ON THE FIELD-REDEFINITION THEOREM IN GRAVITATIONAL THEORIES

M.D. Pollock

L.D. Landau Institute for Theoretical Physics, Russian Academy of Sciences
Ulitsa Kosygina 2, Moscow 117940, Russia†
and
V.A. Steklov Mathematical Institute, Russian Academy of Sciences
Ulitsa Gubkina 8, Moscow 119991, Russia†

mdp30@cam.ac.uk

(Received August 11, 2007)

Version corrected according to Erratum, Acta Phys. Pol. B **46**, 2569 (2015)

The gravitational sector of classical Lagrangian theories can generally be expressed in the form of a power series

$$\mathcal{L} = \sqrt{-g}\left[-\frac{1}{2}\kappa^{-2}R + \sum_{n=2}^{\infty}\left(a_n\mathcal{R}^n + \tilde{a}_n\partial^2\mathcal{R}^n\right)\right],$$

where κ^2 is the gravitational coupling and R is the Ricci scalar. By means of a metric field-redefinition $g_{ij} \to (1+\beta R)g_{ij} + \gamma R_{ij} + \delta R_{ik}R_j^k + \ldots$, the quadratic terms $\mathcal{R}^2$ can be removed completely (due to the Gauss–Bonnet identity) and the cubic and higher-order terms $\mathcal{R}^n$ partially, only those terms constructed solely from the Riemann tensor R_{ijkl} remaining invariant. It has been shown by Lawrence, however, that the implementation of this procedure at a specific order n inevitably gives rise to ghosts at the next and higher orders $n' \geq n+1$, in the sense that a term $\mathcal{R}^n$ in $\mathcal{L}$ is replaced by terms $\mathcal{R}^{n-m}(\partial^2\mathcal{R})^m$, for example. Classically, these ghosts may lead to instabilities, and it is therefore necessary to investigate the stability of the theory to linear perturbations, both before and after the metric has been transformed. In the cosmological Friedmann space-time $ds^2 = dt^2 - a_0^2 e^{2\alpha(t)} d\boldsymbol{x}^2$ which describes the Universe, where t is comoving time and $a_0 e^{\alpha(t)}$ is the radius function of the three-space $d\boldsymbol{x}^2$, assumed flat, we find, by examining the characteristic equation, that the low-energy solution invariably possesses exponentially growing (and decaying) modes, after carrying out the field redefinition, irrespective of whether such modes were present initially. Therefore, it is not expedient to redefine the metric in this background, which, rather, should be considered as fixed. We discuss the relevance of this result for the heterotic superstring theory, particularly with regard to the vacuum solutions obtained previously from the effective Lagrangian including terms $n \leq 4$, and to the terms $\mathcal{R}^2$.

PACS numbers: 04.50.+h, 11.25.Db, 11.30.Pb, 98.80.Hw

† Temporary address.

1. Introduction

In the general theory of relativity, physically meaningful quantities are invariant under arbitrary coordinate transformations

$$x^i \to x'^i(x^j)\,, \tag{1}$$

as a result of which the metric transforms according to

$$g_{ij}(x^k) \to g'_{ij}(x'^k) = \frac{\partial x'^k}{\partial x^i}\frac{\partial x'^l}{\partial x^j} g_{kl}(x^m)\,. \tag{2}$$

It is also possible, however, to transform the metric without changing the coordinates at all, via a renormalization or redefinition of g_{ij},

$$g_{ij}(x^k) \to g'_{ij}(x^k) = g_{ij}(x^k) + h_{ij}(x^k)\,. \tag{3}$$

Under the second type of transformation, the Ricci scalar R, in particular, no longer remains invariant, and hence there results an internal rearrangement of the Lagrangian.

Let us consider a classical theory for which the Lagrangian function of the gravitational sector can be written, without loss of generality, as the power series

$$\mathcal{L} = \sqrt{-g}\left[-\frac{1}{2}\kappa^{-2}R + \sum_{n=2}^{\infty}\left(a_n\mathcal{R}^n + \tilde{a}_n\partial^2\mathcal{R}^n\right)\right]\,, \tag{4}$$

where $\kappa^2 \equiv 8\pi G_{\mathrm{N}}$ is the gravitational coupling, $G_{\mathrm{N}} \equiv M_{\mathrm{P}}^{-2}$ being the Newton constant and M_{P} the Planck mass, $g = \det g_{ij}$, the $\mathcal{R}^n$ are specified combinations of the Riemann tensor R_{ijkl} and its contractions to the Ricci tensor $R_{ij} \equiv R_{ikj}{}^{k}$ and $R \equiv R_k{}^k$, and the constants a_n and $\tilde{a}_n$ denote the coefficients of these terms. Under the metric redefinition (3), $\mathcal{L}$ transforms into

$$\begin{aligned}\mathcal{L}' = \quad & \sqrt{-g'}\Bigg\{ -\frac{1}{2}\kappa^{-2}R + \sum_{n=2}^{\infty}\Bigg\{ a'_n\mathcal{R}^n + \tilde{a}'_n\partial^2\mathcal{R}^n \\ & + \sum_{m=0}^{n}\mathcal{R}^{n-m}\Bigg[b_{mn}\left(\partial\mathcal{R}\right)^{2m} + b'_{mn}\left(\partial^2\mathcal{R}\right)^m \\ & + \sum_{m'=0}^{m} b''_{mm'n}\left(\partial\mathcal{R}\right)^{2m'}\left(\partial^2\mathcal{R}\right)^{m-m'} + \partial^2\Big(\tilde{b}_{mn}\left(\partial\mathcal{R}\right)^{2m} + \tilde{b}'_{mn}\left(\partial^2\mathcal{R}\right)^m \\ & + \sum_{m'=0}^{m}\tilde{b}''_{mm'n}\left(\partial\mathcal{R}\right)^{2m'}\left(\partial^2\mathcal{R}\right)^{m-m'}\Big)\Bigg]\Bigg\} + \ldots\Bigg\}\,. \end{aligned} \tag{5}$$

At this stage, the choice of metric would seem arbitrary, as first noted by 't Hooft and Veltman [1, 2], who calculated the 1-loop divergence in the Einstein theory of gravity and showed that the resulting quadratic terms $\mathcal{R}^2$ could all be removed, essentially by making the field redefinition

$$g_{ij} \to g'_{ij} = (1 + \beta R) g_{ij} + \gamma R_{ij} \,. \tag{6}$$

Note that it is impossible to construct a contribution to g'_{ij} involving R_{ijkl} which is linear in $\mathcal{R}$, because it has four indices, and therefore [2] $\mathcal{R}^2$ can only be made to vanish after rewriting $R_{ijkl}R^{ijkl}$ in terms of $R_{ij}R^{ij}$ and R^2 via the Euler-characteristic density

$$\mathcal{R}^2_{\mathrm{E}} \equiv R_{ijkl}R^{ijkl} - 4R_{ij}R^{ij} + R^2 \,, \tag{7}$$

which produces a topological invariant. (If there is a coupling to a matter field ϕ of the form $\phi^2\mathcal{R}^2$, this term can no longer be removed, since $\sqrt{-g}\phi^2\mathcal{R}^2_{\mathrm{E}}$ is not a total divergence [2].) Restricting the discussion to four dimensions, at higher orders $n \geq 3$ there is no analogue of the Gauss–Bonnet identity (7), and consequently the $\mathcal{R}^n$ cannot be completely removed by the field redefinition (3). When $n = 3$, for example, all terms constructed from at the most two factors of the Riemann tensor can be removed via the transformation

$$g_{ij} \to g'_{ij} = (1+\beta R+\gamma R^2+\delta R_{kl}R^{kl}) g_{ij} + \epsilon R_{ij} + \zeta R_{ik}R_j{}^k + \eta R_{ikjl}R^{kl} + \theta R_{iklm}R_j{}^{klm} \,, \tag{8}$$

while the terms cubic in R_{ijkl} remain invariant.

2. The meaning of ghosts

Although necessary for the renormalization of the gravitational theory based on the Einstein–Hilbert Lagrangian $L_1 = -R/2\kappa^2$, the quadratic (and higher-order), higher-derivative terms $\mathcal{R}^n$, $n \geq 2$, generally contain ghosts, first identified by Stelle [3] and subsequently discussed in detail by Barth and Christensen [4], and one of the chief motivations for applying the field redefinition (3) is to remove these particles, which classically tend to be a source of instabilities. It was shown by Lawrence [5], however, that if one goes one loop further to order $(n + 1)$, new ghosts may appear, and with them new instabilities, which might seem to invalidate the procedure. In fact this is true even if the n-th order Lagrangian contains no ghosts or instabilities at all, as we shall see below by consideration of the theory with quadratic Lagrangian

$$L_2^{(0)} = A_0 R^2 \,. \tag{9}$$

For then only the ghost-free spin-0 field is present, with mass $M_0^2 = 1/12A_0\kappa^2$, while the ghost-containing spin-2 field associated with the Lagrangian

$$L_2^{(2)} = A_2\left(\frac{1}{3}R^2 - R_{ij}R^{ij}\right), \tag{10}$$

for which the mass is $M_2^2 = 1/2A_2\kappa^2$, is absent (that is, $M_2^2 \to \infty$ when $A_2 \to 0$).

The significance of these ghosts therefore has to be clarified. If we consider the quadratic theory, the ghost-free scalar sector gives rise to the propagator [3, 4]

$$G_0(k) = \frac{1}{k^2}, \tag{11}$$

while the spin-2 propagator is

$$G_2(k) = \frac{1}{k^2} - \frac{1}{(k^2 + M_2^{-2})}. \tag{12}$$

There is no violation of unitarity due to the scalar propagator (11), provided that $A_0 \geq 0$, but, as has often been discussed in the past, unitarity cannot be completely maintained for the spin-2 sector. If M_2 is real, the second term in Eq. (12) necessarily represents a ghost, that can only be removed through substitution of the causal particle by a tachyon with $M_2^2 < 0$, which has the effect of excluding the ghost below the critical momentum $k_c = 1/|M_2|$. At higher momenta $k > k_c$, the ghost returns and is present as a tachyon. If $|M_2|$ is sufficiently large, however, one can argue, following Weinberg [6], that neither ghost nor tachyon is excited at low energies $k^2 \ll |M_2^2|$, and that the violation of unitarity can consequently be ignored.

Classically, an important consideration is the stability of the theory to perturbations, restricted in the first instance to be linear. Although unitarity may be approximately conserved at low energies, when the ghosts and tachyons can be ignored quantum-field theoretically, it is not necessarily the case that the low-energy stability criterion still holds in the presence of higher-derivative terms. Subsequent to the earlier analysis by Ruzmaĭkina and Ruzmaikin [7,8], this question was discussed by Barrow and Ottewill [9], as an application of singular perturbation theory. From the results of Bailin *et al.* [11], it becomes clear [12] that the stability of the theory is strictly linked to the absence of tachyons — that is, a tachyonic Lagrangian will give rise to classical instabilities. This can be seen for the theory $L_1 + L_2^{(0)}$, which we shall consider in more detail below with regard to the field redefinition (3). Here, stability is determined by the sign of A_0 alone [7–9], which distinguishes the stable, tachyon-free region $A_0 \geq 0$ from the unstable, tachyonic region $A_0 < 0$. (The instability of the Kerr metric in this theory was discussed by Hersh and Ove [10].)

3. The theory $L = -R/2\kappa^2 + A_0 R^2$

Application of the transformation (3) to an arbitrary Lagrangian function is rather complicated. The most immediate problem, already evident in the field redefinition (6), is the impossibility of inverting the metric g'_{ij} to obtain g'^{ij} by analytical methods when γ is non-zero. In order to have a working example, we shall therefore set $\gamma = 0$, retaining only the quadratic term $A_0 R^2$ in L, whereupon expression (6) reduces to the conformal transformation

$$g_{ij} \to g'_{ij} = \mathrm{e}^{2v} g_{ij}\,, \tag{13}$$

where the conformal factor is

$$\mathrm{e}^{2v} = 1 + \beta R\,. \tag{14}$$

Under the transformation (13), the Ricci scalar changes according to

$$R \to R' = \mathrm{e}^{-2v}\left[R - 6(\nabla v)^2 - 6\Box v\right]\,. \tag{15}$$

We then find, starting from the primed metric and discarding divergences, that the theory

$$\mathcal{L}' = \sqrt{-g'}\left(-\frac{1}{2}\kappa^{-2}R' + A_0 R'^2\right) \tag{16}$$

is transformed into

$$\begin{aligned}\mathcal{L} \;=\; \sqrt{-g}\Bigg\{ -\frac{1}{2}\kappa^{-2}R + \left(A_0 - \frac{1}{2}\kappa^{-2}\beta\right)R^2 + 3\beta\Bigg[\frac{(A_0 - \frac{1}{4}\kappa^{-2}\beta)}{(1+\beta R)} \\ + \frac{A_0}{(1+\beta R)^2}\Bigg](\nabla R)^2 + \frac{9\beta^2 A_0}{(1+\beta R)^2}\left[\Box R - \frac{\beta(\nabla R)^2}{2(1+\beta R)}\right]^2\Bigg\}.\end{aligned} \tag{17}$$

Thus, the field redefinition (13) can be used to remove the term in R^2 by setting

$$\beta = 2\kappa^2 A_0\,, \tag{18}$$

in which case the Lagrangian (17) reads

$$\begin{aligned}\mathcal{L} \;=\; \kappa^{-2}\sqrt{-g}\Bigg\{ -\frac{1}{2}R + \frac{3}{4}\beta^2\left[\frac{3+\beta R}{(1+\beta R)^2}\right](\nabla R)^2 \\ + \frac{9\beta^3}{2(1+\beta R)^2}\left[\Box R - \frac{\beta(\nabla R)^2}{2(1+\beta R)}\right]^2\Bigg\}.\end{aligned} \tag{19}$$

At low curvatures, such that $\beta|R| \ll 1$, expression (19) reduces to the power series

$$\begin{aligned}\mathcal{L} &= \kappa^{-2}\sqrt{-g}\Bigg\{ -\frac{1}{2}R + \frac{9}{4}\beta^2(\nabla R)^2 + \frac{3}{4}\beta^3\left[6(\Box R)^2 - 5R(\nabla R)^2\right] \\ &\quad + \frac{3}{4}\beta^4 R\left[7R^2(\nabla R)^2 - 6\Box R(\nabla R)^2 - 12R(\Box R)^2\right] \\ &\quad + \frac{9}{8}\beta^5\left[-6R^3(\nabla R)^2 + 12R\Box R\left[R\Box R + (\nabla R)^2\right] + (\nabla R)^4\right] + \ldots\Bigg\} . \end{aligned} \tag{20}$$

Specializing to the cosmological Friedmann space-time

$$ds^2 = dt^2 - a_0^2 \mathrm{e}^{2\alpha(t)} d\boldsymbol{x}^2 , \tag{21}$$

where t is comoving time and $a_0 \mathrm{e}^{\alpha(t)}$ is the radius function of the three-space $d\boldsymbol{x}^2$, whose curvature is k, we have, denoting d/dt by $^\bullet$,

$$\begin{aligned} R &= -6\left(\ddot\alpha + 2\dot\alpha^2 + ka^{-2}\right) , \\ \dot R &= -6\left(\dddot\alpha + 4\dot\alpha\ddot\alpha - 2ka^{-2}\dot\alpha\right) , \\ \ddot R &= -6\left[\ddddot\alpha + 4(\ddot\alpha^2 + \dot\alpha\dddot\alpha) - 2ka^{-2}(\ddot\alpha - 2\dot\alpha^2)\right] , \\ (\nabla R)^2 &\equiv \dot R^2 = 36\left[\dddot\alpha^2 + 8\dot\alpha\ddot\alpha\dddot\alpha + 16\dot\alpha^2\ddot\alpha^2 - 4ka^{-2}(\dot\alpha\dddot\alpha + 4\dot\alpha^2\ddot\alpha) + 4k^2a^{-4}\dot\alpha^2\right] \end{aligned}$$

and

$$\Box R \equiv 3\dot\alpha\dot R + \ddot R = -6\left[\ddddot\alpha + 7\dot\alpha\dddot\alpha + 12\dot\alpha^2\ddot\alpha + 4\ddot\alpha^2 - 2ka^{-2}(\ddot\alpha + \dot\alpha^2)\right] . \tag{22}$$

After discarding a divergence, the Lagrangian density can be written as

$$\begin{aligned} \mathcal{L} &= 3\kappa^{-2}a_0^3\mathrm{e}^{3\alpha}\Bigg\{ -\dot\alpha^2 + ka_0^{-2}\mathrm{e}^{-2\alpha} + 27\beta^2\frac{\left[1 - 2\beta(\ddot\alpha + 2\dot\alpha^2 + ka^{-2}\right]}{\left[1 - 6\beta(\ddot\alpha + 2\dot\alpha^2 + ka^{-2}\right]^2} \\ &\quad \times\left[\dddot\alpha^2 + 8\dot\alpha\ddot\alpha\dddot\alpha + 16\dot\alpha^2\ddot\alpha^2 - 4ka^{-2}(\dot\alpha\dddot\alpha + 4\dot\alpha^2\ddot\alpha) + 4k^2a^{-4}\dot\alpha^2\right] \\ &\quad + \mathcal{O}(\beta^3)(\text{8th-order terms}^{\cdots\cdots\cdot})\Bigg\} . \end{aligned} \tag{23}$$

In the form (23), $\mathcal{L}$ is intractable, and therefore we simplify this expression by setting $k = 0$ and assuming the low-energy régime $\beta|R| \ll 1$, whereupon, ignoring the terms $\mathcal{O}(\beta^3)$, we have

$$\mathcal{L} \approx A\left[-\dot\alpha^2 + b(\dddot\alpha^2 + 4\dot\alpha^2\ddot\alpha^2 - 4\ddot\alpha^3)\right]\mathrm{e}^{3\alpha} , \tag{24}$$

after defining the constants

$$A = 3\kappa^{-2}a_0^3 , \qquad b = 27\beta^2 . \tag{25}$$

The Hamiltonian density for the approximate theory (24), which vanishes due to reparametrization invariance, also being the $\binom{0}{0}$ component of the vacuum Einstein equations, is given, see Whittaker [13], upon application of the method of Ostrogradsky [14], by

$$\mathcal{H}=\dddot{\alpha}\,\frac{\partial\mathcal{L}}{\partial\dddot{\alpha}}+\ddot{\alpha}\left[\frac{\partial\mathcal{L}}{\partial\ddot{\alpha}}-\left(\frac{\partial\mathcal{L}}{\partial\dddot{\alpha}}\right)^{\bullet}\right]+\dot{\alpha}\left[\frac{\partial\mathcal{L}}{\partial\dot{\alpha}}-\left(\frac{\partial\mathcal{L}}{\partial\ddot{\alpha}}\right)^{\bullet}+\left(\frac{\partial\mathcal{L}}{\partial\dddot{\alpha}}\right)^{\bullet\bullet}\right]-\mathcal{L}=0\,, \tag{26}$$

where

$$\begin{aligned}
\dot{\alpha}\frac{\partial\mathcal{L}}{\partial\dot{\alpha}} &= 2A\left(-\dot{\alpha}^2+4b\dot{\alpha}^2\ddot{\alpha}^2\right)\mathrm{e}^{3\alpha}\,,\\
\ddot{\alpha}\frac{\partial\mathcal{L}}{\partial\ddot{\alpha}} &= 4Ab\left(2\dot{\alpha}^2\ddot{\alpha}^2-3\ddot{\alpha}^3\right)\mathrm{e}^{3\alpha}\,, &(27)\\
-\dot{\alpha}\left(\frac{\partial\mathcal{L}}{\partial\ddot{\alpha}}\right)^{\bullet} &= 4Ab\left(6\dot{\alpha}\ddot{\alpha}\dddot{\alpha}+5\dot{\alpha}^2\ddot{\alpha}^2-2\dot{\alpha}^3\dddot{\alpha}-6\dot{\alpha}^4\ddot{\alpha}\right)\mathrm{e}^{3\alpha}\,, &(28)\\
\dddot{\alpha}\frac{\partial\mathcal{L}}{\partial\dddot{\alpha}} &= 2Ab\dddot{\alpha}^2\mathrm{e}^{3\alpha}\,,\\
-\ddot{\alpha}\left(\frac{\partial\mathcal{L}}{\partial\dddot{\alpha}}\right)^{\bullet} &= -2Ab\ddot{\alpha}\left(\ddddot{\alpha}+3\dot{\alpha}\dddot{\alpha}\right)\mathrm{e}^{3\alpha} &(29)
\end{aligned}$$

and

$$\dot{\alpha}\left(\frac{\partial\mathcal{L}}{\partial\dddot{\alpha}}\right)^{\bullet\bullet}=2Ab\left(\dot{\alpha}\overset{\cdot\cdot\cdot\cdot\cdot}{\alpha}+6\dot{\alpha}^2\ddddot{\alpha}+9\dot{\alpha}^3\dddot{\alpha}+3\dot{\alpha}\ddot{\alpha}\dddot{\alpha}\right)\mathrm{e}^{3\alpha}\,. \tag{30}$$

Substituting expressions (27)–(30) into the Hamiltonian (26), we thus obtain

$$\begin{aligned}
\mathcal{H} &= A\Big\{-\dot{\alpha}^2+b\Big[2(\dot{\alpha}\overset{\cdot\cdot\cdot\cdot\cdot}{\alpha}-\ddot{\alpha}\ddddot{\alpha})+4\dot{\alpha}^2(3\ddddot{\alpha}+8\ddot{\alpha}^2)\\
&\qquad +24\ddot{\alpha}(\dot{\alpha}\dddot{\alpha}-\dot{\alpha}^4)+10\dot{\alpha}^3\dddot{\alpha}-8\ddot{\alpha}^3+\dddot{\alpha}^2\Big]\Big\}\mathrm{e}^{3\alpha}=0\,.
\end{aligned} \tag{31}$$

4. The linearized stability analysis

In order to examine the stability of the metric to linear perturbations, we write

$$\alpha(t)=\alpha_0(t)+\delta(t)\,, \tag{32}$$

where $\alpha_0(t)$ is a solution to the unperturbed field equations and $|\overset{(n)}{\delta}(t)|\ll|\overset{(n)}{\alpha}_0(t)|$, $n=1,2,3,4$ or 5 counting the number of time derivatives, so that

$$\left[\overset{(n)}{\alpha}(t)\right]^m\approx\left[\overset{(n)}{\alpha}_0(t)\right]^m+m\left[\overset{(n)}{\alpha}_0(t)\right]^{m-1}\overset{(n)}{\delta}(t)\,. \tag{33}$$

Upon substitution of the perturbed metric (32) into the vacuum Hamiltonian (31), assuming the matter energy-density $\rho \equiv T^0_0 \equiv \kappa^{-2} H_{\rm m}$ to be unperturbed if non-vanishing, we obtain the differential equation obeyed by $\delta(t)$,

$$\mathcal{A}\dot{\delta} + \mathcal{B}\ddot{\delta} + \mathcal{C}\dddot{\delta} + \mathcal{D}\ddddot{\delta} + \mathcal{E}\overset{.....}{\delta} = 0\,, \tag{34}$$

in which the coefficients are given by

$$\begin{aligned}
\mathcal{A} &= -2\dot{\alpha}_0 + 2b\left[\overset{.....}{\alpha}_0 + 12\left(\dot{\alpha}_0\ddddot{\alpha}_0 + \ddot{\alpha}_0\dddot{\alpha}_0\right) + 15\dot{\alpha}_0^2\dddot{\alpha}_0 + 32\dot{\alpha}_0\ddot{\alpha}_0^2 - 48\dot{\alpha}_0^3\ddot{\alpha}_0\right]\,,\\
\mathcal{B} &= 2b\left[-\ddddot{\alpha}_0 + 12\left(\dot{\alpha}_0\dddot{\alpha}_0 - \dot{\alpha}_0^4 - \ddot{\alpha}_0^2\right) + 32\dot{\alpha}_0^2\ddot{\alpha}_0\right]\,,\\
\mathcal{C} &= 2b\left(\dddot{\alpha}_0 + 12\dot{\alpha}_0\ddot{\alpha}_0 + 5\dot{\alpha}_0^3\right)\,,\\
\mathcal{D} &= 2b\left(6\dot{\alpha}_0^2 - \ddot{\alpha}_0\right)
\end{aligned}$$

and

$$\mathcal{E} = 2b\dot{\alpha}_0\,. \tag{35}$$

Now, let us assume that the perturbations can be parametrized as

$$\delta(t) = \delta_0 {\rm e}^{\lambda t}\,, \tag{36}$$

where $1/|\lambda|$ is the growth or decay time, according as λ is positive or negative, Eq. (34) then reducing to the quintic

$$\mathcal{A}\lambda + \mathcal{B}\lambda^2 + \mathcal{C}\lambda^3 + \mathcal{D}\lambda^4 + \mathcal{E}\lambda^5 = 0\,. \tag{37}$$

The root $\lambda = 0$ is trivial, since $\delta(t) = \delta_0$ simply describes a constant gauge transformation of $\alpha_0(t)$, and therefore the characteristic equation is essentially the quartic

$$\mathcal{A} + \mathcal{B}\lambda + \mathcal{C}\lambda^2 + \mathcal{D}\lambda^3 + \mathcal{E}\lambda^4 = 0\,. \tag{38}$$

As previously [15], we are chiefly concerned with the low-energy cosmological Friedmann space-time, assuming a perfect-fluid source with energy-density ρ and pressure $p = (\gamma - 1)\rho$, where γ is the adiabatic index. Far from the Planck era, at $t \gg t_{\rm P}$, where $t_{\rm P} \equiv M_{\rm P}^{-1} = 5.38 \times 10^{-44}$ s is the Planck time, the solution takes the form

$$\begin{aligned}
\dot{\alpha}_0 &= 2/3\gamma t\,, \qquad \ddot{\alpha}_0 = -2/3\gamma t^2\,, \qquad \dddot{\alpha}_0 = 4/3\gamma t^3\,,\\
\ddddot{\alpha}_0 &= -4/\gamma t^4 \qquad \text{and} \qquad \overset{.....}{\alpha}_0 = 16/\gamma t^5\,,
\end{aligned} \tag{39}$$

substitution of which into expressions (35) for the coefficients yields

$$\mathcal{A} = -4/3\gamma t + \mathcal{O}(bt^{-5})\,, \qquad \mathcal{B} \sim bt^{-4}\,, \qquad \mathcal{C} \sim bt^{-3}\,,$$

$$\mathcal{D} \sim bt^{-2} \qquad \text{and} \qquad \mathcal{E} = 4b/3\gamma t\,. \tag{40}$$

To lowest order in t^{-1}, Eq. (38) therefore reads $\lambda^4 \approx 1/b$, but this is actually not quite the correct answer.

To explain why, let us examine the last term in the Lagrangian (19), which, from expressions (22), when $k = 0$, is

$$\mathcal{L}_3+\mathcal{L}_4+\mathcal{L}_5=\frac{3A\beta^3}{2}\left[\Box R-\frac{1}{2}\beta\,(\nabla R)^2\right]^2 = 54A\beta^3\Big[\ddddot{\alpha}+7\dot{\alpha}\dddot{\alpha}+12\dot{\alpha}^2\ddot{\alpha}+4\ddot{\alpha}^2 \\ -3\beta\left(\dddot{\alpha}^2+8\dot{\alpha}\ddot{\alpha}\dddot{\alpha}+16\dot{\alpha}^2\ddot{\alpha}^2\right)\Big]^2 \mathrm{e}^{3\alpha}=54A\beta^3\Big[\ddddot{\alpha}^2+14\dot{\alpha}\dddot{\alpha}\,\ddddot{\alpha}+\ldots\Big]\mathrm{e}^{3\alpha}\,. \tag{41}$$

At $t \gg t_{\mathrm{P}}$, we find that $\mathcal{L}_1 \gg \mathcal{L}_2 \gg \mathcal{L}_3 \gg \mathcal{L}_4 \gg \mathcal{L}_5$ classically, where the suffix counts the power of βR, while it turns out that not only $\mathcal{L}_2$ but also $\mathcal{L}_3$ become important at the perturbative level. For expression (41) differs from the Lagrangian (23) [which also contains some terms of order $(\beta R)^3$ with up to three derivative $\dddot{\alpha}$] by the presence of an additional fourth-derivative term in $\ddddot{\alpha}$, due to $\Box R$, necessitating enhancement of the Hamiltonian (26) to

$$\mathcal{H} = \ddddot{\alpha}\,\frac{\partial\mathcal{L}}{\partial\ddddot{\alpha}} + \dddot{\alpha}\left[\frac{\partial\mathcal{L}}{\partial\dddot{\alpha}} - \left(\frac{\partial\mathcal{L}}{\partial\ddddot{\alpha}}\right)^{\bullet}\right] + \ddot{\alpha}\left[\frac{\partial\mathcal{L}}{\partial\ddot{\alpha}} - \left(\frac{\partial\mathcal{L}}{\partial\dddot{\alpha}}\right)^{\bullet} + \left(\frac{\partial\mathcal{L}}{\partial\ddddot{\alpha}}\right)^{\bullet\bullet}\right] \\ +\dot{\alpha}\left[\frac{\partial\mathcal{L}}{\partial\dot{\alpha}} - \left(\frac{\partial\mathcal{L}}{\partial\ddot{\alpha}}\right)^{\bullet} + \left(\frac{\partial\mathcal{L}}{\partial\dddot{\alpha}}\right)^{\bullet\bullet} - \left(\frac{\partial\mathcal{L}}{\partial\ddddot{\alpha}}\right)^{\bullet\bullet\bullet}\right] - \mathcal{L} = 0\,. \tag{42}$$

Substituting $\mathcal{L}_3$ into expression (42), we find that the resulting contribution $\mathcal{H}_3$ to the total Hamiltonian density contains sixth- and seventh-derivative terms $\overset{\cdots\cdots}{\alpha}$ and $\overset{\cdots\cdots\cdot}{\alpha}$,

$$\mathcal{H}_3 = 4Ab\beta\Big[\left(\ddot{\alpha}-9\dot{\alpha}^2\right)\overset{\cdots\cdots}{\alpha} - \dot{\alpha}\,\overset{\cdots\cdots\cdot}{\alpha} + \text{ignorable 8th-order terms} \\ \text{with 5 or less derivatives}\ {}^{\bullet}\ \text{on a single}\ \alpha\Big]\mathrm{e}^{3\alpha}\,. \tag{43}$$

The point is that the second term in $\mathcal{H}_3$ contains $\overset{\cdots\cdots\cdot}{\alpha}$ multiplied by a single factor of $\dot{\alpha}$, which gives rise perturbatively to an additional contribution $\dot{\alpha}_0\,\overset{\cdots\cdots\cdot}{\delta}$ in Eq. (34). Going through the analysis again with $\alpha(t)$ given by Eq. (32), we obtain the characteristic equation, valid at low energies,

$$\mathcal{A} + \mathcal{E}\lambda^4 + \mathcal{G}\lambda^6 = 0\,, \tag{44}$$

where the additional coefficient is, from Eqs. (25) and (39),

$$\mathcal{G} = -4\beta b\dot{\alpha}_0(t) = -8b^{3/2}/9\sqrt{3}\gamma t\,. \tag{45}$$

By change of variable, Eq. (44) can be written as

$$2y^3 - 3y^2 + 1 = 0\,, \tag{46}$$

where

$$y = \sqrt{b/3}\lambda^2 \,, \tag{47}$$

the three roots of which are all real,

$$y = -\frac{1}{2}, 1, 1 \,, \tag{48}$$

implying that

$$\lambda = (3/b)^{1/4}(\pm\,\mathrm{i}/\sqrt{2}, \pm 1, \pm 1)\,. \tag{49}$$

Eq. (44) is invariant under the transformation $\lambda \to -\lambda$, and thus possesses modes that grow exponentially on the time scale

$$\tau = (b/3)^{1/4} = (48\pi|A_0|)^{1/2} t_{\mathrm{P}} \tag{50}$$

independently of the sign of A_0, since $b = 108\kappa^4 A_0^2$.

We are now in a position to make a comparison with the earlier stability analysis applied directly to the theory (16). It was first shown in Refs. [7,8] that this theory is stable only if the coefficient A_0 is positive semi-definite. This result was extended in Ref. [9] to the more general theory in which $A_0 R^2$ is replaced by an arbitrary function $f(R)$ of the scalar curvature, it being found that the stability to linear perturbations is determined entirely, at low energies, by the term quadratic in R. The stable case $A_0 \geq 0$ corresponds to a positive semi-definite scalar mass-squared $M_0^2 \geq 0$, the theory then being ghost-free (due to the absence of the term $R_{ij}R^{ij}$ in L) and tachyon-free (due to the positivity of M_0^2). The freedom from ghosts persists through the transformation defined by Eqs. (13), (14) and (18), which produces a kinetic-energy term $(\nabla R)^2$ with positive semi-definite coefficient $\approx 9\beta^2/4\kappa^2$ at low energies, plus a term $\left[\Box R - \beta(\nabla R)^2/2(1+\beta R)\right]^2$ with positive semi-definite coefficient $9\beta^3/2\kappa^2((1+\beta R)^2$.

The situation here, however, is slightly different from that of Refs. [7–9]. In place of the quadratic characteristic equation with imaginary roots — given by Eq. (81) of Ref. [15] in the case of the superstring theory considered there — we have the sextic Eq. (44), which necessarily includes two real, positive roots, implying instability of the metric. Whereas the question of stability for higher-derivative Lagrangians of the type $L = f(R)$ — or more generally any function $L = f(R, R_{ij}, R_{ijkl})$ — is decided at low energies by the quadratic term $\mathcal{R}^2$, because no derivative of α higher than third degree $\dddot{\alpha}$ ever occurs, the transformation (13), (14) leads to the occurrence of the fourth-degree derivative $\ddddot{\alpha}$.

To conclude, the instability of the theory (19), manifest in the solutions (49) referred to the approximation (20), argues against the viability of making the field redefinition (13), from the classical point of view, in the Friedmann space-time (21).

5. Superstrings

The field-theory limit of the heterotic superstring theory of Gross *et al.* [16–18] is a ten-dimensional supergravity theory [19, 20] (subject to stringy modifications beyond lowest order in the Regge slope parameter α'), in which the space-time metric can be written in the diagonal form

$$\hat{g}_{AB}(X^C) = \mathrm{diag}\left[A_{\mathrm{r}}^{-1}(x^k)g_{ij}(x^l), B_{\mathrm{r}}(x^k)\bar{g}_{\mu\nu}(y^\xi)\right], \tag{51}$$

where A_{r} is the inverse tree-level gauge coupling g_{s}^{-2} and B_{r} is the radius-squared of the internal space $\bar{g}_{\mu\nu}$. In the preceding sections, we have dealt with the field redefinition (3) as applied to the four-metric $g_{ij}(x^k)$, and in principle the question arises, whether the field-redefinition procedure is also applicable to the internal metric, which is independent of time $t \equiv x^0$, thus not being susceptible to temporal growth.

Symmetry considerations play an important rôle, and requiring the low-energy, four-dimensional theory to be free of anomalies means that the cosmological constant Λ has to vanish, which follows if the internal space is Ricci-flat,

$$\bar{R}_{\mu\nu} = 0, \tag{52}$$

since $\hat{R} \to A_{\mathrm{r}}R + B_{\mathrm{r}}^{-1}\bar{R}$ under dimensional reduction. This immediately reduces the field redefinition analogous to Eq. (6) to an identity,

$$\bar{g}'_{\mu\nu}(y^\xi) = \bar{g}_{\mu\nu}(y^\xi), \tag{53}$$

and therefore such redefinitions are not usually given consideration.

In addition to the Einstein–Hilbert term $Y_1 \equiv \int d^6y\sqrt{\bar{g}}\bar{R}$, the integral over the internal space contains higher-derivative terms, for example

$$Y_2 \equiv \int d^6y\sqrt{\bar{g}}\bar{R}_{\mu\nu\xi o}\bar{R}^{\mu\nu\xi o}, \quad Y_4 \equiv \int d^6y\sqrt{\bar{g}}\bar{\mathcal{R}}^4_{\mu\nu\xi o} \tag{54}$$

at quadratic and quartic order, originating from the corresponding ten-dimensional terms $\hat{\mathcal{R}}^2_{\mathrm{E}}$ and $\hat{\mathcal{R}}^4$, respectively. In six dimensions, the Euler-characteristic density $\sqrt{\bar{g}}\bar{\mathcal{R}}^2_{\mathrm{E}}$ is not a divergence and therefore the term Y_2 remains after imposing Eq. (52), having to be cancelled by an equal and opposite contribution $F_2 \equiv \int d^6y\sqrt{\bar{g}}\bar{F}_{\mu\nu}\bar{F}^{\mu\nu}$ from the gauge field $\bar{F}_{\mu\nu}$. This is most directly achieved by identifying the spin connection with the gauge connection, as first advocated by Pauli [21] in early discussions of dimensional reduction — see O'Raifeartaigh [22] — and described in detail by Candelas *et al.* [23], who showed in the superstring context how a Ricci-flat Kähler manifold $\bar{g}_{\mu\nu}$ is necessary for the preservation of $N = 1$ supersymmetry in four dimensions.

With regard to the known gravitational terms up to quartic order $\bar{\mathcal{R}}^4$, the cubic term $\bar{\mathcal{R}}^3$ is absent, there being no initial cubic ten-dimensional term $\hat{\mathcal{R}}^3$ [24], while reduction of $\hat{\mathcal{R}}^4$ yields only the term $R\bar{\mathcal{R}}^3$ cubic in $\bar{\mathcal{R}}$, which simply renormalizes the gravitational constant [25,26]. The internal Calabi–Yau space must be highly curved, however, in order that the six-dimensional Euler characteristic $\bar{\chi}$ be non-vanishing, with $\bar{\chi} = -6$ for a three-generation theory [27,28] and consequently we would expect *a priori* that the quartic contribution Y_4 to Λ is by no means ignorable, even when Eq. (52) is satisfied. Nor is it at all obvious that Y_4 can be cancelled against a quartic gauge contribution $\int d^6y\sqrt{\bar{g}}\mathcal{F}^4$, when the gauge field has already been adjusted to cancel the quadratic term.

The field-redefinition theorem is of no avail in solving this problem, because metric transformations of the type (8) can only remove higher-order terms possessing a factor of at least one Ricci tensor or Ricci scalar, of the form $(\bar{\mathcal{R}}^{\mu\nu})^{(n-1)}\bar{R}_{\mu\nu}$ or $\bar{\mathcal{R}}^{(n-1)}\bar{R}$, both of which vanish anyway by Eq. (52), terms of the type $\bar{\mathcal{R}}^4_{\mu\nu\xi o}$, involving only the Riemann tensor, remaining invariant.

It therefore seems that we have to impose supersymmetry in order to ensure the vanishing of the net Λ, by suitable adjustment of all the fields in the theory, bosonic as well as fermionic. The field equations for $\bar{g}_{\mu\nu}$ admit both Minkowski space and de Sitter space as maximally symmetric vacuum solutions [29], but if the theory is to contain chiral fermions at the four-dimensional level, a much less symmetric solution is obligatory.

The foregoing considerations determine both the space-time metric g_{ij} and the internal-space metric $\bar{g}_{\mu\nu}$, and hence all the geometrical quantities derived from them. One of the purposes of this analysis was to investigate the status of the four-dimensional, gravitational vacuum solutions derived from the terms $-R/2\kappa^2 + \mathcal{R}^2 + \alpha'^2\mathcal{R}^4$ in the Lagrangian L, obtaining from $\hat{R}$ and $\hat{\mathcal{R}}^4$ in $\hat{L}$. For the curved de Sitter space with cosmological constant [15,29]

$$\Lambda = \left[\frac{18}{337\zeta(3)+1/2}\right]^{1/3} A_{\rm r}^{-1}\kappa^{-2}\,, \tag{55}$$

where $\zeta(3) \equiv 1.202$ is the Riemann zeta function, would be susceptible to change under a transformation of the type (8). If $g_{ij}(x^k)$ is kept fixed, however, the solution (55) becomes immutable, as seemingly required by general covariance, subject only to modifications due to the unknown higher-derivative terms $\mathcal{R}^n$, $n \geq 5$, the effect of which has yet to be calculated.

By the same line of reasoning, the quadratic four-dimensional terms arise entirely from the ten-dimensional quartic term, due to the reduction $\hat{\mathcal{R}}^4 \to \mathcal{R}^2\bar{\mathcal{R}}^2$, where [30]

$$\mathcal{R}^2 = B(R^2 - R_{ij}R^{ij}) \tag{56}$$

and the constant $B \approx 1$. As we have argued previously [26, 31], the combination (56) — which would also change under the transformation (8) — is an expression of supersymmetry, in the non-linear formulation of Volkov and Akulov [32–34]. The corresponding fermionic Lagrangian contains the quadratic interaction terms

$$\mathcal{T}^2 = T^2 - T_{ij}T^{ij}\,, \tag{57}$$

where T_{ij} is the fermionic contribution to the energy-momentum tensor and $T \equiv T_i{}^i$, the two expressions (56) and (57) only agreeing in four dimensions [31], after application of the Einstein equations.

Further, the presence in L of the quadratic term $\mathcal{R}^2$ with coefficient B of order unity is important in cosmology, since it leads, via the indeterminacy principle [35], to metric and density fluctuations in the Universe of approximately the required magnitude and spectrum to explain the existence of galaxies [36, 37].

This paper was written at the University of Cambridge, Cambridge, England.

REFERENCES

[1] G. 't Hooft, *Nucl. Phys.* **B62**, 444 (1973).

[2] G. 't Hooft, M. Veltman, *Ann. Inst. Henri Poincaré* **A20**, 69 (1974).

[3] K.S. Stelle, *Phys. Rev.* **D16**, 953 (1977).

[4] N.H. Barth, S.M. Christensen, *Phys. Rev.* **D28**, 1876 (1983).

[5] A.M. Lawrence, *Phys. Lett.* **B217**, 243 (1989).

[6] S. Weinberg, in *General Relativity, an Einstein Centenary Survey*, S.W. Hawking, W. Israel eds., Cambridge University Press, Cambridge 1979.

[7] T.V. Ruzmaǐkina, A.A. Ruzmaikin, *Zh. Eksp. Teor. Fiz.* **57**, 680 (1969).

[8] T.V. Ruzmaǐkina, A.A. Ruzmaikin, *Sov. Phys. JETP* **30**, 372 (1970).

[9] J.D. Barrow, A.C. Ottewill, *J. Phys.* **A16**, 2757 (1983).

[10] J. Hersh, R. Ove, *Phys. Lett.* **B156**, 305 (1985).

[11] D. Bailin, A. Love, R. Wong, *Phys. Lett.* **B165**, 270 (1985).

[12] M.D. Pollock, *Int. J. Mod. Phys.* **D3**, 569 (1994).

[13] E.T. Whittaker, *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies*, 4th edition, Cambridge University Press, Cambridge 1937, pp. 265–7.

[14] M. Ostrogradsky, *Mém. Acad. Imp. Sci. St.-Pétersbourg*, Série VI, **4**, 385 (1850).

[15] M.D. Pollock, *Int. J. Mod. Phys.* **D15**, 845 (2006); *Erratum ibid.* **22**, 1392001(E) (2013).

[16] D.J. Gross, J.A. Harvey, E. Martinec, R. Rohm, *Phys. Rev. Lett.* **54**, 502 (1985).

[17] D.J. Gross, J.A. Harvey, E. Martinec, R. Rohm, *Nucl. Phys.* **B256**, 253 (1985).

[18] D.J. Gross, J.A. Harvey, E. Martinec, R. Rohm, *Nucl. Phys.* **B267**, 75 (1986).

[19] G.F. Chapline, N.S. Manton, *Phys. Lett.* **B120**, 105 (1983).

[20] E. Bergshoeff, M. de Roo, B. de Wit, P. van Nieuwenhuizen, *Nucl. Phys.* **B195**, 97 (1982).

[21] W. Pauli, unpublished.

[22] L. O'Raifeartaigh, *The Dawning of Gauge Theory*, Princeton University Press, Princeton 1997.

[23] P. Candelas, G.T. Horowitz, A. Strominger, E. Witten, *Nucl. Phys.* **B258**, 46 (1985).

[24] D.J. Gross, E. Witten, *Nucl. Phys.* **B277**, 1 (1986).

[25] M.D. Pollock, *Phys. Lett.* **B411**, 68 (1997).

[26] M.D. Pollock, *Int. J. Mod. Phys.* **A16**, 3217 (2001).

[27] B.R. Greene, K.H. Kirklin, P.J. Miron, G.G. Ross, *Phys. Lett.* **B180**, 69 (1986).

[28] B.R. Greene, K.H. Kirklin, P.J. Miron, G.G. Ross, *Nucl. Phys.* **B292**, 606 (1987).

[29] M.D. Pollock, *Int. J. Mod. Phys.* **D16**, 591 (2007); *Erratum ibid.* **23**, 1492001(E) (2014).

[30] M.D. Pollock, *Int. J. Mod. Phys.* **A7**, 4149 (1992).

[31] M.D. Pollock, *Int. J. Mod. Phys.* **A21**, 373 (2006).

[32] D.V. Volkov, V.P. Akulov, *Pis'ma Zh. Ehksp. Teor. Fiz.* **16**, 621 (1972).

[33] D.V. Volkov, V.P. Akulov, *JETP Lett.* **16**, 438 (1972).

[34] D.V. Volkov, V.P. Akulov, *Phys. Lett.* **B46**, 109 (1973).

[35] W. Heisenberg, *Z. Phys.* **43**, 172 (1927).

[36] M.D. Pollock, *Mod. Phys. Lett.* **A10**, 539 (1995).

[37] M.D. Pollock, *Int. J. Mod. Phys.* **D15**, 1487 (2006).