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DECOUPLING OF KINEMATICAL TIME DILATION AND GRAVITATIONAL TIME DILATION IN PARTICULAR GEOMETRIES

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Two different forms of time dilation, namely, the kinematical time dilation of special relativity and gravitational red shift are *coupled* together during observations of a system travelling through a gravitational field. In the case of a Schwarzschild geometry these two effects are decoupled and in consequence they factorise. Such a factorization is not a universal feature. We define here a necessary and sufficient criterion for time dilation and gravitational red-shift decoupling. This property is manifested in a particular form of the frequency shift in a Schwarzschild geometry.

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Two types of time-dilation-like effects: a special relativistic one ("time dilation") of kinematical origin and a gravitational red-shift (in the case of a light signal received by a distant observer) are expected to be coupled in a case of an arbitrary motion of a frame within a gravitational field. In this paper we discuss particular geometries where these time-dilation-contributions are factorizable. We start by discussing a case set in a Schwarzschild geometry.

Let us consider the case of an in-falling observer in an isotropic, static gravitational field, *i.e.* in a Schwarzschild geometry,

$$\begin{aligned} d\tau^2 &= \left(1-\frac{r_{\rm S}}{r}\right)dt^2-\left(1-\frac{r_{\rm S}}{r}\right)^{-1}dr^2-r^2d\theta^2-r^2\sin^2\theta\,d\varphi^2 \\ &\equiv g_{tt}dt^2+g_{rr}dr^2+g_{\theta\theta}d\theta^2+g_{\varphi\varphi}d\varphi^2\,. \end{aligned} \quad (1)$$

In the case of radial free fall, the velocity components

$$\tilde{u} = (u^t, u^r, 0, 0)\,, \qquad u^t = \frac{dt}{d\tau}\,, \qquad u^r = \frac{dr}{d\tau}\,,$$

are found from energy conservation:

$$g_{tt} u^t = A\,, \tag{2}$$
$$u^r = \sqrt{A^2 - g_{tt}}\,. \tag{3}$$

The corresponding Killing vector is

$$\eta^\alpha = \delta_t^\alpha\,. \tag{4}$$

The parameter A in (1) is positive and,

(a) $A = 1$ corresponds to the case of free fall from infinity,

(b) $0 < A < 1$ corresponds to free fall from a finite distance from the gravitational centre,

(c) $A > 1$ corresponds to the case of an in-falling body, beginning at infinity with velocity
$$V_\infty = \sqrt{\frac{A^2 - 1}{A^2}}$$
(see below).

The velocity time-component (2) describes the dilation of (coordinate) time t with respect to the proper time τ measured by the in-falling observer. It turns out to be composed of three time-dilation-like factors of distinct origins.

To show this let us identify the following observers: apart from an in-falling observer, (IFO), we will consider a distant, inertial observer (IO) (placed at infinity), and a local observer (LSO), static with respect to the gravitational field. The standard gravitational red-shift, is described by a factor we term here γ_{g} (*g-factor*):

$$\gamma_{\mathrm{g}} = \frac{1}{\sqrt{g_{tt}}}\,. \tag{5}$$

Then the residual factor on the right hand side of expression (2)

$$\gamma_{\mathrm{s}} = \frac{A}{\sqrt{g_{tt}}} \tag{6}$$

that we term the s-*factor*, originates from the motion of an in-falling body. As an in-falling body reaches a point r, with a corresponding component of the metric tensor $g_{tt}(r)$, its velocity $v_{\rm IFO}$, as measured by the LSO is (see, *e.g.* [1]),

$$v_{\rm IFO} = \sqrt{\frac{A^2 - g_{tt}(r)}{A^2}}\,. \tag{7}$$

In fact, the LSO measures the energy of an object O that passes nearby, by taking the scalar product of the momentum vector of O, $\tilde{p}_{\rm O}$, and his own four-velocity $\tilde{U}_{\rm LSO}$:

$$E = \tilde{p}_{\rm O} \circ \tilde{U}_{\rm LSO}\,. \tag{8}$$

This unit time-like vector is defined by the absolute standard of rest, *i.e.* Killing vector (4),

$$\tilde{U}_{\rm LSO} = (\eta^\alpha \eta_\alpha)^{-1/2}\tilde{\eta} = \frac{1}{\sqrt{g_{tt}}}\tilde{\eta}\,, \tag{9}$$

and the energy is:

$$E = g_{\alpha\beta} U^\alpha_{\rm LSO} p^\beta_{\rm O} = \sqrt{g_{tt}}\, p^t_{\rm O}\,. \tag{10}$$

Expressing energy (10) in the form (see [2]),

$$E = \frac{m_{\rm O}}{\sqrt{1 - v^2_{\rm O}}}\,, \tag{11}$$

where $m_{\rm O}$ denotes the rest mass of an object O, one finds the squared velocity as measured by the LSO

$$v_{\rm O} = \sqrt{1 - \frac{1}{g_{tt}(u^t_{\rm O})^2}}\,. \tag{12}$$

Inserting (2) into (12) one obtains velocity of IFO as given by (7). Thus, the corresponding s-factor (6), turns out to be of kinematical origin [1],

$$\gamma_{\rm s} = \frac{1}{\sqrt{1 - v^2_{\rm IFO}}} = \frac{A}{\sqrt{g_{tt}}} \tag{13}$$

and factorization of (6) corresponds to a decoupling of the two types of time dilations: the gravitational one and the kinematical one,

$$u^t = \gamma_{\rm s}\gamma_{\rm g}\,. \tag{14}$$

In the case of a body thrown towards the gravitational centre, *i.e.* $A > 1$, one can notice that $\gamma_{\rm s}$ factorizes further due to two kinematical contributions. The first one relates to the motion at infinity with velocity v_∞,

$$\gamma_\infty = \frac{1}{\sqrt{1-v_\infty^2}} = A \iff v_\infty^2 = \frac{A^2-1}{A^2}\,. \tag{15}$$

The second factor corresponds to free fall within the gravitational field. In fact, for the case of free fall from the rest at infinity, $A = 1$, one finds the velocity as measured by the LSO (7) (*e.g.* [2]),

$$v_{\rm r} = \sqrt{1-g_{tt}(r)} \tag{16}$$

and the factor $\gamma_{\rm r}$ is,

$$\gamma_{\rm r} = \frac{1}{\sqrt{1-v_{\rm r}^2}} = \frac{1}{\sqrt{g_{tt}(r)}}\,. \tag{17}$$

Finally $\gamma_{\rm s}$ in (14) is decomposed into (17))

$$\gamma_{\rm s} = \gamma_\infty \gamma_{\rm r}\,.$$

Therefore, in the case of radial free fall in Schwarzschild geometry, one obtains a (coordinate) time-dilation-factorization into kinematical:

- $\gamma_\infty = A$ — due to the initial velocity (at infinity),
- $\gamma_{\rm r} = 1/\sqrt{g_{tt}(r)}$ — represents a time-dilation-factor due to the work performed by the gravitational field

and gravitational,

- $\gamma_{\rm g} = 1/\sqrt{g_{tt}}$ — an intrinsic red-shift due to the gravitational field itself

contributions.

This result, factorization of the velocity time component, derived here for a radial free fall, has a deeper meaning. In fact, it holds for *arbitrary* motion within a Schwarzschild spacetime. Indeed, let us consider an object O following arbitrary trajectory, characterized by velocity four vector,

$$u_{\rm O}^\alpha \cdot u_{{\rm O}\alpha} = 1\,. \tag{18}$$

One finds its speed with respect to the stationary observer from (*cf.* (10), (11))

$$\frac{1}{\sqrt{1-v_{\rm O}^2}} = \sqrt{g_{tt}}\,u_{\rm O}^t \equiv \gamma_{\rm s}\,. \tag{19}$$

Therefore, in arbitrary case velocity time component factorizes into gravitational and kinematical contributions:

$$u^t_{\rm O} = \gamma_{\rm g}\gamma_{\rm s}\,, \tag{20}$$

where the velocity $v_{\rm O}$ is measured with respect to the local stationary observer.

The interesting feature is that in the case of a Kerr geometry factorization (20) of gravitational and kinematical contributions also holds for the particular situation of arbitrary motion along the *axis of symmetry* but does not hold for more general case. Taking into account the fact of decoupling of the time dilation contributions of different origins, namely kinematical one and gravitational shift, arising in highly symmetrical situations, one can ask how general is such an effect? Or in other words: what is the status of the statement that time dilation factorizes into g- and s- factors?

To answer this question, one can consider the case of an observer O travelling across an arbitrary static gravitational field where the Killing vector is given by (4). The static observer, LSO, measures O's velocity (see (9)) as:

$$U^\alpha_{\rm LSO}u_{{\rm O}\alpha} = \frac{1}{\sqrt{g_{tt}}}g_{t\beta}u^\beta_{\rm O} \equiv \frac{1}{\sqrt{1-v^2_{\rm O}}}\,. \tag{21}$$

Therefore, for the metric $g_{t\alpha} = g_{tt}\delta^t_\alpha$ in which the time coordinate is orthogonal to the spatial dimensions, *or* for such a fall that $g_{t\beta}u^\beta_{\rm O} = g_{tt}u^t_{\rm O}$ (this is the case of a motion along the symmetry axis in Kerr metric), one can find a factorization of the time dilation:

$$u^t_{\rm O} = \frac{1}{\sqrt{g_{tt}}}\frac{1}{\sqrt{1-v^2_{\rm O}}} \equiv \gamma_{\rm g}\gamma_{\rm s}\,. \tag{22}$$

The final conclusion is that in the case of particular symmetries, or for particular types of motion, a distant (inertial) observer would find time dilation to occur in a factorized form (22). The velocity in the $\gamma_{\rm s}$-factor corresponds to the one measured by the local static observer.

What are the possible experimental consequences of this effect? The most obvious, non-trivial observation, is a frequency shift (generalized Doppler effect). One can find that in the case of a radial fall in Schwarzschild geometry, light signal sent by an IO is received by an IFO as a red-shifted one

$$\frac{\omega_{\rm IFO}}{\omega_{\rm IO}} \equiv \frac{1}{\sqrt{g_{tt}}}\frac{\sqrt{1-V^2_{\rm IFO}}}{1+V_{\rm IFO}}\,,$$

where the first, blue-shift-like factor is of gravitational origin, and the other two red-shift type factors are of kinematical origin. This result is discussed elsewhere [3]. Other possible applications are related to satellite navigation systems [4–6], where the Schwarzschild spacetime is a useful approach.

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