to Eq. (8), consider

$$\dot{x}_t = \left[r - (\sigma^2/2\lambda)\right] x + x\xi_t \,, \tag{17}$$

where ξ_t is the asynchronous binary noise (12). Because

$$x_t = x_0 e^{(r-\sigma^2/2\lambda)t} \exp\left[\int_0^t \xi_s ds\right] , \tag{18}$$

we need

$$\Phi(t) = \left\langle \exp\left[\int_0^t \xi_s ds\right]\right\rangle \tag{19}$$

in order to compute certain averages. Let

$$\Psi(t) = \left\langle \xi_t \exp\left[\int_0^t \xi_s ds\right]\right\rangle . \tag{20}$$

Then $\dot{\Phi} = \Psi$ and $\dot{\Psi} = -2\lambda\Psi + \sigma^2\Phi$, where the latter equation follows from Shapiro–Loginov formula [16]. The solution of

$$\ddot{\Phi} + 2\lambda\dot{\Phi} - \sigma^2\Phi = 0 \tag{21}$$

satisfying $\Phi(0) = 1$, $\dot{\Phi}(0) = 0$ is

$$\Phi(t) = \left(\frac{1}{2} + \frac{1}{2q}\right) e^{\lambda(q-1)t} + \left(\frac{1}{2} - \frac{1}{2q}\right) e^{-\lambda(q+1)t} , \tag{22}$$

where $q = \sqrt{1 + \sigma^2/\lambda^2}$. For sufficiently long time and $(2D/\lambda =)\ \sigma^2/\lambda^2 \ll 1$

$$\Phi(t) \approx \left(1 - \frac{1}{4}\frac{\sigma^2}{\lambda^2} + \ldots\right) \exp\left(\frac{\sigma^2 t}{2\lambda} - \frac{\sigma^4 t}{8\lambda^3} + \ldots\right) \tag{23}$$

and thus

$$\langle x_t \rangle \approx x_0(1 - D\tau)e^{(r-D^2\tau)t} , \tag{24}$$

where $\tau = 1/2\lambda$ and $D = \sigma^2/2\lambda$. Comparing to Eq. (16) the case (24) seems even worse, because, among the similar initial provision $D\tau$, the intrinsic growth rate is decreased from r to $r - D^2\tau$, or instantaneous losses are generated in discounted prices. On the other hand the "buy and hold" strategy, for selfevident reasons, is quite inappropriate for this case. Let us assume $r = 0$, which is equivalent to use the discounted prices. Then Eq. (18) shows that the *realization* of x_t consists of the periods of exponential decay $x \sim e^{-(|\sigma|+D)\Delta t}$ separated by the periods of exponential growth $x \sim e^{(|\sigma|-D)\Delta t}$ (if $|\sigma| < 2\lambda$; otherwise the price always falls). The length of the periods is random with the average equal to $1/\lambda$. Moreover, the trajectory of x_t is continuous. "Playing with trend" one buys the stock at the beginning of a growth period and sells immediately when the move changes the direction. The distribution of waiting times for DM is given by $p(t) = \lambda e^{-\lambda t}$ and the corresponding price $x(t) = x_0 e^{(|\sigma|-\sigma^2/2\lambda)t}$, so the expected return per one cycle of an investment is

$$\bar{x} = \int_0^\infty p(t)x(t)dt = x_0 \frac{2}{(1-|\sigma|/\lambda)^2+1}\,. \tag{25}$$

Note that at the GWN-limit, $|\sigma| = \sqrt{2\lambda D}$, $\lambda \to \infty$, the r.h.s. of Eq. (25) is (still) equal x_0, which reflects the *fairness* of the ideal market. The ratio $\bar{x}/x_0 > 1$ if $|\sigma|/\lambda < 2$ (or $D < 2\lambda$). The maximum $\bar{x}/x_0 = 2$ corresponds to $|\sigma| = \lambda = 2D$. Thus, in spite of the general decreasing tendency (24), the market described by Eq. (17) provides easy earn opportunities.

5. Remarks

The GBM model (B&S equation), written in the Stratonovich form (8), can be easily generalized by an appropriate replacement of the driving noise. The B&S equation with a color noise remains exactly solvable. The general conclusion is the following. The limit of zero correlation-time corresponds to the ordinary B&S model, when the discounted price $\tilde{x}_t = e^{-rt}x_t$ is a martingale and the expected future price is $\langle x_t\rangle = x_0 e^{rt}$. On the correlated market ($\tau > 0$) the expected price is lowered: $\langle x_t\rangle \approx x_0(1-D\tau)e^{rt}$, Eq. (16), for GCN and $\langle x_t\rangle \approx x_0(1-D\tau)e^{(r-D^2\tau)t}$, Eq. (24), for DM, respectively. Thus the long-time investment is not particularly recommended. On the other hand, in the presence of correlations the historical prices contain a certain information which can be used to improve the investment. In the case of DM it is easy to identify and use the short-time trends to get a certain earn, as shows Eq. (25).

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