Vol. 39 (2008) ACTA PHYSICA POLONICA B No 6

THERMODYNAMICS OF CHARGED BTZ BLACK HOLES AND EFFECTIVE STRING THEORY

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(Received February 23, 2007)

In this paper we study the first law of thermodynamics for the (2+1) dimensional charged BTZ black hole considering a pair of thermodynamical systems constructed with the two horizons of this solution. We show that these two systems are similar to the right and left movers of string theory and that the temperature associated with the black hole is the harmonic mean of the temperatures associated with these two systems.

PACS numbers: 04.70.Dy, 04.70.Bw, 11.25.–w

1. Introduction

Bekenstein and Hawking showed that black holes have non-zero entropy and that they emit a thermal radiation that is proportional to its surface gravity at the horizon. When the black hole has other properties as angular momentum $\boldsymbol{J}$ and electric charge Q, these quantities are related with the mass through the identity

$$dM = TdS + \varOmega dJ + \varPhi dQ\,, \tag{1}$$

where $\varOmega = \frac{\partial M}{\partial J}$ is the angular velocity and $\varPhi = \frac{\partial M}{\partial Q}$ is the electric potential. This relation is called *the first law of black hole thermodynamics* [1,2]. When the black hole has two horizons, it is known that it is possible to associate a first law with each of them. The outer horizon is related with the Hawking radiation while the inner horizon is related with the absorption process [3–5].

In this paper we will apply the method described by Wu [6] to describe the thermodynamics of the charged BTZ black hole in (2+1) dimensions and relate it with the effective string theory and D-brane description of black holes. It is well known that a characteristic of effective strings or D-branes is that their spectrum can be expressed as right and left moving excitations. The entropy of the system is given as the sum of the two contributions,

$$S = S_{\rm R} + S_{\rm L}\,, \tag{2}$$

while the temperature associated with the system is the harmonic mean of the temperatures of the modes,

$$\frac{2}{T_{\rm H}} = \frac{1}{T_{\rm R}} + \frac{1}{T_{\rm L}}\,. \tag{3}$$

Therefore, we will define two thermodynamical systems as the sum and the difference of the two horizons associated with the charged BTZ black hole. These systems resemble the R and L moving modes of string theory and will provide a way to show how the Hawking temperature $T_{\rm H}$ associated with the BTZ black hole can be interpreted as the harmonic mean of the temperature of the R and L parts. We will also identify the electric potential associated with each of the systems and their relation with the black hole properties.

2. The charged BTZ black hole

The charged BTZ black hole [7] is a solution of $(2+1)$ dimensional gravity with a negative cosmological constant $\Lambda = -\frac{1}{l^2}$. Its line element can be written as

$$ds^2 = -\Delta dt^2 + \frac{dr^2}{\Delta} + r^2 d\varphi^2\,, \tag{4}$$

where the lapse function is

$$\Delta = -M + \frac{r^2}{l^2} - \frac{Q^2}{2}\ln\left(\frac{r}{l}\right)\,. \tag{5}$$

This solution has two horizons given by the condition $\Delta = 0$,

$$M = \frac{r_\pm^2}{l^2} - \frac{Q^2}{2}\ln\left(\frac{r_\pm}{l}\right)\,. \tag{6}$$

The Bekenstein–Hawking entropy associated with the black hole is twice the perimeter of the outer horizon,

$$S = 4\pi r_+\,, \tag{7}$$

and therefore, the mass can be written as

$$M = \frac{S^2}{(4\pi)^2 l^2} - \frac{Q^2}{2} \ln\left(\frac{S}{4\pi l}\right) . \tag{8}$$

The Bekenstein–Smarr integral mass formula is [10]

$$M = \frac{1}{2}TS + \frac{1}{2}\Phi Q + \frac{1}{4}Q^2 , \tag{9}$$

$$M = \kappa\mathcal{P} + \frac{1}{2}\Phi Q + \frac{1}{4}Q^2 , \tag{10}$$

where $\mathcal{P} = \frac{P}{2\pi}$ is the "reduced" perimeter and κ is the surface gravity. Thus, the Hawking–Bekenstein entropy can be written as

$$S = 4\pi\mathcal{P} , \tag{11}$$

and the mass (8) is given by

$$M = \frac{\mathcal{P}^2}{l^2} - \frac{Q^2}{2} \ln\left(\frac{\mathcal{P}}{l}\right) . \tag{12}$$

Finally, the differential form of the first law for this black hole takes the form

$$dM = 2\kappa d\mathcal{P} + \Phi dQ . \tag{13}$$

As discussed before [4, 8], we can associate a thermodynamics to both outer and inner horizons. The four laws associated with these horizons describe the Hawking radiation process as well as the absorption process. Therefore, the integral and differential mass formulae can be written for the two horizons,

$$M = \frac{{\mathcal{P}_\pm}^2}{l^2} - \frac{Q^2}{2} \ln\left(\frac{\mathcal{P}_\pm}{l}\right) , \tag{14}$$

$$dM = 2\kappa_\pm d\mathcal{P}_\pm + \Phi_\pm dQ . \tag{15}$$

From these relations, is easy to see that the surface gravity and electrostatic potential at the two horizons are

$$\kappa_\pm = \frac{1}{2} \left.\frac{\partial M}{\partial \mathcal{P}_\pm}\right|_Q = \frac{\mathcal{P}_\pm}{l^2} - \frac{Q^2}{4P_\pm} = \frac{r_\pm}{l^2} - \frac{Q^2}{4r_\pm} , \tag{16}$$

$$\Phi_\pm = \left.\frac{\partial M}{\partial Q}\right|_{\mathcal{P}_\pm} = -Q\ln\left(\frac{\mathcal{P}_\pm}{l}\right) = -Q\ln\left(\frac{r_\pm}{l}\right) , \tag{17}$$

while the entropy and temperature associated with each horizon are

$$\begin{aligned} S_\pm &= 4\pi\mathcal{P}_\pm\,, &(18)\\ T_\pm &= \frac{\kappa_\pm}{2\pi}\,. &(19)\end{aligned}$$

From Eq. (14) we can obtain to important relations,

$$\begin{aligned} M &= \frac{\mathcal{P}_+^2+\mathcal{P}_-^2}{2l^2} - \frac{Q^2}{4}\ln\left(\frac{\mathcal{P}_+\mathcal{P}_-}{l^2}\right)\,, &(20)\\ \frac{\mathcal{P}_+^2-\mathcal{P}_-^2}{l^2} &= \frac{Q^2}{2}\ln\left(\frac{\mathcal{P}_+}{\mathcal{P}_-}\right)\,. &(21)\end{aligned}$$

Now, using the inner and outer horizons we will define two independent thermodynamical systems. Following Wu [6], the R-system will have a reduced perimeter corresponding to the sum of the inner and outer perimeters while L-system corresponds to the difference of these perimeters,

$$\begin{aligned} \mathcal{P}_\mathrm{R} &= \mathcal{P}_+ + \mathcal{P}_-\,, &(22)\\ \mathcal{P}_\mathrm{L} &= \mathcal{P}_+ - \mathcal{P}_-\,. &(23)\end{aligned}$$

It is important to note that each of these systems carry two hairs (M,Q), but we will show that the electric potential is different for each of them. However, to begin, we will obtain the thermodynamical relations for these systems and then, we will relate them with the thermodynamics of the charged BTZ black hole and its Hawking temperature.

3. R- and L-system thermodynamics

First, consider the surface gravity for these systems,

$$\kappa_\mathrm{R} = \frac{1}{2}\left(\frac{\partial M}{\partial\mathcal{P}_\mathrm{R}}\right)_Q = \frac{1}{2}\left[\left(\frac{\partial M}{\partial\mathcal{P}_+}\right)_Q\left(\frac{\partial\mathcal{P}_+}{\partial\mathcal{P}_\mathrm{R}}\right)_Q + \left(\frac{\partial M}{\partial\mathcal{P}_-}\right)_Q\left(\frac{\partial\mathcal{P}_-}{\partial\mathcal{P}_\mathrm{R}}\right)_Q\right]\,, \quad (24)$$

$$\kappa_\mathrm{L} = \frac{1}{2}\left(\frac{\partial M}{\partial\mathcal{P}_\mathrm{L}}\right)_Q = \frac{1}{2}\left[\left(\frac{\partial M}{\partial\mathcal{P}_+}\right)_Q\left(\frac{\partial\mathcal{P}_+}{\partial\mathcal{P}_\mathrm{L}}\right)_Q + \left(\frac{\partial M}{\partial\mathcal{P}_-}\right)_Q\left(\frac{\partial\mathcal{P}_-}{\partial\mathcal{P}_\mathrm{L}}\right)_Q\right]\,. \quad (25)$$

Since $\mathcal{P}_\mathrm{R} = \mathcal{P}_+ + \mathcal{P}_-$, we get

$$\frac{\partial\mathcal{P}_+}{\partial\mathcal{P}_\mathrm{R}} + \frac{\partial\mathcal{P}_-}{\partial\mathcal{P}_\mathrm{R}} = 1\,, \quad (26)$$

and Eq. (21) gives

$$\kappa_+\frac{\partial\mathcal{P}_+}{\partial\mathcal{P}_\mathrm{R}} = \kappa_-\frac{\partial\mathcal{P}_-}{\partial\mathcal{P}_\mathrm{R}}\,. \quad (27)$$

Since $\mathcal{P}_{\rm L} = \mathcal{P}_+ - \mathcal{P}_-$, we have

$$\frac{\partial \mathcal{P}_+}{\partial \mathcal{P}_{\rm L}} - \frac{\partial \mathcal{P}_-}{\partial \mathcal{P}_{\rm L}} = 1\,, \tag{28}$$

and using the Eq. (21) we obtain

$$\kappa_+ \frac{\partial \mathcal{P}_+}{\partial \mathcal{P}_{\rm L}} = \kappa_- \frac{\partial \mathcal{P}_-}{\partial \mathcal{P}_{\rm L}}\,. \tag{29}$$

Hence, using Eqs. (26) and (27) and Eqs. (28) and (29) we can write

$$\frac{\partial \mathcal{P}_+}{\partial \mathcal{P}_{\rm R,L}} = \frac{\kappa_-}{\kappa_- \pm \kappa_+}\,, \tag{30}$$

$$\frac{\partial \mathcal{P}_-}{\partial \mathcal{P}_{\rm R}} = \frac{\kappa_+}{\kappa_- \pm \kappa_+}\,. \tag{31}$$

On the other hand, using Eq. (20) we obtain

$$\left(\frac{\partial M}{\partial \mathcal{P}_+}\right)_Q = \frac{\mathcal{P}_+}{l^2} - \frac{Q^2}{4\mathcal{P}_+} = \kappa_+\,, \tag{32}$$

$$\left(\frac{\partial M}{\partial \mathcal{P}_-}\right)_Q = \frac{\mathcal{P}_-}{l^2} - \frac{Q^2}{4\mathcal{P}_-} = \kappa_-\,. \tag{33}$$

Therefore, putting Eqs. (30)–(33) into (24) and (25) we obtain

$$\kappa_{\rm R,L} = \frac{\kappa_+\kappa_-}{\kappa_- \pm \kappa_+}\,, \tag{34}$$

or

$$\frac{1}{\kappa_{\rm R,L}} = \frac{1}{\kappa_+} \pm \frac{1}{\kappa_-}\,. \tag{35}$$

This equation shows that the temperature for the R- and L-systems satisfies

$$\frac{1}{T_{\rm R,L}} = \frac{1}{T_+} \pm \frac{1}{T_-}\,, \tag{36}$$

while the entropy can be written as

$$S_{\rm R,L} = 4\pi \mathcal{P}_{\rm R,L} = 4\pi\,(\mathcal{P}_+ \pm \mathcal{P}_-) = S_+ \pm S_-\,. \tag{37}$$

On the other side, the electric potential for each system is obtained by the expressions

$$\begin{aligned} \Phi_{\rm R} = & \left(\frac{\partial M}{\partial Q}\right)_{\mathcal{P}_{\rm R}} = \left(\frac{\partial M}{\partial \mathcal{P}_+}\right)_{\mathcal{P}_{\rm R},Q} \left(\frac{\partial \mathcal{P}_+}{\partial Q}\right)_{\mathcal{P}_{\rm R}} \\ & + \left(\frac{\partial M}{\partial \mathcal{P}_-}\right)_{\mathcal{P}_{\rm R},Q} \left(\frac{\partial \mathcal{P}_-}{\partial Q}\right)_{\mathcal{P}_{\rm R}} + \left(\frac{\partial M}{\partial Q}\right)_{\mathcal{P}_+,\mathcal{P}_-,Q}\,, \end{aligned} \tag{38}$$

$$\begin{aligned}\Phi_{\rm L} = \quad & \left(\frac{\partial M}{\partial Q}\right)_{\mathcal{P}_{\rm L}} = \left(\frac{\partial M}{\partial \mathcal{P}_+}\right)_{\mathcal{P}_{\rm L},Q}\left(\frac{\partial \mathcal{P}_+}{\partial Q}\right)_{\mathcal{P}_{\rm L}} \\ & + \left(\frac{\partial M}{\partial \mathcal{P}_-}\right)_{\mathcal{P}_{\rm L},Q}\left(\frac{\partial \mathcal{P}_-}{\partial Q}\right)_{\mathcal{P}_{\rm L}} + \left(\frac{\partial M}{\partial Q}\right)_{\mathcal{P}_+,\mathcal{P}_-,Q} . \end{aligned} \tag{39}$$

This time, from $\mathcal{P}_{\rm R} = \mathcal{P}_+ + \mathcal{P}_-$ we obtain

$$\left(\frac{\partial \mathcal{P}_+}{\partial Q}\right)_{\mathcal{P}_{\rm R}} + \left(\frac{\partial \mathcal{P}_-}{\partial Q}\right)_{\mathcal{P}_{\rm R}} = 0\,, \tag{40}$$

and Eq. (21) gives

$$\left(\frac{\partial \mathcal{P}_+}{\partial Q}\right)_{\mathcal{P}_{\rm R}} = \frac{1}{2}\frac{(\Phi_+ - \Phi_-)}{\kappa_+ + \kappa_-}\,. \tag{41}$$

Similarly, $\mathcal{P}_{\rm L} = \mathcal{P}_+ - \mathcal{P}_-$ gives

$$\left(\frac{\partial \mathcal{P}_+}{\partial Q}\right)_{\mathcal{P}_{\rm L}} + \left(\frac{\partial \mathcal{P}_-}{\partial Q}\right)_{\mathcal{P}_{\rm L}} = 0\,, \tag{42}$$

and using the Eq. (21) is easily to obtain

$$\left(\frac{\partial \mathcal{P}_+}{\partial Q}\right)_{\mathcal{P}_{\rm L}} = \frac{1}{2}\frac{(\Phi_+ - \Phi_-)}{\kappa_+ - \kappa_-}\,. \tag{43}$$

On the other hand, using Eq. (20) we obtain

$$\left(\frac{\partial M}{\partial Q}\right)_{\mathcal{P}_{\rm R},\mathcal{P}_+,\mathcal{P}_-} = \frac{1}{2}\left(\Phi_+ + \Phi_-\right)\,, \tag{44}$$

$$\left(\frac{\partial M}{\partial Q}\right)_{\mathcal{P}_{\rm L},\mathcal{P}_+,\mathcal{P}_-} = \frac{1}{2}\left(\Phi_+ + \Phi_-\right)\,. \tag{45}$$

Therefore, putting Eqs. (32), (33), (41), (43)–(45) into (38) and (39), we find the electric potential

$$\Phi_{\rm R} = \frac{(\Phi_+ + \Phi_-)}{2} + \frac{\kappa_+ - \kappa_-}{\kappa_+ + \kappa_-}\frac{(\Phi_+ - \Phi_-)}{2}\,, \tag{46}$$

$$\Phi_{\rm L} = \frac{(\Phi_+ + \Phi_-)}{2} + \frac{\kappa_+ + \kappa_-}{\kappa_+ - \kappa_-}\frac{(\Phi_+ - \Phi_-)}{2}\,. \tag{47}$$

Thus, the integral and differential mass formulae for the R- and L-systems can be written as

$$\begin{aligned} M &= \kappa_{\rm R,L}\mathcal{P}_{\rm R,L} + \frac{1}{2}\Phi_{\rm R,L}Q + \frac{1}{4}Q^2\,, & (48)\\ dM &= 2\kappa_{\rm R,L}d\mathcal{P}_{\rm R,L} + \Phi_{\rm R,L}dQ\,, & (49)\end{aligned}$$

that corresponds to what is expected as a generalization of Eqs. (10) and (13).

4. Relationship between the R, L-systems and the BTZ thermodynamics

The thermodynamic laws of the R, L-systems are related with the BTZ black hole thermodynamics. Eq. (35),

$$\frac{1}{\kappa_{\mathrm{R,L}}} = \frac{1}{\kappa_+} \pm \frac{1}{\kappa_-}\,, \tag{50}$$

that corresponds exactly with the relation found by Wu [6] for Kerr–Newman black hole and Larranaga [8] for the BTZ black hole. This relation is in direct correspondence to effective string theory and D-brane physics.

Since temperature is proportional to surface gravity, we have a similar expression obtained for temperatures in Eq. (36),

$$\frac{1}{T_{\mathrm{R,L}}} = \frac{1}{T_+} \pm \frac{1}{T_-}\,. \tag{51}$$

This last relation give us immediately an expression for the Hawking temperature associated with the BTZ black hole, that corresponds to the temperature of the outer horizon,

$$T_{\mathrm{H}} = T_+ = \frac{\kappa_+}{2\pi}\,, \tag{52}$$

in terms of the temperatures of the R and L systems. The relation is

$$\frac{2}{T_{\mathrm{H}}} = \frac{1}{T_{\mathrm{R}}} + \frac{1}{T_{\mathrm{L}}}\,, \tag{53}$$

which shows that the Hawking temperature is again the harmonic mean of the R and L temperatures.

Now, lets play some attention to the electric potential. Note that using Eq. (35) we can rewrite the R and L electric potentials given by (46) and (47), as

$$\Phi_{\mathrm{R,L}} = \frac{(\Phi_+ + \Phi_-)}{2} + \frac{\kappa_{\mathrm{R,L}}}{\kappa_{\mathrm{L,R}}} \frac{(\Phi_+ - \Phi_-)}{2}\,. \tag{54}$$

This equation is exactly the same found by Wu [6] for the Kerr–Newman black hole, showing again that the effective string theory thermodynamics seems to be a universal picture holding also in 2+1 gravity.

5. Conclusion

In this paper we have shown that the thermodynamics of the (2+1) dimensional charged BTZ black hole can be constructed from two independent thermodynamical systems that resemble the right and left modes of string theory. If one assume that the effective strings have the same mass and electric charge that the charged BTZ black hole, there is a correspondence between the R and L modes thermodynamics and the thermodynamics of the horizons.

We have shown that the Hawking temperature associated with the black hole is obtained as the harmonic mean of the temperatures associated with the R and L systems, just as in the case of stringy thermodynamics. Moreover, equation (54) shows that the electric potential of the R and L systems is related with the electric potential of the inner and outer horizons with the same equation obtained by Wu [6] for the Kerr–Newman black hole.

All these facts suggest that there is a deep connection between string theory and D-branes with black holes physics that seems to hold in many cases, not only in General Relativity but also in 2+1 gravity. Therefore, it is really interesting to investigate if this relation can give some clue for the understanding of the origin of black hole entropy.

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