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EXACT TREE-LEVEL QCD SPIN-AMPLITUDES AND THEIR GAUGE INVARIANT SUB-STRUCTURES* **

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We discuss possible separations into gauge invariant parts of exact, massive, tree-level spin amplitudes of processes involving two quarks, two gluons and a color-neutral current. We search for forms compatible with parton shower languages, without applying approximations or restrictions on phase space regions. Special emphasis will be put on the isolation of parts corresponding to the running coupling constant and parts necessary for the construction of evolution kernels for individual splittings. Our representation is quite universal: any color-neutral current can be used, in particular our approach is not restricted to vector currents only.

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1. Introduction

One of the essential steps in the construction of any algorithm for multi-particle final states is the appropriate analysis of the phase space parametrization. In the PHOTOS Monte Carlo [1] for multi-photon production, such an exact phase space parametrization is embodied in an iterative algorithm, the details of which are best described in [2], but the control of the relative size of sub-samples of distinct numbers of final state particles requires a precise knowledge of the matrix elements including virtual corrections. In the KKMC Monte Carlo, the approach to phase space generation is different, but

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the necessity to control matrix elements and their truncation is again essential [3,4]. For the latter, iterative procedures for parts of amplitudes, which are at the foundation of exponentiation [4,5] and structure functions [6–10], were exploited. In particular the description of the dominantly s-channel process $e^+e^- \to \nu_e\bar{\nu}_e\gamma\gamma$ where t-channel W-exchange diagrams with gauge boson couplings contribute to matrix elements provided an interesting example [11]. These studies were motivated by practical reasons, but also pointed at quite astonishing properties of tree-level spin amplitudes, namely that they can be separated into gauge invariant parts in a semi-automated way, easy to apply in the Kleiss–Stirling methods [12,13].

Here, we want to handle the question whether similar techniques can be used for QCD. Similar analyses are, of course, already included in existing works, such as for example at the foundation of parton shower algorithms (new or well established) [14–22] or for other, fundamental or phenomenological applications [23–30]. The common idea between all of these papers is to separate approximate or exact results into parts, often with the help of iteration. In this context, we also want to mention the existence of possible limitations in such strategies, if applied to parton shower applications [31].

As was done for QED, we will study possible ways of separating amplitudes, but now for the process of two quarks and two gluons entering a color-neutral current. First, we try to identify sub-structures of the amplitude which would be useful for possible applications. Then we would like to verify to which degree they can be used to localize parts of the amplitude related to *e.g.* evolution kernels and the running coupling constant. We expect such expressions to be identifiable at least in approximations valid in certain regions of phase space, dominant for specific purposes, like in applications using p_{T} ordered phase space, but we hope to localize them partly already at the level of exact amplitudes.

The text is organized as follows. In Sec. 2 we present our notation and general strategy to organize the amplitude. The treatment of different possibilities for this organization concerning the color part of the amplitude is distributed over Sec. 3 and Sec. 4. In Sec. 5 we explore certain limits, in which some parts of the amplitude can be dropped out. Sec. 6, finally, is the summary.

2. Notation and general strategy

The exact spin amplitude for the process $q\bar{q} \to Jgg$ can be written as an expansion in combinations of the SU(N_{c})-generators, for example the combinations $\{T^a, T^b\}$ and $[T^a, T^b]$. Here, T^a is the color generator associated with gluon number 1, which has momentum k_1 and polarization vector e_1, and T^b is the color generator associated with gluon number 2, which has momentum k_2 and polarization vector e_2. Another option would be to use, for example, the combinations T^aT^b and T^bT^a. But let us stick to the former example for now, and return to this issue later. The amplitude can then be expressed as

$$\mathcal{M}^{a,b} = \frac{1}{2}\bar{v}(p)\left(\left[T^a,T^b\right]I^{[1,2]} + \left\{T^a,T^b\right\}I^{\{1,2\}}\right)u(q)\,, \tag{1}$$

where $\bar{v}(p)$ and $u(q)$ are the spinors associated with the anti-quark, which has momentum p, and the quark, which has momentum q, respectively. We do not specify the spin or color states for the quark fields, any choice can be used. This type of separation of the spin amplitude into gauge invariant parts is known [32] and exploited since long time. The main task is now to calculate the coefficients $I^{[1,2]}$ and $I^{\{1,2\}}$. Our expressions will contain the object $\not{J}$, representing the color neutral current, and we want to stress that they are valid under the condition that this object is constructed from color neutral objects like $(v + a\gamma^5)$, γ^μ, $\not{p}$, $\not{k}_1$, *etc.*, although our notation might seem to indicate stronger limitations.

The first step in this calculation is the construction of the relevant graphs, depicted in Fig. 1, from the Feynman rules. Next, we choose to replace explicit mass terms by fermion momenta using the Dirac equation. It is not difficult to extract expressions of the type mentioned before for $I^{[1,2]}$ and $I^{\{1,2\}}$ at this point. Thanks to properties of the gamma-matrices, we can choose how their contractions with four-momenta are ordered, at the expense of the appearance or disappearance of certain scalar products of those four-momenta. We choose to order them such that factors $\not{J}\not{e}_{1,2}$, $\not{e}_{1,2}\not{J}$ or $\not{k}_{1,2}\not{k}_{2,1}$, $\not{e}_{1,2}\not{e}_{2,1}$ are not present. This is for the convenience of possible use of the Kleiss–Stirling method for coding final expressions similarly as in [3].

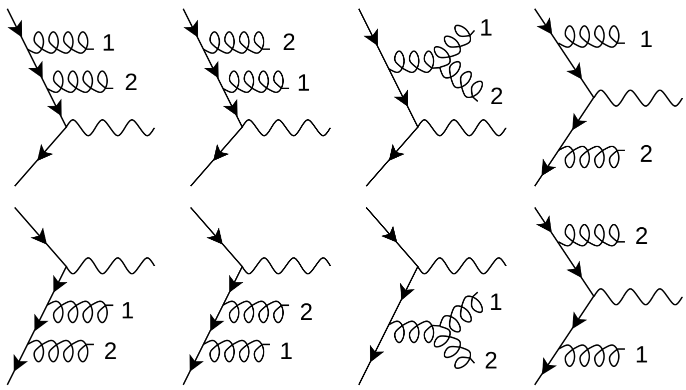

Fig. 1. Feynman graphs for the process $q\bar{q} \to Jgg$.

The game is now to compactify expressions, or organize them in an easily interpretable way, by factorizing certain sums of terms. The most important guideline we use is that each term, consisting of factors which consist themselves of sums of terms, should be manifestly gauge invariant. In this process, we allow for the introduction of what we call *subtraction terms*, that is terms subtracted at one place of the complete expression, and added at another place. They can help to make certain factorizations possible, which look desirable but for which some terms appear to be missing. Such subtraction terms were also used in our earlier QED-studies of the same type [11]. In fact, the terms were constructed from parts of lower-order amplitudes and were an important element in the attempt to define a semi-automated method to obtain expressions for spin amplitudes also used in Monte Carlo programs. For the two-photon case, the following subtraction terms were used:

$$S_{1,q}^{\{1,2\}} = \frac{1}{2}\not{J}\left(\frac{q\cdot e_1}{q\cdot k_1}\frac{q\cdot e_2}{q\cdot k_1+q\cdot k_2}+\frac{q\cdot e_1}{q\cdot k_1+q\cdot k_2}\frac{q\cdot e_2}{q\cdot k_2}\right), \tag{2}$$

$$S_{2,q}^{\{1,2\}} = \frac{1}{2}\not{J}\left(\frac{\not{k}_2\not{e}_2}{q\cdot k_2}\frac{q\cdot e_1}{q\cdot k_1}+\frac{\not{k}_1\not{e}_1}{q\cdot k_1}\frac{q\cdot e_2}{q\cdot k_2}\right), \tag{3}$$

and similar terms with q replaced by p. As mentioned before, these terms are constructed from parts of the one-photon amplitude, which is given by

$$\mathcal{M}^a = \bar{v}(p)T^a I^{(1)} u(q)\,, \tag{4}$$

with

$$I^{(1)} = \not{J}\left[\left(\frac{p\cdot e_1}{p\cdot k_1}-\frac{q\cdot e_1}{q\cdot k_1}\right)\right]-\frac{1}{2}\left[\frac{\not{e}_1\not{k}_1}{p\cdot k_1}\right]\not{J}+\frac{1}{2}\not{J}\left[\frac{\not{k}_1\not{e}_1}{q\cdot k_1}\right]. \tag{5}$$

Note that each segment encapsulated by square brackets is manifestly gauge invariant. As in the case of QED, we will use subtraction terms (2) and (3). In addition, however, we will also use

$$S_{3,q}^{\{1,2\}} = -\frac{1}{2}\not{J}\frac{k_1\cdot k_2}{q\cdot k_1+q\cdot k_2-k_1\cdot k_2}\frac{k_1\cdot e_2\, k_2\cdot e_1}{k_1\cdot k_2 k_1\cdot k_2} \tag{6}$$

and a similar term with q replaced by p. Note that, in contrast to (2) and (3), subtraction term (6) is *not* constructed from elements present in the single bremsstrahlung amplitude given in Eq. (5). In particular, it introduces an artificial singularity $1/(k_1\cdot k_2)$ which would be worrisome for QED applications. We, however, are not concerned with that, since it points at the singularity of an intermediate gluon virtuality, with which we have to deal anyway in a QCD amplitude.

We already mentioned the examples of an expansion in $[T^a, T^b]$ and $\{T^a, T^b\}$, and an expansion in T^aT^b and T^bT^a. We will deal with both cases in the next sections. The most compact expressions for the amplitude will be obtained with a mixed choice, in which all four combinations $[T^a, T^b]$, $\{T^a, T^b\}$, T^aT^b and T^bT^a of color generators are used. The coefficients for each term is not unique in this case of course, but we will be able to find a choice which seems to be optimal.

3. QED-inspired picture

In this section, we investigate expressions for the two-gluon amplitude obtained when expanded in its color contents as in Eq. (1). We start our analysis by collecting terms proportional to the anti-commutator of the color generators. It can be considered the less complicated one, because it does not include terms originating from the triple-gluon vertex. Note that it is identical with the expression for the QED amplitude of the process $e^+e^- \to \nu_\mu\bar{\nu}_\mu\gamma\gamma$ as described in [11]. This is possible, because the anti-commutator projects out the triple-gauge coupling.

The choice of the subtraction terms affects the final form of the results. In case of QED we could limit ourselves to the gauge invariant parts available from the single photon (gluon) amplitude, leading to a seemingly unique result. The case of QCD is more complex, and the choice of subtraction terms is rather motivated by aesthetic considerations.

3.1. $\{T^a, T^b\}$-part

The part of the amplitude proportional to $\{T^a, T^b\}$ can be represented as sum of twelve individually gauge invariant parts:

$$\begin{aligned} I^{\{1,2\}} &= I_1^{\{1,2\}} + I_{2l}^{\{1,2\}} + I_{2r}^{\{1,2\}} + I_3^{\{1,2\}} + I_{4l}^{\{1,2\}} + I_{4r}^{\{1,2\}} \\ &\quad + I_{5lA}^{\{1,2\}} + I_{5lB}^{\{1,2\}} + I_{5rA}^{\{1,2\}} + I_{5rB}^{\{1,2\}} + I_{6l}^{\{1,2\}} + I_{6r}^{\{1,2\}} \,, \end{aligned} \tag{7}$$

where

$$I_1^{\{1,2\}} = \frac{1}{2}\not{J}\left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{q\cdot e_1}{q\cdot k_1}\right)\left(\frac{p\cdot e_2}{p\cdot k_2} - \frac{q\cdot e_2}{q\cdot k_2}\right), \tag{8}$$

$$I_{2l}^{\{1,2\}} = -\frac{1}{4}\left[\left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{q\cdot e_1}{q\cdot k_1}\right)\frac{\not{e}_2\not{k}_2}{p\cdot k_2} + \left(\frac{p\cdot e_2}{p\cdot k_2} - \frac{q\cdot e_2}{q\cdot k_2}\right)\frac{\not{e}_1\not{k}_1}{p\cdot k_1}\right]\not{J}, \tag{9}$$

$$I_{2r}^{\{1,2\}} = \frac{1}{4}\not{J}\left[\left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{q\cdot e_1}{q\cdot k_1}\right)\frac{\not{k}_2\not{e}_2}{q\cdot k_2} + \left(\frac{p\cdot e_2}{p\cdot k_2} - \frac{q\cdot e_2}{q\cdot k_2}\right)\frac{\not{k}_1\not{e}_1}{q\cdot k_1}\right], \tag{10}$$

$$I_3^{\{1,2\}} = -\frac{1}{8}\left(\frac{\not{e}_1\not{k}_1}{p\cdot k_1}\not{J}\frac{\not{k}_2\not{e}_2}{q\cdot k_2} + \frac{\not{e}_2\not{k}_2}{p\cdot k_2}\not{J}\frac{\not{k}_1\not{e}_1}{q\cdot k_1}\right), \tag{11}$$

$$I_{4l}^{\{1,2\}} = \frac{1}{8}\frac{1}{p\cdot k_1 + p\cdot k_2 - k_1\cdot k_2}\left(\frac{\not{e}_1\not{k}_1\not{e}_2\not{k}_2}{p\cdot k_1} + \frac{\not{e}_2\not{k}_2\not{e}_1\not{k}_1}{p\cdot k_2}\right)\not{J}, \tag{12}$$

$$I_{4r}^{\{1,2\}} = \frac{1}{8}\not{J}\frac{1}{q\cdot k_1 + q\cdot k_2 - k_1\cdot k_2}\left(\frac{\not{k}_2\not{e}_2\not{k}_1\not{e}_1}{q\cdot k_1} + \frac{\not{k}_1\not{e}_1\not{k}_2\not{e}_2}{q\cdot k_2}\right), \tag{13}$$

$$I_{5lA}^{\{1,2\}} = \frac{1}{2}\not{J}\frac{k_1\cdot k_2}{p\cdot k_1 + p\cdot k_2 - k_1\cdot k_2}\left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{k_2\cdot e_1}{k_2\cdot k_1}\right)\left(\frac{p\cdot e_2}{p\cdot k_2} - \frac{k_1\cdot e_2}{k_1\cdot k_2}\right), \tag{14}$$

$$I_{5lB}^{\{1,2\}} = -\frac{1}{2}\not{J}\frac{1}{p\cdot k_1 + p\cdot k_2 - k_1\cdot k_2}\left(\frac{k_1\cdot e_2 k_2\cdot e_1}{k_1\cdot k_2} - e_1\cdot e_2\right), \tag{15}$$

$$I_{5rA}^{\{1,2\}} = \frac{1}{2}\not{J}\frac{k_1\cdot k_2}{q\cdot k_1 + q\cdot k_2 - k_1\cdot k_2}\left(\frac{q\cdot e_1}{q\cdot k_1} - \frac{k_2\cdot e_1}{k_2\cdot k_1}\right)\left(\frac{q\cdot e_2}{q\cdot k_2} - \frac{k_1\cdot e_2}{k_1\cdot k_2}\right), \tag{16}$$

$$I_{5rB}^{\{1,2\}} = -\frac{1}{2}\not{J}\frac{1}{q\cdot k_1 + q\cdot k_2 - k_1\cdot k_2}\left(\frac{k_1\cdot e_2 k_2\cdot e_1}{k_1\cdot k_2} - e_1\cdot e_2\right), \tag{17}$$

$$I_{6l}^{\{1,2\}} = -\frac{1}{4}\frac{k_1\cdot k_2}{p\cdot k_1 + p\cdot k_2 - k_1\cdot k_2}\left[\left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{k_2\cdot e_1}{k_1\cdot k_2}\right)\frac{\not{e}_2\not{k}_2}{p\cdot k_2} + \left(\frac{p\cdot e_2}{p\cdot k_2} - \frac{k_1\cdot e_2}{k_1\cdot k_2}\right)\frac{\not{e}_1\not{k}_1}{p\cdot k_1}\right]\not{J}, \tag{18}$$

$$I_{6r}^{\{1,2\}} = -\frac{1}{4}\not{J}\frac{k_1\cdot k_2}{q\cdot k_1 + q\cdot k_2 - k_1\cdot k_2}\left[\left(\frac{q\cdot e_1}{q\cdot k_1} - \frac{k_2\cdot e_1}{k_1\cdot k_2}\right)\frac{\not{k}_2\not{e}_2}{q\cdot k_2} + \left(\frac{q\cdot e_2}{q\cdot k_2} - \frac{k_1\cdot e_2}{k_1\cdot k_2}\right)\frac{\not{k}_1\not{e}_1}{q\cdot k_1}\right]. \tag{19}$$

Note that contrary to [11] (formula (27)), where the amplitude is separated into seven terms $\mathcal{M}_1$ to $\mathcal{M}_7$, we have allowed here a new object $e_1\cdot e_2$, which from the point of view of fermionic spin amplitudes and Kleiss–Stirling methods [12] can be considered to be rather ugly. It will appear anyway in the commutator part discussed in the next section as a consequence of the triple gauge coupling. Furthermore, we separated the expressions $\mathcal{M}_2$ and $\mathcal{M}_4$ of [11] into several new parts, partly because of the new subtraction term (6). Such a separation is inappropriate for QED applications, since the new gauge invariant parts, even though compact, are more singular in the soft-photon limit than their sum, and would form an obstacle to exponentiation. For QCD, however, this is not so much an issue because of the triple-gluon vertex and such a separation will be useful later. That is why we allow the structure of apparent singularities to be larger than necessary.

3.2. $[T^a, T^b]$*-part*

For the commutator part, we introduce subtraction terms mainly in order to separate terms depending simultaneously on both momenta p and q into simple and intuitively clear expressions. The part of the amplitude proportional to $[T^a, T^b]$ is again represented as a sum of individually gauge invariant parts and reads:

$$\begin{aligned} I^{[1,2]} &= I_1^{[1,2]} + I_{2l}^{[1,2]} + I_{2r}^{[1,2]} + I_3^{[1,2]} + I_{4l}^{[1,2]} + I_{4r}^{[1,2]} + I_{5lA}^{[1,2]} + I_{5lB}^{[1,2]} \\ &\quad + I_{5rA}^{[1,2]} + I_{5rB}^{[1,2]} + I_{6l}^{[1,2]} + I_{6r}^{[1,2]} + I_{7lA}^{[1,2]} + I_{7lB}^{[1,2]} + I_{7rA}^{[1,2]} + I_{7rB}^{[1,2]} \,, \end{aligned} \tag{20}$$

where

$$I_1^{[1,2]} = -\frac{1}{4}\not{J}\left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{q\cdot e_1}{q\cdot k_1}\right)\left(\frac{p\cdot e_2}{p\cdot k_2} + \frac{q\cdot e_2}{q\cdot k_2} - 2\frac{k_1\cdot e_2}{k_1\cdot k_2}\right) \tag{21}$$

$$+\frac{1}{4}\not{J}\left(\frac{p\cdot e_2}{p\cdot k_2} - \frac{q\cdot e_2}{q\cdot k_2}\right)\left(\frac{p\cdot e_1}{p\cdot k_1} + \frac{q\cdot e_1}{q\cdot k_1} - 2\frac{k_2\cdot e_1}{k_1\cdot k_2}\right) , \tag{22}$$

$$I_{2l}^{[1,2]} = \frac{1}{4}\left[\left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{q\cdot e_1}{q\cdot k_1}\right)\frac{\not{e}_2\not{k}_2}{p\cdot k_2} - \left(\frac{p\cdot e_2}{p\cdot k_2} - \frac{q\cdot e_2}{q\cdot k_2}\right)\frac{\not{e}_1\not{k}_1}{p\cdot k_1}\right]\not{J}, \tag{23}$$

$$I_{2r}^{[1,2]} = \frac{1}{4}\not{J}\left[\left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{q\cdot e_1}{q\cdot k_1}\right)\frac{\not{k}_2\not{e}_2}{q\cdot k_2} - \left(\frac{p\cdot e_2}{p\cdot k_2} - \frac{q\cdot e_2}{q\cdot k_2}\right)\frac{\not{k}_1\not{e}_1}{q\cdot k_1}\right] , \tag{24}$$

$$I_3^{[1,2]} = \frac{1}{8}\left(-\frac{\not{e}_1\not{k}_1}{p\cdot k_1}\not{J}\frac{\not{k}_2\not{e}_2}{q\cdot k_2} + \frac{\not{e}_2\not{k}_2}{p\cdot k_2}\not{J}\frac{\not{k}_1\not{e}_1}{q\cdot k_1}\right) , \tag{25}$$

$$I_{4l}^{[1,2]} = \frac{1}{8}\frac{1}{p\cdot k_1 + p\cdot k_2 - k_1\cdot k_2}\left(\frac{\not{e}_1\not{k}_1\not{e}_2\not{k}_2}{p\cdot k_1} - \frac{\not{e}_2\not{k}_2\not{e}_1\not{k}_1}{p\cdot k_2}\right)\not{J}, \tag{26}$$

$$I_{4r}^{[1,2]} = \frac{1}{8}\not{J}\frac{1}{q\cdot k_1 + q\cdot k_2 - k_1\cdot k_2}\left(\frac{\not{k}_1\not{e}_1\not{k}_2\not{e}_2}{q\cdot k_2} - \frac{\not{k}_2\not{e}_2\not{k}_1\not{e}_1}{q\cdot k_1}\right) , \tag{27}$$

$$I_{5lA}^{[1,2]} = -\frac{1}{2}\not{J}\frac{p\cdot k_1 - p\cdot k_2}{p\cdot k_1 + p\cdot k_2 - k_1\cdot k_2}\left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{k_2\cdot e_1}{k_2\cdot k_1}\right)\left(\frac{p\cdot e_2}{p\cdot k_2} - \frac{k_1\cdot e_2}{k_1\cdot k_2}\right) , \tag{28}$$

$$I_{5lB}^{[1,2]} = \frac{1}{2}\not{J}\frac{p\cdot k_1 - p\cdot k_2}{p\cdot k_1 + p\cdot k_2 - k_1\cdot k_2}\left(\frac{k_1\cdot e_2}{k_1\cdot k_2}\frac{k_2\cdot e_1}{k_1\cdot k_2} - \frac{e_1\cdot e_2}{k_1\cdot k_2}\right) , \tag{29}$$

$$I_{5rA}^{[1,2]} = -\frac{1}{2}\not{J}\frac{q\cdot k_2 - q\cdot k_1}{q\cdot k_1 + q\cdot k_2 - k_1\cdot k_2}\left(\frac{q\cdot e_1}{q\cdot k_1} - \frac{k_2\cdot e_1}{k_2\cdot k_1}\right)\left(\frac{q\cdot e_2}{q\cdot k_2} - \frac{k_1\cdot e_2}{k_1\cdot k_2}\right) , \tag{30}$$

$$I_{5rB}^{[1,2]} = \frac{1}{2}\not{J}\frac{q\cdot k_2 - q\cdot k_1}{q\cdot k_1 + q\cdot k_2 - k_1\cdot k_2}\left(\frac{k_1\cdot e_2}{k_1\cdot k_2}\frac{k_2\cdot e_1}{k_1\cdot k_2} - \frac{e_1\cdot e_2}{k_1\cdot k_2}\right) , \tag{31}$$

$$\begin{aligned} I_{6l}^{[1,2]} &= \frac{1}{4}\frac{k_1\cdot k_2}{p\cdot k_1 + p\cdot k_2 - k_1\cdot k_2}\left[+\left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{k_2\cdot e_1}{k_1\cdot k_2}\right)\frac{\not{e}_2\not{k}_2}{p\cdot k_2}\right. \\ &\quad \left. -\left(\frac{p\cdot e_2}{p\cdot k_2} - \frac{k_1\cdot e_2}{k_1\cdot k_2}\right)\frac{\not{e}_1\not{k}_1}{p\cdot k_1}\right]\not{J}, \end{aligned} \tag{32}$$

$$
\begin{aligned}
I_{6r}^{[1,2]} &= \frac{1}{4}\not{J}\frac{k_1\cdot k_2}{q\cdot k_1+q\cdot k_2-k_1\cdot k_2}\Bigg[-\left(\frac{q\cdot e_1}{q\cdot k_1}-\frac{k_2\cdot e_1}{k_1\cdot k_2}\right)\frac{\not{k}_2\not{e}_2}{q\cdot k_2} \\
&\quad +\left(\frac{q\cdot e_2}{q\cdot k_2}-\frac{k_1\cdot e_2}{k_1\cdot k_2}\right)\frac{\not{k}_1\not{e}_1}{q\cdot k_1}\Bigg], \qquad (33)\\
I_{7lA}^{[1,2]} &= \frac{1}{2}\frac{1}{p\cdot k_1+p\cdot k_2-k_1\cdot k_2}\Bigg[-\left(\frac{p\cdot e_1}{p\cdot k_1}-\frac{k_2\cdot e_1}{k_1\cdot k_2}\right)\not{e}_2\not{k}_2 \\
&\quad +\left(\frac{p\cdot e_2}{p\cdot k_2}-\frac{k_1\cdot e_2}{k_1\cdot k_2}\right)\not{e}_1\not{k}_1\Bigg]\not{J}, \qquad (34)\\
I_{7lB}^{[1,2]} &= \frac{1}{4}\frac{1}{k_1\cdot k_2}\,\frac{1}{p\cdot k_1+p\cdot k_2-k_1\cdot k_2}\Big(\not{e}_2\not{k}_2\not{e}_1\not{k}_1-\not{e}_1\not{k}_1\not{e}_2\not{k}_2\Big)\not{J}, \qquad (35)\\
I_{7rA}^{[1,2]} &= \frac{1}{2}\not{J}\frac{1}{q\cdot k_1+q\cdot k_2-k_1\cdot k_2}\Bigg[+\left(\frac{q\cdot e_1}{q\cdot k_1}-\frac{k_2\cdot e_1}{k_1\cdot k_2}\right)\not{k}_2\not{e}_2 \\
&\quad -\left(\frac{q\cdot e_2}{q\cdot k_2}-\frac{k_1\cdot e_2}{k_1\cdot k_2}\right)\not{k}_1\not{e}_1\Bigg], \qquad (36)\\
I_{7rB}^{[1,2]} &= -\frac{1}{4}\not{J}\frac{1}{k_1\cdot k_2}\,\frac{1}{q\cdot k_1+q\cdot k_2-k_1\cdot k_2}\left(\not{k}_1\not{e}_1\not{k}_2\not{e}_2-\not{k}_2\not{e}_2\not{k}_1\not{e}_1\right). \qquad (37)
\end{aligned}
$$

This time, we do not have a good motivation for the particular form. We mainly tried to keep it analogous to $I^{\{1,2\}}$ term by term. Only the terms with a subscript starting with 7 do not have an analogue in $I^{\{1,2\}}$. The subtraction terms

$$
\begin{aligned}
S_{1,q}^{[1,2]} &= \not{J}\frac{k_2\cdot e_1}{k_1\cdot k_2}\left(\frac{p\cdot e_2}{p\cdot k_2}-\frac{q\cdot e_2}{q\cdot k_2}\right), \qquad (38)\\
S_{2,q}^{[1,2]} &= -\not{J}\left(\frac{\not{k}_2\not{e}_2}{q\cdot k_2}\frac{q\cdot e_1}{q\cdot k_1}-\frac{\not{k}_1\not{e}_1}{q\cdot k_1}\frac{q\cdot e_2}{q\cdot k_2}\right), \qquad (39)\\
S_{3,q}^{[1,2]} &= \frac{1}{2}\not{J}\frac{1}{q\cdot k_1+q\cdot k_2-k_1\cdot k_2}\left(-\frac{2q\cdot e_2 k_2\cdot e_1}{q\cdot k_2}+\frac{2q\cdot e_1 k_1\cdot e_2}{q\cdot k_1}\right) \qquad (40)
\end{aligned}
$$

are quite analogous to the ones used in the previous subsection and we will not elaborate on them further. The terms $S_{1,p}^{[1,2]}$, $S_{2,p}^{[1,2]}$ and $S_{3,p}^{[1,2]}$ are chosen analogously.

4. Picture of consecutive gluon emission

Now, we try to reorganize the expressions by incorporating an ordering of the gluons, a strategy which has proved its usefulness in a broad spectrum of applications and calculations concerning QCD.

4.1. Color-ordered amplitudes

Instead of an expansion in the commutator and the anti-commutator of color generators, let us simply use generator products to express

$$\mathcal{M}^{a,b} = \frac{1}{2}\bar{v}(p)\left(T^aT^bI^{(1,2)} + T^bT^aI^{(2,1)}\right)u(q)\,. \tag{41}$$

The so-called color-ordered amplitudes are obtained from the coefficients $I^{(1,2)}$ and $I^{(2,1)}$ by including the spinors $\bar{v}(p)$ and $u(q)$. Thanks to this ordering the expressions shorten significantly. For the T^aT^b-part, we find

$$I^{(1,2)} = \left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{k_2\cdot e_1}{k_2\cdot k_1} - \frac{\not{e}_1\not{k}_1}{2p\cdot k_1}\right)\not{J}\left(\frac{\not{k}_2\not{e}_2}{2q\cdot k_2} + \frac{k_1\cdot e_2}{k_1\cdot k_2} - \frac{q\cdot e_2}{q\cdot k_2}\right) \tag{42}$$

$$+\frac{p\cdot k_2}{p\cdot k_1 + p\cdot k_2 - k_1\cdot k_2}\left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{k_2\cdot e_1}{k_2\cdot k_1} - \frac{\not{e}_1\not{k}_1}{2p\cdot k_1}\right)$$

$$\times\left(\frac{p\cdot e_2}{p\cdot k_2} - \frac{k_1\cdot e_2}{k_1\cdot k_2} - \frac{\not{e}_2\not{k}_2}{2p\cdot k_2}\right)\not{J} \tag{43}$$

$$+\not{J}\frac{q\cdot k_1}{q\cdot k_1 + q\cdot k_2 - k_1\cdot k_2}\left(\frac{q\cdot e_1}{q\cdot k_1} - \frac{k_2\cdot e_1}{k_2\cdot k_1} - \frac{\not{k}_1\not{e}_1}{2q\cdot k_1}\right)$$

$$\times\left(\frac{q\cdot e_2}{q\cdot k_2} - \frac{k_1\cdot e_2}{k_1\cdot k_2} - \frac{\not{k}_2\not{e}_2}{2q\cdot k_2}\right) \tag{44}$$

$$+\not{J}\left(1 - \frac{p\cdot k_2}{p\cdot k_1 + p\cdot k_2 - k_1\cdot k_2} - \frac{q\cdot k_1}{q\cdot k_1 + q\cdot k_2 - k_1\cdot k_2}\right)$$

$$\times\left(\frac{k_1\cdot e_2}{k_1\cdot k_2}\frac{k_2\cdot e_1}{k_1\cdot k_2} - \frac{e_1\cdot e_2}{k_1\cdot k_2}\right) \tag{45}$$

$$-\frac{1}{4}\frac{1}{p\cdot k_1 + p\cdot k_2 - k_1\cdot k_2}\left(\frac{\not{e}_1\not{k}_1\not{e}_2\not{k}_2 - \not{e}_2\not{k}_2\not{e}_1\not{k}_1}{k_1\cdot k_2}\right)\not{J} \tag{46}$$

$$-\frac{1}{4}\not{J}\frac{1}{q\cdot k_1 + q\cdot k_2 - k_1\cdot k_2}\left(\frac{\not{k}_1\not{e}_1\not{k}_2\not{e}_2 - \not{k}_2\not{e}_2\not{k}_1\not{e}_1}{k_1\cdot k_2}\right)\,. \tag{47}$$

Each term in the above expressions is individually gauge invariant. The coefficient $I^{(2,1)}$ is obtained by a permutation of the momenta and polarization vectors of the two gluons in $I^{(1,2)}$.

4.2. Mixed representation

Now we use a mixed, and therefore over-defined, basis. Thanks to such a representation not only the exact results, but also expressions dominant in some regions of the phase space (obtained simply by truncation) will be more compact. The decomposition of the amplitude looks as follows

$$\mathcal{M}^{a,b} = \frac{1}{2}\bar{v}(p)\Big(T^aT^bI^{(1,2)}_{\rm mix} + T^bT^aI^{(2,1)}_{\rm mix} + [T^a,T^b]I^{[1,2]}_{\rm mix} + \{T^a,T^b\}I^{\{1,2\}}_{\rm mix}\Big)u(q)\,. \quad (48)$$

The first coefficient is given by

$$\begin{aligned} I^{(1,2)}_{\rm mix} &= \left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{k_2\cdot e_1}{k_2\cdot k_1} - \frac{\not{e}_1\not{k}_1}{2p\cdot k_1}\right)\not{J}\left(\frac{\not{k}_2\not{e}_2}{2q\cdot k_2} + \frac{k_1\cdot e_2}{k_1\cdot k_2} - \frac{q\cdot e_2}{q\cdot k_2}\right) \\ &+\frac{p\cdot k_2}{p\cdot k_1 + p\cdot k_2 - k_1\cdot k_2}\left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{k_2\cdot e_1}{k_2\cdot k_1} - \frac{\not{e}_1\not{k}_1}{2p\cdot k_1}\right) \\ &\times\left(\frac{p\cdot e_2}{p\cdot k_2} - \frac{k_1\cdot e_2}{k_1\cdot k_2} - \frac{\not{e}_2\not{k}_2}{2p\cdot k_2}\right)\not{J} \\ &+\not{J}\frac{q\cdot k_1}{q\cdot k_1 + q\cdot k_2 - k_1\cdot k_2}\left(\frac{q\cdot e_1}{q\cdot k_1} - \frac{k_2\cdot e_1}{k_2\cdot k_1} - \frac{\not{k}_1\not{e}_1}{2q\cdot k_1}\right) \\ &\times\left(\frac{q\cdot e_2}{q\cdot k_2} - \frac{k_1\cdot e_2}{k_1\cdot k_2} - \frac{\not{k}_2\not{e}_2}{2q\cdot k_2}\right)\,, \end{aligned} \quad (49)$$

and coefficient $I^{(2,1)}_{\rm mix}$ is again obtained by a permutation of the momenta and polarization vectors of the gluons in the above expression. The other coefficients are given by

$$\begin{aligned} I^{[1,2]}_{\rm mix} &= \frac{1}{2}\not{J}\left(\frac{p\cdot k_1 - p\cdot k_2}{p\cdot k_1 + p\cdot k_2 - k_1\cdot k_2} + \frac{q\cdot k_2 - q\cdot k_1}{q\cdot k_1 + q\cdot k_2 - k_1\cdot k_2}\right) \\ &\times\left(\frac{k_1\cdot e_2}{k_1\cdot k_2}\frac{k_2\cdot e_1}{k_1\cdot k_2} - \frac{e_1\cdot e_2}{k_1\cdot k_2}\right) \\ &-\frac{1}{4}\frac{1}{p\cdot k_1 + p\cdot k_2 - k_1\cdot k_2}\left(\frac{\not{e}_1\not{k}_1\not{e}_2\not{k}_2 - \not{e}_2\not{k}_2\not{e}_1\not{k}_1}{k_1\cdot k_2}\right)\not{J} \\ &-\frac{1}{4}\not{J}\frac{1}{q\cdot k_1 + q\cdot k_2 - k_1\cdot k_2}\left(\frac{\not{k}_1\not{e}_1\not{k}_2\not{e}_2 - \not{k}_2\not{e}_2\not{k}_1\not{e}_1}{k_1\cdot k_2}\right)\,, \end{aligned} \quad (50)$$

$$\begin{aligned} I^{\{1,2\}}_{\rm mix} &= -\frac{1}{2}\not{J}\left(\frac{1}{p\cdot k_1 + p\cdot k_2 - k_1\cdot k_2} + \frac{1}{q\cdot k_1 + q\cdot k_2 - k_1\cdot k_2}\right) \\ &\times\left(\frac{k_1\cdot e_2\; k_2\cdot e_1}{k_1\cdot k_2} - e_1\cdot e_2\right)\,. \end{aligned} \quad (51)$$

The choice for the above representation is not unique, and at this point it seems to be based mainly on aesthetic grounds. Again each term in the above expressions is individually gauge invariant. In order to justify our choice a bit further, we change our strategy in the construction of the expressions. So far, the only manipulations we performed consisted of the reorganization of terms, and the introduction of our so-called subtraction terms. All these terms were written explicitly in terms of momenta and polarization vectors. Now, we will leave this path, and start to introduce new objects useful to compactify the expression even further. The objects are the momentum of the virtual gluon

$$k^\mu_{1+2} = k^\mu_1 + k^\mu_2 \,, \tag{52}$$

and the four-vectors

$$\begin{aligned} e^\mu_{1+2} &= \frac{k^\mu_1 - k^\mu_2}{2\,k_1\cdot k_2}\left(\frac{e_1\cdot k_2\; e_2\cdot k_1}{k_1\cdot k_2} - e_1\cdot e_2\right) , \\ \hat{e}^\mu_{1+2} &= \frac{k^\mu_1 - k^\mu_2}{2\,k_1\cdot k_2}\,\frac{\frac{\mathrm{i}}{4}\,\mathrm{Tr}(\gamma^5 \not{e}_1 \not{k}_1 \not{e}_2 \not{k}_2)}{k_1\cdot k_2} \,, \end{aligned} \tag{53}$$

which represent the effective polarizations of the virtual gluon piled together with its propagator[1]. Notice, that the expressions are gauge-invariant. Using the fact that

$$\frac{\not{e}_1\not{k}_1\not{e}_2\not{k}_2 - \not{e}_2\not{k}_2\not{e}_1\not{k}_1}{2k_1\cdot k_2} = (\not{e}_{1+2} + \mathrm{i}\gamma^5 \not{\hat{e}}_{1+2})\not{k}_{1+2} \,, \tag{54}$$

we immediately see that we can write

$$\begin{aligned} I^{[1,2]}_{\mathrm{mix}} &= \not{J}\left(\frac{p\cdot e_{1+2}}{p\cdot k_{1+2} - k_1\cdot k_2} - \frac{q\cdot e_{1+2}}{q\cdot k_{1+2} - k_1\cdot k_2}\right) \\ &\quad -\frac{1}{2}\left[\frac{(\not{e}_{1+2} + \mathrm{i}\gamma^5\not{\hat{e}}_{1+2})\not{k}_{1+2}}{p\cdot k_{1+2} - k_1\cdot k_2}\right]\not{J} + \frac{1}{2}\not{J}\left[\frac{\not{k}_{1+2}(\not{e}_{1+2} - \mathrm{i}\gamma^5\not{\hat{e}}_{1+2})}{q\cdot k_{1+2} - k_1\cdot k_2}\right] . \end{aligned} \tag{55}$$

Notice the similarity of this expression with Eq. (5) for the single-gluon emission. In the limit when $k_1\cdot k_2$ becomes zero, whether it be a soft or a collinear limit, the quantities

$$\left(\frac{e_1\cdot k_2 e_2\cdot k_1}{k_1\cdot k_2} - e_1\cdot e_2\right) \qquad \text{and} \qquad \frac{\frac{\mathrm{i}}{4}\,\mathrm{Tr}(\gamma^5\not{e}_1\not{k}_1\not{e}_2\not{k}_2)}{k_1\cdot k_2} \tag{56}$$

[1] Proper units of energy will appear only after including the appropriate factor from the phase space Jacobian (this is usually done, only after some assumption is made on phase space regions to be considered). Note also, that e_{1+2} is parallel to $\hat{e}_{1+2}$ and has a significant component along the k_{1+2}-direction. This is another reason to be cautious with their physical interpretation.

stay both finite. In particular, when the polarization vectors e_1 and e_2 become parallel, $\hat{e}_{1+2}$, absent anyway in the first line of (55), becomes zero, indicating that it contributes negligibly compared to e_{1+2}, and making the (55) even more similar to (5). Also,

$$\left(\frac{p}{p\cdot k_1 + p\cdot k_2 - k_1\cdot k_2} - \frac{q}{q\cdot k_1 + q\cdot k_2 - k_1\cdot k_2}\right)\cdot\frac{k_1 - k_2}{k_1\cdot k_2} \tag{57}$$

remains finite when k_1 and k_2 become collinear. This property of the expression present in the first term of (50) is a technical manifestation of the absence of longitudinal gluons, but this time for the virtual gluon. Again, the similarity to (5) manifests itself. There, the cancellation is exact, and is a consequence of gauge invariance in combination with the fact that the gluon is on mass-shell.

4.3. Properties of the amplitudes

Let us analyze the building blocks which appear at the exact amplitude level in the expressions given in Sec. 4.1 and Sec. 4.2. In [33] (Sec. 13.1) it was shown in a pedagogical manner, that for example in case k_1 becomes collinear with p the factor

$$\left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{\not{e}_1\not{k}_1}{2p\cdot k_1}\right) \tag{58}$$

gives, after phase space integration over the appropriate region, the Altarelli–Parisi kernel. It can thus be understood as its precursor at the spin amplitude level. Like this, of course, such an expression makes no sense, since it is gauge-dependent. Only a gauge invariant-object like

$$\left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{k_2\cdot e_1}{k_2\cdot k_1} - \frac{\not{e}_1\not{k}_1}{2p\cdot k_1}\right) \tag{59}$$

can be interpreted unambiguously. It can be understood as a precursor at the spin amplitude level for the emission of k_1 from a dipole-source spanned by p and k_2, but this time including the effect due to the fermion spin as well. Not only this assures gauge invariance, but also gives a scale for the appropriate logarithm if the integration would be completed. At this point, it seems straightforward to interpret expressions like (59) present in (49) as factors for the emission from incoming fermions and with angle distributions controlled by the direction of the other gluon, completing at the same time the second arm of the dipole. This interpretation is based mainly on the visual appearance of the expressions, ignoring the structure of singular terms and/or interferences, and therefore might be misleading. On the other hand, we want to note that our observations are similar to the ones which can be found in the literature, where indeed such interpretation was performed for the sake of constructing parton shower models, even though usually some approximations were used then.

Let us look at (59), as present *e.g.* in the first line of (49). Independently whether we interpret this factor as the description of the emission of a gluon or not, the momentum entering J from the left side is $p - k_1$. If the collinear limit can be used, and thus to a good approximation J varies slowly with an eventual virtuality $(p - k_1)^2$, then the effective intermediate state of a fermion of momentum $p - k_1$ can be used to simplify the description. A similar line of arguments is used in [33] (and since long in any formulation of the PDF evolution) and we will not repeat it here. Note that the intermediate effective gluon definition introduced in Sec. 4.2 is valid all over phase space, even in phase space regions where it does not have any physical meaning anymore. We do not have such descriptions for an intermediate effective fermion available at this moment: we would have to stick to the low virtuality limit, and thus have to introduce an approximation.

Let us shift our attention to the second term of (49). For the simplified picture of the previous paragraph to work, one of the effective incoming momenta must be a collinear projection of $p - k_1 - k_2$ (or $q - k_1 - k_2$), while the other one simply remains q or p. No ambiguity appears in this respect. Regarding the factors $(q{\cdot}k_1)/(q{\cdot}k_1 + q{\cdot}k_2 - k_2{\cdot}k_1)$ and $(p{\cdot}k_1)/(p{\cdot}k_1 + p{\cdot}k_2 - k_2{\cdot}k_1)$ we want to mention that, in case of collinear configurations, they are indispensable for the redefinition of the spinor normalization used in the definition of evolution kernels. As we do not want to limit ourselves to such configurations, but want to use expressions valid all over the phase space, definitions of spinors for intermediate, seemingly on mass-shell, fermion states should remain valid everywhere as well. We do not have such definition available.

Let us now turn our attention to (50). It is rather tempting to interpret its p- and q-dependent parts as the real-emission contributions to the running of the coupling constant (of the single gluon emission). The most singular parts of these contributions cancel each other partly as we can see in Eq. (57). Such a partial result may already provide a hint on the possible optimal choice of scale to be used as an argument of the running coupling constant.

The role of the remaining parts of the amplitude, expression (51), is less clear, although it resembles the running coupling constant part. It may equally well be interpreted as a genuine second-order part of the amplitude, which cannot be interpreted in the language suitable for resummation at all. Note that, from the point of view of $pq \to k_1 k_2 J$ kinematics, this contribution is less singular than any of the previously discussed ones.

5. Picture of ordered gluon emission in dipole language

In the previous chapters we have collected several forms for exact spin amplitudes. Now, we concentrate on cases of special limits, usually associated with some type of ordering, and for which iterative descriptions such as BFKL [34,35], DGLAP [6,7,36] or CCFM [37–39] are valid and are known to be useful. We will attempt to define amplitudes which are dominant in certain regions of phase space, but nonetheless valid *all* over phase space, and which differ from the complete amplitude only by gauge invariant and analytically available expressions.

5.1. x ordering and soft gluon limit (BFKL)

Let us restrict our attention to the region of phase space where $\sqrt{s} \gg k_1^0 \gg k_2^0$ and $k_1^0 k_2^0 \simeq k_1 \cdot k_2$ (all in the reaction rest frame). As a consequence, we can use the conditions $p\cdot k_1 \gg p\cdot k_2 \gg k_1\cdot k_2$ and/or $q\cdot k_1 \gg q\cdot k_2 \gg k_1\cdot k_2$ for the localization of dominant terms. Such a choice is consistent with the BFKL approximation [34,35]. Under these constraints, the T^aT^b-part (49) of our expression reduces to

$$\begin{aligned} I_{\rm mix}^{(1,2)} &= \not{J}\left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{k_2\cdot e_1}{k_2\cdot k_1}\right)\left(\frac{k_1\cdot e_2}{k_1\cdot k_2} - \frac{q\cdot e_2}{q\cdot k_2}\right) \\ &\quad + \not{J}\left(\frac{q\cdot e_1}{q\cdot k_1} - \frac{k_2\cdot e_1}{k_2\cdot k_1}\right)\left(\frac{q\cdot e_2}{q\cdot k_2} - \frac{k_1\cdot e_2}{k_1\cdot k_2}\right) . \end{aligned} \tag{60}$$

All other parts are smaller. A similar expression is obtained for $I_{\rm mix}^{(2,1)}$. After some short manipulations, and since the contributions from Eq. (50) and Eq. (51) are also negligible thanks to the conditions, the full amplitude reduces to:

$$\begin{aligned} \mathcal{M}_{\rm BFKL}^{a,b} &= \frac{1}{2}\bar{v}(p)\not{J}u(q)\left[T^aT^b\left(\frac{q\cdot e_1}{q\cdot k_1} - \frac{p\cdot e_1}{p\cdot k_1}\right)\left(\frac{q\cdot e_2}{q\cdot k_2} - \frac{k_1\cdot e_2}{k_1\cdot k_2}\right)\right. \\ &\quad \left. + T^bT^a\left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{q\cdot e_1}{q\cdot k_1}\right)\left(\frac{p\cdot e_2}{p\cdot k_2} - \frac{k_1\cdot e_2}{k_1\cdot k_2}\right)\right] . \end{aligned} \tag{61}$$

A picture of linked dipoles is manifest in this expression. It is also worth mentioning that (50), suspected to contribute to the running of the coupling constant, can be neglected under our conditions. The obtained amplitude (61) is a well defined part of the exact one, and at the same time is consistent with the BFKL approximation. It is easy to implement into a Monte Carlo program incorporated with exact Lorentz invariant phase space as in QED [1,2] for the solution based on the exact matrix element and full phase space coverage. Note that the truncated amplitude is given in the same kinematical formulation as the complete exact amplitude for two-gluon emission. It can be calculated at any point in phase space, and not only where the approximation is justified. Re-installation of the exact amplitude into a calculation based on sufficiently refined dipole language can be accommodated for by a well-defined weight. We want to mention, that the picture of linked dipoles is used with success in Ariadne parton shower [40].

5.2. $p_{\rm T}$ ordering or DGLAP

Also $p_{\rm T}$ ordering can be formulated in terms of properties of Lorentz invariant expressions, allowing for an easy identification of the parts of the amplitudes which may be simply dropped out. In the following, we will discuss the two kinematical cases of emissions into one, or two (opposite) hemispheres. We will devote a separate subsection to the discussion of the running coupling constant contribution.

5.2.1. Dominant parts for emissions in one hemisphere

We assume first that $p\cdot k_1 \gg p\cdot k_2$ or $q\cdot k_1 \gg q\cdot k_2$, but we accept configurations where $p\cdot k_1 \simeq k_1\cdot k_2$ or $q\cdot k_1 \simeq k_1\cdot k_2$. In practice, such conditions mean that k_2, p and q are basically parallel to each other from the point of view of the direction of k_1. The 'macroscopic' size of $k_1\cdot k_2$ implicitly removes the phase space regions where the gluons become collinear. Such a case must be treated separately. Under our conditions and after some manipulations, the T^aT^b part of the amplitude takes the form:

$$\begin{aligned} I^{(1,2)}_{\rm mix} &= -\frac{k_1\cdot k_2}{q\cdot k_1 - k_1\cdot k_2}\left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{k_2\cdot e_1}{k_2\cdot k_1} - \frac{\not{e}_1\not{k}_1}{2p\cdot k_1}\right)\not{J}\left(\frac{\not{k}_2\not{e}_2}{2q\cdot k_2} + \frac{k_1\cdot e_2}{k_1\cdot k_2} - \frac{q\cdot e_2}{q\cdot k_2}\right) \\ &\quad + \frac{q\cdot k_1}{q\cdot k_1 - k_1\cdot k_2}\left\{\not{J}\left(\frac{q\cdot e_1}{q\cdot k_1} - \frac{k_2\cdot e_1}{k_2\cdot k_1} - \frac{\not{k}_1\not{e}_1}{2q\cdot k_1}\right)\right. \\ &\quad \left. - \left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{k_2\cdot e_1}{k_2\cdot k_1} - \frac{\not{e}_1\not{k}_1}{2p\cdot k_1}\right)\not{J}\right\}\left(\frac{q\cdot e_2}{q\cdot k_2} - \frac{k_1\cdot e_2}{k_1\cdot k_2} - \frac{\not{k}_2\not{e}_2}{2q\cdot k_2}\right), \end{aligned} \tag{62}$$

and the T^bT^a part becomes

$$\begin{aligned} I^{(2,1)}_{\rm mix} &= -\frac{k_1\cdot k_2}{p\cdot k_1 - k_2\cdot k_1}\left(\frac{p\cdot e_2}{p\cdot k_2} - \frac{k_1\cdot e_2}{k_1\cdot k_2} - \frac{\not{e}_2\not{k}_2}{2p\cdot k_2}\right)\not{J}\left(\frac{\not{k}_1\not{e}_1}{2q\cdot k_1} + \frac{k_2\cdot e_1}{k_2\cdot k_1} - \frac{q\cdot e_1}{q\cdot k_1}\right) \\ &\quad + \frac{p\cdot k_1}{p\cdot k_1 - k_2\cdot k_1}\left(\frac{p\cdot e_2}{p\cdot k_2} - \frac{k_1\cdot e_2}{k_1\cdot k_2} - \frac{\not{e}_2\not{k}_2}{2p\cdot k_2}\right) \\ &\quad \times\left\{\left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{k_2\cdot e_1}{k_2\cdot k_1} - \frac{\not{e}_1\not{k}_1}{2p\cdot k_1}\right)\not{J} + \not{J}\left(\frac{\not{k}_1\not{e}_1}{2q\cdot k_1} + \frac{k_2\cdot e_1}{k_2\cdot k_1} - \frac{q\cdot e_1}{q\cdot k_1}\right)\right\}. \end{aligned} \tag{63}$$

The above expressions can easily be understood in the language of evolution kernels. The factors $(q\cdot k_1)/(q\cdot k_1 - k_2\cdot k_1)$ and $(p\cdot k_1)/(p\cdot k_1 - k_2\cdot k_1)$ can be understood as redefinition of the spinor normalization, again similarly as explained in [33]. Note that the contents of the curly brackets in the last two formulas are free of terms proportional to $(k_1\cdot e_2)/(k_2\cdot k_1)$ and $(k_2\cdot e_1)/(k_2\cdot k_1)$. They represent simple dipoles spanned on the p,q-pair. In contrast with the previous section, correcting terms with $\not{k}_1$ and/or $\not{k}_2$ in the numerator remain. This is closely related to the necessity/possibility to introduce effective momenta flowing into J, once the language of PDFs is introduced.

The coefficient for the running coupling constant (55) seems to survive in its near complete form. In our limit and because of cancellations it develops an extra power of $k_1\cdot k_2$ in the eikonal part, and a factor of $\sqrt{k_1\cdot k_2}$ in the terms proportional to k_{1+2}. This will cancel the singularity of the virtual gluon once the amplitude is squared and a partial integration over the phase space is performed. Because of this, an interpretation in the language of the running coupling constant is prevented. It can be treated as a non-singular correction, and thus ignored.

For now, we can conclude that our amplitude, with properties consistent with p_{T} ordering, is more compact than the exact one. It is gauge invariant and valid all over phase space. Such a simple, amplitude-level, expression can be used at the intermediate step for the definition of a parton shower algorithm, or to better understand the already existing ones. As in the previous BFKL case, an explicit form for the weight necessary to reinstall the exact distribution based on the two-gluon amplitude is available. It was mentioned already earlier how to identify emission kernels already at the spin amplitude level. It seems that the explanations included in [33] can be applied here as well.

5.2.2. Dominant parts for simultaneous emissions from p and q

Now, let us assume that $p\cdot k_2 \gg p\cdot k_1$ and $q\cdot k_1 \gg q\cdot k_2$. Under such conditions, k_1 is basically parallel to p, and k_2 to q. It is obvious that contribution (42) dominates over all other contributions of the complete spin amplitude. If in addition $p\cdot k_2 \gg q\cdot k_1$, then we can replace $(k_1\cdot e_2)/(k_1\cdot k_2)$ with $(p\cdot e_2)/(p\cdot k_2)$ and get

$$I^{(1,2)} = \left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{k_2\cdot e_1}{k_2\cdot k_1} - \frac{\not{e}_1\not{k}_1}{2p\cdot k_1}\right)\not{J}\left(\frac{\not{k}_2\not{e}_2}{2q\cdot k_2} + \frac{p\cdot e_2}{p\cdot k_2} - \frac{q\cdot e_2}{q\cdot k_2}\right). \tag{64}$$

This expression is again of the dipole form. At the same time it is part of the expression discussed previously. This means, that the replacement of $(k_1\cdot e_2)/(k_1\cdot k_2)$ with $(p\cdot e_2)/(p\cdot k_2)$, which is potentially dangerous and valid only in this particular region of phase space, is not necessary, except for our proof here. Difficulties, as the ones to be discussed in Sec. 5.3, can be avoided.

5.2.3. Case when k_1 and k_2 may become parallel

So far, we have implicitly excluded the phase space region contributing to the running coupling constant from our discussion. The p_{T} ordering makes it unfavorable for the two gluons to become parallel one to another. Therefore, we have to look at regions of phase space where the virtuality of the gluon may approach zero separately. For that purpose, we will assume that the overall p_{T} of the virtual gluon is small, but larger than $k_1{\cdot}k_2$. We have found such an approach in studies presented in [41, 42]. We will consider the configuration when $q{\cdot}k_1 \gg p{\cdot}k_1$, $q{\cdot}k_2 \gg p{\cdot}k_2$, $p{\cdot}k_1 \gg k_1{\cdot}k_2$ and $p{\cdot}k_1 \sim p{\cdot}k_2$. Such choice represents the splitting of the anti-quark with momentum p into a virtual fermion line entering J, and a single gluon of small virtuality and (moderately) small p_{T}. Under such circumstances, dominant contributions from (49) take the form:

$$\begin{aligned} I^{(1,2)}_{\mathrm{mix}} = & \left(\frac{p{\cdot}e_1}{p{\cdot}k_1} - \frac{k_2{\cdot}e_1}{k_2{\cdot}k_1} - \frac{\not{e}_1\not{k}_1}{2p{\cdot}k_1}\right)\not{J}\left(\frac{\not{k}_2\not{e}_2}{2q{\cdot}k_2} + \frac{k_1{\cdot}e_2}{k_1{\cdot}k_2} - \frac{q{\cdot}e_2}{q{\cdot}k_2}\right) \\ & + \frac{p{\cdot}k_2}{p{\cdot}k_1 + p{\cdot}k_2}\left(\frac{p{\cdot}e_1}{p{\cdot}k_1} - \frac{k_2{\cdot}e_1}{k_2{\cdot}k_1} - \frac{\not{e}_1\not{k}_1}{2p{\cdot}k_1}\right)\left(\frac{p{\cdot}e_2}{p{\cdot}k_2} - \frac{k_1{\cdot}e_2}{k_1{\cdot}k_2} - \frac{\not{e}_2\not{k}_2}{2p{\cdot}k_2}\right)\not{J} \\ & + \not{J}\frac{q{\cdot}k_1}{q{\cdot}k_1 + q{\cdot}k_2}\left(\frac{q{\cdot}e_1}{q{\cdot}k_1} - \frac{k_2{\cdot}e_1}{k_2{\cdot}k_1} - \frac{\not{k}_1\not{e}_1}{2q{\cdot}k_1}\right)\left(\frac{q{\cdot}e_2}{q{\cdot}k_2} - \frac{k_1{\cdot}e_2}{k_1{\cdot}k_2} - \frac{\not{k}_2\not{e}_2}{2q{\cdot}k_2}\right) . \end{aligned} \tag{65}$$

The analogous form for $I^{(2,1)}_{\mathrm{mix}}$ is obtained by a permutation of the momenta and polarization vectors of the gluons in the above expression. The dominant contributions of (50) take the form:

$$\begin{aligned} I^{[1,2]}_{\mathrm{mix}} = & \frac{1}{2}\not{J}\left(\frac{p{\cdot}k_1 - p{\cdot}k_2}{p{\cdot}k_1 + p{\cdot}k_2} + \frac{q{\cdot}k_2 - q{\cdot}k_1}{q{\cdot}k_1 + q{\cdot}k_2}\right)\left(\frac{k_1{\cdot}e_2}{k_1{\cdot}k_2}\frac{k_2{\cdot}e_1}{k_1{\cdot}k_2} - \frac{e_1{\cdot}e_2}{k_1{\cdot}k_2}\right) \\ & - \frac{1}{4}\frac{1}{p{\cdot}k_1 + p{\cdot}k_2}\left(\frac{\not{e}_1\not{k}_1\not{e}_2\not{k}_2 - \not{e}_2\not{k}_2\not{e}_1\not{k}_1}{k_1{\cdot}k_2}\right)\not{J} \\ & - \frac{1}{4}\not{J}\frac{1}{q{\cdot}k_1 + q{\cdot}k_2}\left(\frac{\not{k}_1\not{e}_1\not{k}_2\not{e}_2 - \not{k}_2\not{e}_2\not{k}_1\not{e}_1}{k_1{\cdot}k_2}\right) . \end{aligned} \tag{66}$$

Contribution (51) becomes non-leading as a whole. Obviously our choice of remaining terms is different from the one of Sec. 5.2.1. Nonetheless one can observe that (65) is included in (62), if for the latter both possibilities $p{\cdot}k_1 \gg p{\cdot}k_2$ and $p{\cdot}k_1 \ll p{\cdot}k_2$ are added together. Expression (66) may contribute to the running coupling constant. The last two terms of (66) are non-leading for our choice of kinematical conditions and could be dropped out. However, it is gauge invariant and rather compact, and it is necessary in case gluons would be collimated with q. Also, one should bear in mind that the first line of (66) is less singular than it may seem, because of a partial cancellation of p and q dependent terms.

5.3. Angular ordering or CCFM style

The CCFM case is definitely more difficult for our approach than the ones discussed so far. We have to use angular ordering to select the dominant terms. Such a choice is less straightforward for the Lorentz invariant representation. It is thus not possible to simply neglect some terms because they would be explicitly smaller than others. In fact, it seems that all terms of our expressions for the amplitude will need to be kept. Some kind of a language exploiting the concept of effective intermediate states is needed. Even if approximated amplitudes would be defined, they would differ from the complete ones quite substantially by the presence of these effective intermediate states. That would definitely be out of scope of our present discussion.

On the other hand, we want to remark that the interpretation of the result using some sort of dipole language will persist in this case as well, as it is already visible in the exact amplitudes. There will simply be more terms to take into account, that is formulas (42), (43), (44) and their symmetric analogies. The fermion normalization redefinition factors $(q\cdot k_1)/(q\cdot k_1 + q\cdot k_2 - k_2\cdot k_1)$ and $(p\cdot k_1)/(p\cdot k_1 + p\cdot k_2 - k_2\cdot k_1)$ are more complicated than in the previously discussed cases. Also the terms contributing to the running of the coupling constant would require a more sophisticated treatment.

6. Summary

We have analyzed different forms of the exact tree level QCD spin amplitude for the process $q\bar{q} \to Jgg$ where J may represent any, color singlet, current. We have found quite well structured representations consisting of sums and products of compact gauge invariant parts. In contrast to previous studies for QED, the present expressions do not seem to be unique. This is understandable, since amplitudes and the structure of singularities are more complex in QCD. In particular, the discussion of singularities for processes where incoming quarks would be interchanged with outgoing gluons should be included in our considerations to constrain ambiguities.

We have used two types of organizations of parts. The first one, discussed in Sec. 3, manifests the relation between QCD and QED amplitudes. In Sec. 4, we provided expressions for the exact spin amplitude more useful for QCD phenomenology. In Sec. 4.3, we attempted to give a physical interpretation for the different parts that can be recognized in these expressions. We found a separation into terms possible to interpret either as responsible for consecutive emissions of gluons or as contribution to the decay of a virtual gluon (running of the coupling constant).

In Sec. 5, we discussed approximations for the amplitude, which consist of BFKL and DGLAP pictures. As a consequence, some parts of our expressions for the exact amplitude could be dropped. The remaining parts are straightforward to interpret, and explicit expressions for the difference with the exact amplitude at any point of phase space exist. They are straightforward to manipulate in any numerical application, for example in parton shower Monte Carlo applications, using positively defined weights.

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