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A NEW INTEGRABLE SYMPLECTIC MAP OF BARGMANN TYPE

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By resorting to the nonlinearization approach, a Bargmann constraint associated with a discrete 3×3 matrix eigenvalue problem is considered. The lattice soliton hierarchy and the bi-Hamiltonian structures are obtained. A new symplectic map of the Bargmann type is obtained by nonlinearization of the discrete eigenvalue problem and its adjoint one. With the help of the generating function, we arrive at the involutive system of conserved integrals of the symplectic map, which is further proved to be completely integrable.

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1. Introduction

In recent years, there have developed several systematic approaches to carry out the principle, pointed by Flaschka, Ablowitz and others [1–3], that the new finite-dimensional integrable systems can be constructed from the infinite-dimensional ones. One of the approaches, nonlinearization of eigenvalue problems on Lax pairs [4–9], has been proved to be a powerful tool for a wide class of continuous soliton equations. The framework of the discrete version of classical integrable systems was found to generate the lattice soliton hierarchies, and then some associated integrable symplectic maps [10–12] were obtained. Since then, the discrete version of classical integrable systems have become the focus of common concern and become the important field of soliton and integrable systems. While the 3×3 matrix spectral problems were few considered before the work of Blaszak *et al.* [13], which using the R-matrix approach to construct the integrable lattice systems and their bi-Hamiltonian structure. Meanwhile, several 3×3 discrete matrix spectral problems were studied in [14–16] by the nonlinearization method.

The nonlinearization technique [17–19] is proved to be a powerful tool for obtaining new finite-dimensional integrable systems from various soliton hierarchies under certain constraints between the potentials and the eigenfunctions. The technique is also effective in the discrete case [20]. In the continuous case, the nonlinearization of the eigenvalue problem gives an integrable system, while in the discrete case it yields an integrable symplectic map. The constraint in nonlinearization method is practically a kind of symmetry constraint, depends on the eigenfunctions of Lax pairs [21,22].

In this paper, we use the nonlinearization approach to study a 3×3 matrix spectral problem, and give a Bargmann constraint between the eigenfunction, the adjoint eigenfunction and potentials. A hierarchy of lattice soliton equations associated with the discrete eigenvalue problem is constructed, as well as their Hamiltonian structures by making use of the trace identity [23] in Sec. 2. In Sec. 3, a new symplectic map of the Bargmann type is obtained by the nonlinearization of the discrete 3×3 eigenvalue problem and its adjoint one. Finally in Sec. 4, we use the generating function approach to calculate the involutivity of integrals, by which the symplectic map of the Bargmann type is further proved to be completely integrable in Liouville sense.

2. The lattice soliton hierarchy and the Hamiltonian structures

We first introduce the definition of shift operator and difference operators needed in the sequel:

$$Ef(n) = f(n+1)\,, \qquad \Delta f(n) = (E-1)f(n)\,, \qquad \Delta^* f(n) = \left(E^{-1}-1\right) f(n)\,.$$

For the sake of convenience, we usually write $f(n) = f$, $f(n+k) = E^k f$, $n, k \in \boldsymbol{Z}$. Consider the discrete 3×3 spectral problem [13]

$$E\psi(\lambda) = U\psi(\lambda)\,, \tag{1}$$

with

$$\psi(\lambda) = \begin{pmatrix} \psi^1(\lambda) \\ \psi^2(\lambda) \\ \psi^3(\lambda) \end{pmatrix}, \qquad U = \begin{pmatrix} a+\lambda & b & 1 \\ c & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

where a, b, c are three potentials and λ is a constant spectral parameter. In order to derive the hierarchy of lattice equations associated with Eq. (1), we first solve the stationary discrete zero-curvature equation:

$$(EV)U - UV = 0\,, \qquad V = (V_{ij})\,, \tag{2}$$

where each entry $V_{ij} = V_{ij}(A, B, D)$ of the 3×3 matrix V is a Laurent expansion of λ:

$$\begin{array}{ll} V_{11} = D\,, & V_{12} = bA + c^{-1}EB\,, \\ V_{13} = A\,, & V_{21} = E^{-1}cA\,, \\ V_{22} = E^2 D - E(a+\lambda)A - \Delta bB\,, & V_{23} = B\,, \\ V_{31} = E^{-1}B\,, & V_{32} = E^{-1}c^{-1}E^{-1}cA - E^{-1}c^{-1}(a+\lambda)B\,, \\ V_{33} = ED - (a+\lambda)A - bB\,, & \end{array}$$

$$A = \sum_{j=0}^{\infty} A_{j-1}\lambda^{-j}\,, \qquad B = \sum_{j=0}^{\infty} B_{j-1}\lambda^{-j}\,, \qquad D = \sum_{j=0}^{\infty} D_{j-1}\lambda^{-j}\,.$$

The stationary discrete zero-curvature Eq. (2) is equivalent to the recursive relation:

$$\begin{aligned} &\left(cEb - bE^{-1}c\right) A_{j-1} + \left(cEc^{-1}E - E^{-1}\right) B_{j-1} + a\triangle D_{j-1} = -\triangle D_j\,, \\ &\left(E - E^{-1}c^{-1}E^{-1}c + b\triangle a\right) A_{j-1} + \left(E^{-1}ac^{-1} - ac^{-1}E + b\triangle b\right) B_{j-1} \\ &= -b\triangle A_j + \left(c^{-1}E - E^{-1}c^{-1}\right) B_j\,, \\ &c\left(1 - E^2\right) aA_{j-1} - cE\triangle bB_{j-1} + c\left(E^3 - 1\right) D_{j-1} = c\left(E^2 - 1\right) A_j\,, \\ &\triangle D_{-1} = 0\,, \\ &-b\triangle A_{-1} + \left(c^{-1}E - vE^{-1}c^{-1}\right) B_{-1} = 0\,, \\ &c\left(E^2 - 1\right) A_{-1} = 0\,. \end{aligned} \tag{3}$$

We define $\{F_j\}$ by the following relation:

$$ED_j = bB_j - \left(1 + E^{-1}\right) cF_j\,. \tag{4}$$

Using Eqs. (3) and Eq. (4), we can obtain

$$KG_{j-1} = JG_j\,, \qquad G_j = \left(A_j, B_j, F_j\right)^T\,, \tag{5}$$

where K, J are called Lenard's operator pair:

$$K = \begin{pmatrix} cEb - bE^{-1}c & -E^{-1} + cEc^{-1}E - a\triangle^* b & a\left(E^{-2} - 1\right)c \\ E - E^{-1}c^{-1}E^{-1}c + b\triangle a & E^{-1}ac^{-1} - ac^{-1}E & b\left(E - E^{-1}\right)c \\ c\left(1 - E^2\right)a & \left(E - E^{-1}\right)b & c\left(E^{-1} - E + E^{-2} - E^2\right)c \end{pmatrix},$$

$$J = \begin{pmatrix} 0 & \triangle^* b & -\left(E^{-2} - 1\right)c \\ -b\triangle & c^{-1}E - E^{-1}c^{-1} & 0 \\ c\left(E^2 - 1\right) & 0 & 0 \end{pmatrix},$$

and it is easy to see that K, J are skew-symmetry. From Eqs. (3) and Eq. (5), as well as Eq. (4), we have

$$G_{-1}=(1,0,0)^T\,,\qquad G_0=\left(-a,E^{-1}c,Eb\right)^T\,.$$

Let $\psi(\lambda)$ satisfy the spectral problem (1) and its auxiliary problem

$$\frac{\partial}{\partial t}\psi(\lambda)=V^{(m)}\psi(\lambda)\,,\tag{6}$$

where

$$V^{(m)}=\left(V_{ij}^{(m)}\right)_{3\times 3}\,,\quad V_{ij}^{(m)}=V_{ij}\left(A^{(m)},B^{(m)},D^{(m)}\right)\,,$$
$$A^{(m)}=\sum_{j=0}^{m}A_{j-1}\lambda^{m-j}\,,\quad B^{(m)}=\sum_{j=0}^{m}B_{j-1}\lambda^{m-j}\,,\quad D^{(m)}=\sum_{j=0}^{m}D_{j-1}\lambda^{m-j}\,.$$

Then the compatibility condition between Eq. (1) and Eq. (6) yields the discrete zero-curvature equation

$$\frac{\partial}{\partial t}U=\left(EV^{(m)}\right)U-UV^{(m)}\,,$$

which is equivalent to

$$\frac{\partial}{\partial t_m}u=KG^{(m)}-\lambda JG^{(m)}\,,\qquad G^{(m)}=\left(A^{(m)},B^{(m)},D^{(m)}\right)^T\,,\tag{7}$$

where $u=(a,b,c)^T$. Eq. (7) implies the discrete soliton equation

$$\frac{\partial}{\partial t_m}u=X_m\,,\qquad m\geq -1\,,\tag{8}$$

here $X_j=JG_j=KG_{j-1}, j\geq 0$. When $m=0$, the evolution equation of the hierarchy (8) is

$$\frac{\partial}{\partial t}\begin{pmatrix}a(n)\\ b(n)\\ c(n)\end{pmatrix}=\begin{pmatrix}b(n+1)c(n)-b(n)c(n-1)\\ b(n)[a(n+1)-a(n)]+\frac{1}{c(n-1)}[c(n-1)-c(n-2)]\\ c(n)[a(n)-a(n+2)]\end{pmatrix}.\tag{9}$$

To establish the Hamiltonian structure of the discrete soliton hierarchy (8), we need to calculate the following quantities which satisfy

$$\mathrm{tr}\left(\hat{V}\frac{\partial U}{\partial a}\right)=A,\qquad \mathrm{tr}\left(\hat{V}\frac{\partial U}{\partial b}\right)=B\,,$$
$$\mathrm{tr}\left(\hat{V}\frac{\partial U}{\partial c}\right)=\frac{1}{c}\left[D-(\lambda+a)A\right]\,,\qquad \mathrm{tr}\left(\hat{V}\frac{\partial U}{\partial \lambda}\right)=A\,,\tag{10}$$

where $\hat{V} = VU^{-1}$. By means of the trace identity [23], we have

$$\left(\frac{\delta}{\delta a}, \frac{\delta}{\delta b}, \frac{\delta}{\delta c}\right) A = \left[\lambda^{-\gamma}\left(\frac{\partial}{\partial \lambda}\right)\lambda^{\gamma}\right]\left(A, B, \frac{1}{c}[D - (\lambda + a)A]\right), \qquad (11)$$

where γ is a constant to be fixed. Equating the coefficients of λ^{-j-1} on both sides of Eq. (11), we obtain

$$\left(\frac{\delta}{\delta a}, \frac{\delta}{\delta b}, \frac{\delta}{\delta c}\right) A_j = (\gamma - j)\left(A_{j-1}, B_{j-1}, \frac{1}{c}\left[D_{j-1} - A_j - aA_{j-1}\right]\right), \quad (12)$$

taking $j = 0$, we arrive at $\gamma = -1$. From the third equation of Eqs. (3) and Eq. (4), we have

$$D_{j-1} - A_j - aA_{j-1} = cF_{j-1}\,. \qquad (13)$$

Hence

$$\left(\frac{\delta}{\delta a}, \frac{\delta}{\delta b}, \frac{\delta}{\delta c}\right) H_j = G_j\,, \qquad H_j = -\frac{1}{j+1}A_{j+1}\,, \qquad (14)$$

which shows that the discrete soliton hierarchy (8) possesses the bi-Hamiltonian structures

$$\frac{\partial u}{\partial t} = X_m = J\left(\frac{\delta}{\delta a}, \frac{\delta}{\delta b}, \frac{\delta}{\delta c}\right)^T H_m = K\left(\frac{\delta}{\delta a}, \frac{\delta}{\delta b}, \frac{\delta}{\delta c}\right)^T H_{m-1}\,, \qquad m \geq 0\,.$$

3. A symplectic map of the Bargmann type

In order to get a symplectic map associated with spectral problem (1), we need to consider its adjoint discrete 3×3 matrix spectral problem

$$E\varphi(\lambda) = \left(U^{-1}\right)^T \varphi(\lambda)\,, \qquad \varphi(\lambda) = \left(\varphi^1(\lambda), \varphi^2(\lambda), \varphi^3(\lambda)\right)^T. \qquad (15)$$

For N mutual distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_N$, the systems associated with Eq. (1) and Eq. (15) can be written in the form

$$\begin{aligned} E\left(q_j^1, q_j^2, q_j^3\right) &= \left(q_j^1, q_j^2, q_j^3\right) U\left(u, \lambda_j\right)^T, \\ E\left(p_j^1, p_j^2, p_j^3\right) &= \left(p_j^1, p_j^2, p_j^3\right) U\left(u, \lambda_j\right)^{-1}, \end{aligned} \qquad (16)$$

where $q_j^i = \psi^i(\lambda_j), p_j^i = \varphi^i(\lambda_j), 1 \leq i \leq 3, 1 \leq j \leq N$, are normalized eigenfunctions. It is readily verified that the functional gradient of the eigenvalue λ_j with respect to the potentials (a, b, c) is

$$\nabla\lambda_j = \begin{pmatrix} \delta\lambda_j/\delta a \\ \delta\lambda_j/\delta b \\ \delta\lambda_j/\delta c \end{pmatrix} = \begin{pmatrix} -q_j^1 p_j^3 \\ -q_j^2 p_j^3 \\ \frac{1}{c}(\lambda_j q_j^1 p_j^3 + a q_j^1 p_j^3 - q_j^1 p_j^1) \end{pmatrix}. \qquad (17)$$

Though a direct calculation, we can get the following equation about the above gradient,

$$K\nabla\lambda_j = \lambda_j J\nabla\lambda_j\,. \tag{18}$$

Now consider the Bargmann constraint

$$G_0 = \sum_{j=1}^{N} \nabla\lambda_j\,, \tag{19}$$

which gives

$$a = \left\langle q^1, p^3\right\rangle\,, \qquad b = \frac{\left\langle q^2, p^2\right\rangle}{\left\langle q^2, p^3\right\rangle}\,, \qquad c = \Omega\,, \tag{20}$$

here

$$\begin{aligned}\Omega \;=\; & -\left\langle (a+\Lambda)q^1 + bq^2 + q^3, p^1 - (a+\Lambda)p^3\right\rangle \\ & +\left(a\left\langle q^1,p^3\right\rangle + \left\langle \Lambda q^1, p^3\right\rangle - \left\langle q^1, p^1\right\rangle\right)\left\langle (a+\Lambda)\,q^1 + bq^2 + q^3, p^2 - bp^3\right\rangle\,,\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ is the standard inner-product in $\mathcal{R}^N$, $q^i = (q^i_1, \cdots, q^i_N)^T$, $p^i = (p^i_1, \cdots, p^i_N)^T$, $1 \le i \le 3$, $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_N)$. Substituting the expressions (20) into Eqs. (16), we obtain a discrete Bargmann system

$$\begin{aligned} Eq^1 &= \left\langle q^1,p^3\right\rangle q^1 + \Lambda q^1 + \frac{\left\langle q^2,p^2\right\rangle}{\left\langle q^2,p^3\right\rangle} q^2 + q^3\,, \\ Eq^2 &= \Omega q^1\,, \\ Eq^3 &= q^2\,, \\ Ep^1 &= p^3\,, \\ Ep^2 &= \Omega^{-1}\left[p^1 - \left\langle q^1,p^3\right\rangle p^3 - \Lambda p^3\right]\,, \\ Ep^3 &= p^2 - \frac{\left\langle q^2,p^2\right\rangle}{\left\langle q^2,p^3\right\rangle} p^3\,. \end{aligned} \tag{21}$$

Through tedious calculations we get

$$\sum_{j=1}^{N}\sum_{i=1}^{3} d\left(Eq^i_j\right) \wedge d\left(Ep^i_j\right) = \sum_{j=1}^{N}\sum_{i=1}^{3} dq^i_j \wedge dp^i_j\,.$$

Therefore Eqs. (21) determine a symplectic map H of the Bargmann type,

$$\left(Eq^1, Eq^2, Eq^3, Ep^1, Ep^2, Ep^3\right) = H\left(q^1, q^2, q^3, p^1, p^2, p^3\right)\,. \tag{22}$$

4. The integrability of the symplectic map

In this section, we will investigate the involutive of conserved integrals for the symplectic map H.

Using Eq. (5), Eq. (18) and the constraint (19), we take the following restriction

$$G_j = \sum_{k=1}^{N} \lambda_k^j \nabla \lambda_k \,, \tag{23}$$

which is a special solution of Eq. (5). Then expression (23) is equivalent to

$$\begin{aligned} A_j &= -\left\langle \Lambda^j q^1, p^3 \right\rangle , \qquad B_j = -\left\langle \Lambda^j q^2, p^3 \right\rangle , \\ F_j &= c^{-1}\left(\left\langle \Lambda^{j+1} q^1, p^3 \right\rangle + a \left\langle \Lambda^j q^1, p^3 \right\rangle - \left\langle \Lambda^j q^1, p^1 \right\rangle \right) . \end{aligned} \tag{24}$$

By substitution the above expressions (24) into Eq. (13), we obtain

$$D_j = -\left\langle \Lambda^j q^1, p^1 \right\rangle , \qquad j \geq 0 \,. \tag{25}$$

Consider a bilinear function Q_λ^{ik} on $\mathcal{R}^N$ and its partial-fraction expression and Laurent expression:

$$Q_\lambda^{ik} = \left\langle (\lambda - \Lambda)^{-1} q^i, p^k \right\rangle = \sum_{j=1}^{N} \frac{q_j^i p_j^k}{\lambda - \lambda_j} = \sum_{m \geq 0} \lambda^{-m-1} \left\langle \Lambda^m q^i, p^k \right\rangle . \tag{26}$$

It is easy to verify that

$$Q_\lambda^{ij}\left(\Lambda^k\right) = \lambda Q_\lambda^{ij}\left(\Lambda^{k-1}\right) - \left\langle \Lambda^{k-1} q^i, p^j \right\rangle , \tag{27}$$

where $Q_\lambda^{ij}(\Lambda^k) = \langle (\lambda - \Lambda)^{-1} \Lambda^k q^i, p^j \rangle$. By virtue of the above notations, and substituting expressions (24) into the Laurent expressions of A, B, D, we get

$$A = 1 - Q_\lambda^{13} , \qquad B = -Q_\lambda^{23} , \qquad D = -Q_\lambda^{11} \,. \tag{28}$$

Now introduce a Lax matrix by

$$\widetilde{V}_\lambda = \mathbb{Q} + \begin{pmatrix} 0 & s_1 & -1 \\ s_2 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} , \tag{29}$$

where $\mathbb{Q} = (Q_\lambda^{ij})_{3\times 3}$ and $s_1 = -\frac{\langle q^2, p^2 \rangle}{\langle q^2, p^3 \rangle}$, $s_2 = \langle q^2, p^3 \rangle$.

Let

$$\mathcal{F}_{\varsigma\lambda} = \det\left(\varsigma I - \widetilde{V}_\lambda\right) = \varsigma^3 - \mathcal{F}_\lambda^{(0)} \varsigma^2 + \mathcal{F}_\lambda^{(1)} \varsigma - \mathcal{F}_\lambda^{(2)} , \tag{30}$$

where I is a 3×3 unit matrix, ς is a parameter. Then from Eq. (30), we can arrive at a set of integrals $\{F_m^{(i)}\}, 0 \leq i \leq 2$, of the discrete Bargmann system (22) (see Appendix). The Poisson bracket of two functions in the symplectic space $(R^{6N} \sum_{i=1}^{3} dp^i \wedge dq^i)$ is defined as

$$\{f,g\}=\sum_{j=1}^{N}\sum_{i=1}^{3}\left(\frac{\partial f}{\partial q_j^i}\frac{\partial g}{\partial p_j^i}-\frac{\partial f}{\partial p_j^i}\frac{\partial g}{\partial q_j^i}\right)=\sum_{i=1}^{3}\left(\left\langle\frac{\partial f}{\partial q^i},\frac{\partial g}{\partial p^i}\right\rangle-\left\langle\frac{\partial f}{\partial p^i},\frac{\partial g}{\partial q^i}\right\rangle\right),$$

which is skew-symmetric, bilinear, satisfies Jacobi identity and Leibnitz rule: $\{fg,h\} = f\{g,h\} + g\{f,h\}$.

We can prove the following assertions:

Theorem 1 *The function $\mathcal{F}_{\xi\mu}$ is invariant along the $\tau_{\varsigma\lambda}$-flow.*

Proof Through tedious calculations, and with the aid of expressions (30) and (A1)–(A4), as well as the identity

$$\left\langle (\mu I - \Lambda)^{-1}\,(\lambda I - \Lambda)^{-1} q^i, p^j \right\rangle = \frac{1}{\mu - \lambda}\left(Q_\lambda^{ij} - Q_\mu^{ij}\right),$$

we obtain $\{\mathcal{F}_{\xi\mu}, \mathcal{F}_{\varsigma\lambda}\} = 0, \forall \lambda, \varsigma, \mu, \xi \in \mathcal{C}$, which implies the derivative of the function $\mathcal{F}_{\xi\mu}$ along the $\tau_{\varsigma\lambda}$-flow is zero.

Theorem 2 *The integrals $\{F_m^{(i)}\}, 0 \leq i \leq 2, m \geq 0$, are in involution in pairs, that is, $\{F_m^{(i)}, F_l^{(j)}\} = 0, 0 \leq i, j \leq 2$, for any $m, l \geq 0$.*

Theorem 3 *The symplectic map of the Bargmann type defined by Eq. (22) is completely integrable in the Liouville sense.*

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Appendix A

For the sake of convenience, we denote $\mathbb{Q}_{ij}$ as the cofactor of element Q_λ^{ij}, and $\mathbb{Q}^*$ as the adjoint of matrix $\mathbb{Q}$. From (26), we know that $\mathbb{Q}_{ij} = \sum_{m\geq 0} \lambda^{-m-2}\mathbb{Q}_{ij,m}$ and $\det \mathbb{Q} = \sum_{m\geq 0} \lambda^{-m-3}|\mathbb{Q}|_m$. Regard the generating function $\mathcal{F}_\lambda^{(2)} = det\widetilde{V}_\lambda$ as a Hamiltonian in the symplectic space (R^{6N},

$\sum_{i=1}^{3} dp^i \wedge dq^i$). Denote the variable of $\mathcal{F}_\lambda^{(2)}$-flow by τ_λ. Through a direct calculation gives the canonical equation of the $\mathcal{F}_\lambda^{(2)}$-flow:

$$\frac{d}{d\tau_\lambda}\begin{pmatrix} q_k^1 \\ q_k^2 \\ q_k^3 \end{pmatrix} = \begin{pmatrix} \partial\mathcal{F}_\lambda^{(2)}/\partial p_k^1 \\ \partial\mathcal{F}_\lambda^{(2)}/\partial p_k^2 \\ \partial\mathcal{F}_\lambda^{(2)}/\partial p_k^3 \end{pmatrix} = W(\lambda,\lambda_k)\begin{pmatrix} q_k^1 \\ q_k^2 \\ q_k^3 \end{pmatrix},$$

$$\frac{d}{d\tau_\lambda}\begin{pmatrix} p_k^1 \\ p_k^2 \\ p_k^3 \end{pmatrix} = \begin{pmatrix} -\partial\mathcal{F}_\lambda^{(2)}/\partial q_k^1 \\ -\partial\mathcal{F}_\lambda^{(2)}/\partial q_k^2 \\ -\partial\mathcal{F}_\lambda^{(2)}/\partial q_k^3 \end{pmatrix} = -W(\lambda,\lambda_k)^T\begin{pmatrix} p_k^1 \\ p_k^2 \\ p_k^3 \end{pmatrix}, \quad \text{(A.1)}$$

where

$$W(\lambda,\mu) = \frac{1}{\lambda-\mu}\mathbb{Q}^* + \begin{pmatrix} 0 & 0 & 0 \\ 0 & w_1 & 0 \\ 0 & w_2 & 0 \end{pmatrix} + \frac{1}{\lambda-\mu}$$

$$\times\begin{pmatrix} \lambda^2+\lambda Q_\lambda^{22}+\lambda Q_\lambda^{33} & -(\lambda s_1+\lambda Q_\lambda^{12}+s_1 Q_\lambda^{33}+Q_\lambda^{32}) & Q_\lambda^{32}+s_1 Q_\lambda^{23}-\lambda Q_\lambda^{13} \\ -(\lambda s_2+\lambda Q_\lambda^{21}+s_2 Q_\lambda^{33}) & \lambda Q_\lambda^{11}+Q_\lambda^{31} & s_2 Q_\lambda^{13}-s_2-Q_\lambda^{21} \\ s_2 Q_\lambda^{32}-\lambda Q_\lambda^{31} & s_1 Q_\lambda^{31} & \lambda Q_\lambda^{11}-s_1 Q_\lambda^{21}-s_2 Q_\lambda^{12}-s_1 s_2 \end{pmatrix},$$

with

$$\begin{aligned} w_1 &= \frac{1}{\langle q^2,p^3\rangle}\left(\lambda Q_\lambda^{21}+\lambda s_2+s_2 Q_\lambda^{33}+\mathbb{Q}_{12}\right), \\ w_2 &= -\frac{\langle q^2,p^2\rangle}{\langle q^2,p^3\rangle^2}\left(\lambda Q_\lambda^{21}+\lambda s_2+s_2 Q_\lambda^{33}+\mathbb{Q}_{12}\right), \\ &\quad +\left(\lambda Q_\lambda^{12}+\lambda s_1+s_1 Q_\lambda^{32}+Q_\lambda^{32}+\mathbb{Q}_{21}\right). \end{aligned}$$

Denote the variable of $\mathcal{F}_{\varsigma\lambda}$-flow by $\tau_{\varsigma\lambda}$, where $\mathcal{F}_{\varsigma\lambda} = det(\varsigma I - \widetilde{V}_\lambda)$, I is a 3×3 unit matrix, ς is a parameter. It is easy to see that $\mathcal{F}_{\varsigma\lambda}$ can be written as

$$\mathcal{F}_{\varsigma\lambda} = \varsigma^3 - \mathcal{F}_\lambda^{(0)}\varsigma^2 + \mathcal{F}_\lambda^{(1)}\varsigma - \mathcal{F}_\lambda^{(2)}, \quad \text{(A.2)}$$

where

$$\begin{aligned} \mathcal{F}_\lambda^{(0)} &= Q_\lambda^{11}+Q_\lambda^{22}+Q_\lambda^{33}+2\lambda, \\ \mathcal{F}_\lambda^{(1)} &= \lambda\left(2Q_\lambda^{11}+Q_\lambda^{22}+Q_\lambda^{33}+\lambda\right)-s_1 s_2-s_1 Q_\lambda^{21}-s_2 Q_\lambda^{12}+Q_\lambda^{31}+\sum_{1\le i\le 3}\mathbb{Q}_{ii}, \end{aligned}$$

$$\begin{aligned}\mathcal{F}_\lambda^{(2)} &= \lambda^2 Q_\lambda^{11} - \lambda\left(s_1 Q_\lambda^{21} + s_2 Q_\lambda^{12} + s_1 s_2\right) - s_2 Q_\lambda^{32} - s_1 s_2 Q_\lambda^{33} - \mathbb{Q}_{13} \\ &\quad + \lambda \mathbb{Q}_{33} + \lambda \mathbb{Q}_{22} - s_1 \mathbb{Q}_{12} - s_2 \mathbb{Q}_{21} + \det \mathbb{Q}\,. \end{aligned} \tag{A.3}$$

Substituting the identity (26) into expressions (A3), we arrive at

$$\begin{aligned}\mathcal{F}_\lambda^{(0)} &= 2\lambda + \sum_{m\geq 0} \lambda^{-m-1} F_m^{(0)}\,, \qquad \mathcal{F}_\lambda^{(1)} = \lambda^2 + \sum_{m\geq 0} \lambda^{-m-1} F_m^{(1)}\,, \\ \mathcal{F}_\lambda^{(2)} &= \sum_{m\geq 0} \lambda^{-m} F_m^{(2)}\,, \end{aligned} \tag{A.4}$$

where

$$\begin{aligned}
F_m^{(0)} &= \left\langle \Lambda^m q^1, p^1 \right\rangle + \left\langle \Lambda^m q^2, p^2 \right\rangle + \left\langle \Lambda^m q^3, p^3 \right\rangle\,, \\
F_0^{(1)} &= 2\left\langle \Lambda q^1, p^1 \right\rangle + \left\langle \Lambda q^2, p^2 \right\rangle + \left\langle \Lambda q^3, p^3 \right\rangle + \frac{\left\langle q^2, p^2 \right\rangle}{\left\langle q^2, p^3 \right\rangle} \left\langle q^2, p^1 \right\rangle \\
&\quad - \left\langle q^2, p^3 \right\rangle \left\langle q^1, p^2 \right\rangle + \left\langle q^3, p^1 \right\rangle\,, \\
F_m^{(1)} &= 2\left\langle \Lambda^{m+1} q^1, p^1 \right\rangle + \left\langle \Lambda^{m+1} q^2, p^2 \right\rangle + \left\langle \Lambda^{m+1} q^3, p^3 \right\rangle + \frac{\left\langle q^2, p^2 \right\rangle}{\left\langle q^2, p^3 \right\rangle} \left\langle \Lambda^m q^2, p^1 \right\rangle \\
&\quad - \left\langle q^2, p^3 \right\rangle \left\langle \Lambda^m q^1, p^2 \right\rangle + \left\langle \Lambda^m q^3, p^1 \right\rangle + \sum_{1\leq i\leq 3} \mathbb{Q}_{ii,m-1}\,, \qquad m > 2\,, \\
F_0^{(2)} &= \left\langle \Lambda q^1, p^1 \right\rangle + \frac{\left\langle q^2, p^2 \right\rangle}{\left\langle q^2, p^3 \right\rangle} \left\langle q^2, p^1 \right\rangle - \left\langle q^2, p^3 \right\rangle \left\langle q^1, p^2 \right\rangle\,, \\
F_1^{(2)} &= \frac{\left\langle q^2, p^2 \right\rangle}{\left\langle q^2, p^3 \right\rangle} \left\langle \Lambda q^2, p^1 \right\rangle - \left\langle q^2, p^3 \right\rangle \left\langle \Lambda q^1, p^2 \right\rangle - \left\langle q^2, p^3 \right\rangle \left\langle q^3, p^2 \right\rangle \\
&\quad + \left\langle q^2, p^2 \right\rangle \left\langle q^3, p^3 \right\rangle + \left\langle q^1, p^1 \right\rangle \left\langle q^2, p^2 \right\rangle - \left\langle q^1, p^2 \right\rangle \left\langle q^2, p^1 \right\rangle \\
&\quad + \left\langle q^1, p^1 \right\rangle \left\langle q^3, p^3 \right\rangle - \left\langle q^1, p^3 \right\rangle \left\langle q^3, p^1 \right\rangle + \left\langle \Lambda^2 q^1, p^1 \right\rangle\,, \\
F_2^{(2)} &= \frac{\left\langle q^2, p^2 \right\rangle}{\left\langle q^2, p^3 \right\rangle} \left\langle \Lambda^2 q^2, p^1 \right\rangle - \left\langle q^2, p^3 \right\rangle \left\langle \Lambda^2 q^1, p^2 \right\rangle - \left\langle q^2, p^3 \right\rangle \left\langle \Lambda q^3, p^2 \right\rangle \\
&\quad + \left\langle q^2, p^2 \right\rangle \left\langle \Lambda q^3, p^3 \right\rangle + \left\langle q^1, p^1 \right\rangle \left\langle \Lambda q^2, p^2 \right\rangle - \left\langle \Lambda q^1, p^2 \right\rangle \left\langle q^2, p^1 \right\rangle \\
&\quad + \left\langle \Lambda q^1, p^1 \right\rangle \left\langle q^2, p^2 \right\rangle - \left\langle q^1, p^2 \right\rangle \left\langle \Lambda q^2, p^1 \right\rangle + \left\langle q^1, p^1 \right\rangle \left\langle \Lambda q^3, p^3 \right\rangle \\
&\quad - \left\langle \Lambda q^1, p^3 \right\rangle \left\langle q^3, p^1 \right\rangle + \left\langle \Lambda q^1, p^1 \right\rangle \left\langle q^3, p^3 \right\rangle - \left\langle \Lambda q^3, p^1 \right\rangle \left\langle q^1, p^3 \right\rangle \\
&\quad - \left\langle q^2, p^1 \right\rangle \left\langle q^3, p^2 \right\rangle + \left\langle q^2, p^3 \right\rangle \left[\left\langle q^1, p^3 \right\rangle \left\langle q^3, p^2 \right\rangle - \left\langle q^1, p^2 \right\rangle \left\langle q^3, p^3 \right\rangle\right] \\
&\quad + \frac{\left\langle q^2, p^2 \right\rangle}{\left\langle q^2, p^3 \right\rangle} \left\langle q^2, p^1 \right\rangle \left\langle q^3, p^3 \right\rangle + \left\langle \Lambda^3 q^1, p^1 \right\rangle\,,
\end{aligned}$$

$$\begin{aligned} F_m^{(2)} &= \left\langle \Lambda^{m+1} q^1, p^1 \right\rangle + \frac{\left\langle q^2, p^2 \right\rangle}{\left\langle q^2, p^3 \right\rangle} \left\langle \Lambda^m q^2, p^1 \right\rangle - \left\langle q^2, p^3 \right\rangle \left\langle \Lambda^m q^1, p^2 \right\rangle \\ &\quad - \left\langle q^2, p^3 \right\rangle \left\langle \Lambda^{m-1} q^3, p^2 \right\rangle + \left\langle q^2, p^2 \right\rangle \left\langle \Lambda^{m-1} q^3, p^3 \right\rangle + \mathbb{Q}_{33,m-1} + \mathbb{Q}_{22,m-1} \\ &\quad - \mathbb{Q}_{13,m-2} + \frac{\left\langle q^2, p^2 \right\rangle}{\left\langle q^2, p^3 \right\rangle} \mathbb{Q}_{12,m-2} - \left\langle q^2, p^3 \right\rangle \mathbb{Q}_{21,m-2} + |\mathbb{Q}|_{m-3}, \quad m \geq 3\,. \end{aligned}$$

In this way, we obtain the integrals $\{F_m^{(i)}\}, 0 \leq i \leq 2$, of the discrete Bargmann system (22).

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