

BRANE COSMOLOGY WITH ${}^{(4)}R$ TERM IN THE BULK

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We consider brane cosmology when the 4D Ricci scalar term is added to the 5D Einstein–Hilbert action and discuss the role that the addition of this term has on the brane-bulk system. The induced brane dynamics is shown to be the usual Einstein dynamics coupled to a modified energy-momentum tensor which is well defined once the 5D Einstein equations are solved in the bulk. The 5D Einstein equations valid everywhere in the bulk, but not in the brane, are projected on the brane. Then making use for the embedding of the brane in the bulk of the Israel junction conditions, modified by a source term coming from the addition of the intrinsic curvature scalar in the bulk action, it is possible to obtain the effective 4D Einstein equations on the brane consistent with the bulk geometry.

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1. Introduction

In the present work we investigate the cosmological evolution of the brane-bulk system in the framework of the Randall–Sundrum (A)dS₅ scenario [1]. The effective 4D gravitational equations in the brane without curvature correction terms were first obtained by Shiromizu, Maeda and Sasaki [2]. These equations have been later recovered and generalized both on the brane and in the bulk taking into account the effect of a general bulk energy-momentum tensor and either the asymmetric embedding [3] or the accelerations of normals [4]. However, even employing more generalized gravitational actions, the derived 4D Einstein equations do not in general form a closed system due to the presence of a Weyl term which can only be specified in terms of the bulk metric, so other equations are to be written down and different procedures arise in splitting the non-Einsteinian terms between bulk and brane [5]. We assume homogeneity and isotropy in the three ordinary spatial dimensions but these symmetries cannot be extended to the extra dimension due to the presence of the brane, so all the physical quantities will depend on time and on the extra dimension. The solutions of

the 5D Einstein equations that we shall obtain will be valid strictly in the bulk. Then we project on the brane at $y = 0$ those equations and making use of the junction conditions, modified by an additional term coming from the ${}^{(4)}R$ curvature correction, as the boundary conditions imposed for the embedding of the brane in the bulk we obtain the effective 4D Einstein equations on the brane consistent with the bulk geometry. The method used here of deriving the 5D solution that is then projected onto the brane to study the brane dynamics was used extensively in the early days of braneworld cosmology by Kanti *et al.* [6]. The paper is organized as follows. In the next section, before giving a brief account of our method, we summarize the results obtained by Kofinas [7] when the ${}^{(4)}R$ term is included in the 5D action. Such a term, that was considered by a number of authors in the literature (see [8,9,10] and references therein), is generically introduced by quantum corrections coming from the bulk gravity and its coupling with matter confined to the brane, moreover its inclusion brings a convenient decomposition of the matter terms. In Section 3 we find the related equations in the brane, assumed infinitely thin and Z_2 symmetric in the bulk. In Section 4 we use the flexibility of the 5D solution to describe some cosmological models in 4D. Finally, in the Appendix we show how the 5D dynamical solution we adopt can be obtained starting from a 5D static solution.

Conventions. Throughout the paper the 5D metric signature is taken to be $(+, +, +, -, \varepsilon)$ where ε can be $+1$ or -1 depending on whether the extra dimension is space-like or time-like, while the choice of the 4D metric signature is $(+, +, +, -)$. The space-times coordinates are labeled $x^i = (r, \vartheta, \varphi)$, $x^4 = t$. The extra coordinate is $x^5 = y$. Bulk indices will be denoted by capital Latin letters and brane indices by lower Greek letters. In what follows we choose units such that $\hbar = c = 1$.

2. Braneworld Einstein field equations

In this section we recall the results obtained by Kofinas [7], which we shall use in the following, giving a brief account of their derivation. Once we have solved the equations in the bulk, the form of the induced equations will allow us finding brane solutions following the methods of General Relativity with a well defined energy-momentum tensor. The starting point in [7] is a three-dimensional brane Σ embedded in a five-dimensional spacetime M . For convenience the coordinate y is chosen such that the hypersurface $y = 0$ coincides with the brane. The total action for the system is taken to be

$$\begin{aligned} \mathcal{S} = & \frac{1}{2\kappa_5^2} \int_M \sqrt{-\varepsilon} {}^{(5)}g \left({}^{(5)}R - 2\Lambda_5 \right) d^5x + \frac{1}{2\kappa_4^2} \int_\Sigma \sqrt{-{}^{(4)}g} \left({}^{(4)}R - 2\Lambda_4 \right) d^4x \\ & + \int_M \sqrt{-\varepsilon} {}^{(5)}g L_5^{\text{mat}} d^5x + \int_\Sigma \sqrt{-{}^{(4)}g} L_4^{\text{mat}} d^4x. \end{aligned} \quad (1)$$

The constants κ_5^2 and κ_4^2 are given by

$$\kappa_5^2 = 8\pi G_5 = M_5^{-3}, \quad \kappa_4^2 = 8\pi G_4 = M_4^{-2}, \quad (2)$$

where M_5 and M_4 are the Planck masses. Varying (1) with respect to the bulk metric g_{AB} one obtains the equations

$$^{(5)}G_A^B = -\Lambda_5 \delta_A^B + \kappa_5^2 \left(^{(5)}T_A^B + {}^{(\text{loc})}T_A^B \delta(y) \right), \quad (3)$$

where

$${}^{(\text{loc})}T_A^B = -\frac{1}{\kappa_4^2} \sqrt{\frac{-^{(4)}g}{-\varepsilon ^{(5)}g}} \left(^{(4)}G_A^B - \kappa_4^2 {}^{(4)}T_A^B + \Lambda_4 h_A^B \right) \quad (4)$$

is the localized energy-momentum tensor of the brane. $^{(5)}G_{AB}$ and $^{(4)}G_{AB}$ denote the Einstein tensors constructed from the bulk and the brane metrics respectively, while the tensor $h_{AB} = g_{AB} - \varepsilon n_A n_B$ is the induced metric on the hypersurfaces $y = \text{constant}$, with n^A the normal unit vector on these

$$n^A = \frac{\delta_5^A}{\Phi}, \quad n_A = (0, 0, 0, 0, \varepsilon \Phi). \quad (5)$$

The scalar Φ which normalizes n^A is known as the lapse function and in a cosmological scenario which we shall consider later, it will depend on t and y . The way the coordinate y has been chosen allows to write the five-dimensional line element, at least in the neighborhood of the brane, as

$$dS^2 = g_{AB} dx^A dx^B = g_{\mu\nu} dx^\mu dx^\nu + \varepsilon \Phi^2 dy^2. \quad (6)$$

Using the methods of canonical analysis [11] the Einstein equations (3) in the bulk are split into the following sets of equations

$$K_{\mu;\nu}^\nu - K_{;\mu} = \varepsilon \kappa_5^2 \Phi ^{(5)}T_\mu^y, \quad (7a)$$

$$K_\nu^\mu K_\mu^\nu - K^2 + \varepsilon ^{(4)}R = 2\varepsilon (\Lambda_5 - \kappa_5^2 ^{(5)}T_y^y), \quad (7b)$$

$$\begin{aligned} \frac{\partial K_\nu^\mu}{\partial y} + \Phi K K_\nu^\mu - \varepsilon \Phi ^{(4)}R_\nu^\mu + \varepsilon g^{\mu\lambda} \Phi_{;\lambda\nu} = & -\varepsilon \kappa_5^2 \Phi \left({}^{(\text{loc})}T_\nu^\mu - \frac{1}{3} {}^{(\text{loc})}T \delta_\nu^\mu \right) \delta(y) \\ & - \varepsilon \kappa_5^2 \Phi ^{(5)}T_\nu^\mu \\ & + \frac{\varepsilon}{3} \Phi \left(\kappa_5^2 ^{(5)}T - 2\Lambda_5 \right) \delta_\nu^\mu, \end{aligned} \quad (7c)$$

where $K_{\mu\nu}$ is the extrinsic curvature of the hypersurfaces $y = \text{constant}$

$$K_{\mu\nu} = \frac{1}{2\Phi} \frac{\partial g_{\mu\nu}}{\partial y}, \quad K_{Ay} = 0. \quad (8)$$

The Israel junction conditions [12] for the singular part in Eq. (7c) are

$$[K_\nu^\mu] = -\varepsilon \kappa_5^2 \Phi_0 \left({}^{(\text{loc})}T_\nu^\mu - \frac{1}{3} {}^{(\text{loc})}T \delta_\nu^\mu \right), \quad (9)$$

where the square brackets mean discontinuity of the quantity across $y = 0$ and Φ_0 represents Φ at $y = 0$. Consequently, considering a Z_2 symmetry on reflection around the brane, (9) becomes

$${}^{(4)}G_\nu^\mu = -\Lambda_4 \delta_\nu^\mu + \kappa_4^2 {}^{(4)}T_\nu^\mu + \frac{2\varepsilon}{r_c} (\overline{K}_\nu^\mu - \overline{K} \delta_\nu^\mu), \quad (10)$$

where $\overline{K}_\nu^\mu = K_\nu^\mu(y = 0^+) = -K_\nu^\mu(y = 0^-)$ and $r_c = \kappa_5^2/\kappa_4^2$ is a crossover term which determines the region of validity of conventional four-dimensional General Relativity. From Eq. (7a) it follows that the tensor ${}^{(4)}T_\nu^\mu$ satisfies the conservation law ${}^{(4)}T_{\nu;\mu}^\mu = 0$ provided ${}^{(5)}T_\mu^y = 0$, which means no exchange of energy between brane and bulk. The quantities \overline{K}_ν^μ are still undetermined and should be obtained from some exact solution of the global five-dimensional spacetime. To determine the equations on the brane one can follow the method suggested in Ref. [2], but this will reveal a difficult task due to the necessity of taking into account the evolution of the Weyl term to close the system of equations. A different approach, as discussed by Binètruy *et al.* [13], is to solve the 5D Einstein equations strictly in the bulk ($y \neq 0$) and then to take the brane into account by using the Israel junction conditions. In this work we shall keep in mind this latter approach but, having added the ${}^{(4)}R$ term, we shall make a different use of the junction conditions. More in detail, we start projecting on the brane the solution obtained in the bulk without considering the distributional part at $y = 0$. This will be done using the geometrical identity

$${}^{(4)}R_{BCD}^A = {}^{(5)}R_{NKL}^M h_M^A h_B^N h_C^K h_D^L + \varepsilon (K_C^A K_{BD} - K_D^A K_{BC}) \quad (11)$$

and taking suitable contractions from the above relation. So it is possible to construct the four and five-dimensional Einstein tensors and to get finally the parallel to the brane equations

$$\begin{aligned} {}^{(4)}G_\nu^\mu &= -\frac{1}{2} \Lambda_5 \delta_\nu^\mu + \frac{2}{3} \kappa_5^2 \left({}^{(5)}\overline{T}_\nu^\mu + \left({}^{(5)}\overline{T}_y^y - \frac{1}{4} {}^{(5)}\overline{T} \right) \delta_\nu^\mu \right) \\ &+ \varepsilon \left(\overline{K} \overline{K}_\nu^\mu - \overline{K}_\lambda^\mu \overline{K}^\lambda \right) + \frac{\varepsilon}{2} \left(\overline{K}_\lambda^\kappa \overline{K}_\kappa^\lambda - \overline{K}^2 \right) \delta_\nu^\mu - g^{\kappa\mu} {}^{(5)}\overline{C}_{\kappa y \nu}^y. \end{aligned} \quad (12)$$

Here ${}^{(5)}C_{\kappa y \nu}^y$ is the “electric” part of the bulk Weyl tensor, while \overline{T} and \overline{C} are the limiting values of those quantities at $y = 0^+$ or 0^- . Once we have solved the Einstein equations strictly in the bulk we can make explicit the

various terms appearing in the right-hand side of (12). Now we impose to the above solution boundary conditions to take into account the physical presence of the brane. This can be done if we consider equations (10) as the boundary conditions imposed for the embedding of the brane in the bulk so we have to equate the two independent equations (10) and (12). In this way, however, we would obtain the “bare” quantities Λ_4 and ${}^{(4)}T_\nu^\mu$ but not the effective quantities as seen by an observer confined to the brane. For a brane observer Eqs. (10) are instead written as the usual Einstein equations

$${}^{(4)}G_\nu^\mu = -\Lambda_{4,\text{eff}} \delta_\nu^\mu + \kappa_4^2 \left({}^{(4)}T_\nu^\mu \right)_{\text{eff}} . \tag{13}$$

So the cosmological constant $\Lambda_{4,\text{eff}}$ and the effective energy-momentum tensors $\left({}^{(4)}T_\nu^\mu \right)_{\text{eff}}$ can be obtained equating the right-hand sides of Eqs. (12) and (13).

3. Dynamics in the brane-bulk system

We consider the 5D metric in the form commonly used in cosmological applications

$$dS^2 = a^2(t, y) d\sigma_k^2 - n^2(t, y) dt^2 + \varepsilon \Phi^2(t, y) dy^2 , \tag{14}$$

where

$$d\sigma_k^2 = \frac{dr^2}{1 - kr^2} + r^2(d\vartheta^2 + \sin^2 \vartheta d\varphi^2) \tag{15}$$

and $k = +1, 0, -1$ is the curvature index. Having specified the form of the metric, we now turn to the 5D Einstein equations (3) considered strictly in the bulk, that is, without the energy-momentum tensor at $y = 0$. These equations can be solved once the structure and the content of the bulk come as a result of a physically acceptable theory in higher dimensions. Exact time-dependent solutions which generalize the static solutions were constructed using diffeomorphism invariance [14,15]. Kehagias and Tamvakis [16] transformed the static Randall–Sundrum (A)dS₅ solution into a dynamical one by considering boosts along the fifth dimension and found a time-dependent 5D solution for a bulk with vacuum energy but otherwise empty and with vanishing Weyl tensor. In the present work we want instead to consider (see (12)) a bulk where the non-localized energy-momentum tensor and the “electric” part of the Weyl tensor are different from zero, so we should start by a well defined bulk matter content described by the tensor ${}^{(5)}T_A^B$ and then solve the field equations. However our aim is to overcome the problem of the brane field equations being non-closed so, to give an illustrative example of our method, we shall proceed in a bit unorthodox way. In order to have a simple and non-trivial dynamical 5D solution we start from

a static Randall–Sundrum (A)dS₅ bulk and we construct, generalizing the transformations in [16] a dynamical 5D line element with non-vanishing 5D Weyl tensor. Subsequently we obtain the correspondent energy-momentum content using the Einstein equations. This manner of proceeding may be justified by the fact that this work is mainly focused on the brane phenomenology of the model. Here we anticipate the main results of the procedure and defer to the Appendix for detailed calculations. Our dynamical line element will be obtained transforming the static Randall–Sundrum (A)dS₅ metric, where the three-space is not necessarily flat but has a curvature index $k = +1, 0, -1$, into a dynamical one by considering boosts along the fifth dimension. Then we take into account the Einstein equations in the bulk, away from the brane at $y = 0$, and require that there is no energy flow from the brane towards the bulk and *vice versa*, which implies ${}^{(5)}G_t^y = 0$. The above constraint is easily satisfied if one chooses wave-like expression for the metric coefficients so, assuming the Z_2 symmetry $y \rightarrow -y$, it follows that $a(t, y)$, $n(t, y)$ and $\Phi(t, y)$ in the line element (14) will be function of $w = t - \lambda|y|$ with λ a dimensionless constant. Metric coefficients in the form of plane waves propagating in the fifth dimension have previously been used in the literature in somewhat different contexts [17,18,19,20]. Finally, the bulk line element away from the brane was found to depend only by the scale factor $a(t, y) = a(t - \lambda|y|)$ in the form

$$dS^2 = a^2 d\sigma_k^2 - \frac{1}{2\gamma^2\lambda^2} \left(\kappa^2 a^2 + \sqrt{\kappa^4 a^4 + 4\varepsilon\gamma^2\lambda^2 (\overset{*}{a})^2} \right) dt^2 + \frac{1}{2\gamma^2} \left(-\kappa^2 a^2 + \sqrt{\kappa^4 a^4 + 4\varepsilon\gamma^2\lambda^2 (\overset{*}{a})^2} \right) dy^2, \quad (16)$$

where κ is the constant scale factor for the extra dimension, the superscribed asterisk $*$ denotes derivative with respect to w and γ is a dimensionless constant which comes from the constraint ${}^{(5)}G_t^y = 0$, namely $\overset{*}{a} = \gamma n \Phi$. It should be noted that $a(t - \lambda|y|) = 0$ corresponds to a scale factor singularity for the 5D model which is similar to those that occur in the 4D Friedmann–Robertson–Walker models. From the following curvature invariants

$${}^{(5)}R = -20\epsilon\kappa^2 + \frac{6k}{a^2(t - \lambda|y|)},$$

$${}^{(5)}R_{AB} {}^{(5)}R^{AB} = 80\epsilon\kappa^4 - \frac{48\epsilon k \kappa^2}{a^2(t - \lambda|y|)} + \frac{12k^2}{a^4(t - \lambda|y|)}, \quad (17)$$

$${}^{(5)}R_{ABCD} {}^{(5)}R^{ABCD} = 40\epsilon\kappa^4 - \frac{24\epsilon k \kappa^2}{a^2(t - \lambda|y|)} + \frac{12k^2}{a^4(t - \lambda|y|)} \quad (18)$$

we see that there is no other singularity except the one which may unavoidably occur if the scale factor of the fifth dimension vanishes in some (t, y) hyperplane.

Now we calculate the 5D Einstein tensor and from the field equations considered strictly in the bulk we obtain the cosmological constant Λ_5 and the energy-momentum tensor ${}^{(5)}T_A^B$ as

$$\Lambda_5 = -6\epsilon\kappa^2, \quad {}^{(5)}T_A^B = \text{diag}(p_B, p_B, p_B, -\rho_B, p_\perp), \quad (19)$$

where

$$\kappa_5^2 p_B = -\frac{k}{a^2(t - \lambda|y|)}, \quad \kappa_5^2 \rho_B = -\kappa_5^2 p_\perp = \frac{3k}{a^2(t - \lambda|y|)} \quad (20)$$

the subscript B referring to the bulk. A comment is needed about the cosmological fluid described by the energy-momentum tensor which arises from the curvature index k in Eq. (15). It obeys an equation of state $p_B = (\gamma - 1)\rho_B$ with barotropic index $\gamma = 2/3$, its pressure and energy density are proportional to k and scale as a^{-2} . All these features lead to the interesting possibility that the energy-tensor (20) can describe a fluid composed of cosmic strings, as discussed by a number of authors [21]. We can use the flexibility of the metric (16) to choose many different 5D scale factors, but clearly each choice must meet the necessary requirements to give models acceptable on physical grounds.

Let us now deal with the brane dynamics. We consider homogeneous and isotropic geometries in the brane so the effective tensor ${}^{(4)}T_\nu^\mu|_{\text{eff}}$ will describe a cosmological fluid endowed with pressure p_{eff} and energy density ρ_{eff} . Equating Eqs. (12) and (13) we obtain

$$-\Lambda_{4,\text{eff}} + \kappa_4^2 p_{\text{eff}} = -\frac{\Lambda_5}{2} - \frac{\epsilon}{\Phi_0^2} \left(\frac{1}{a} \frac{\partial a}{\partial y} \right)_0 \left(\frac{1}{a} \frac{\partial a}{\partial y} + \frac{2}{n} \frac{\partial n}{\partial y} \right)_0 - \frac{k}{a_0^2}, \quad (21)$$

$$-\Lambda_{4,\text{eff}} - \kappa_4^2 \rho_{\text{eff}} = -\frac{\Lambda_5}{2} - \frac{3\epsilon}{\Phi_0^2} \left(\frac{1}{a} \frac{\partial a}{\partial y} \right)_0^2 - \frac{3k}{a_0^2}. \quad (22)$$

The effective cosmological constant $\Lambda_{4,\text{eff}}$ may be different from $\Lambda_5/2 = -3\epsilon\kappa^2$, its value being modified by possible additive constant terms contained in (21) and (22). The Einstein tensor ${}^{(4)}G_\nu^\mu$ which appears in the left-hand sides of (12) and (13) is constructed from the brane metrics

$$ds^2 = \tilde{a}^2(t) d\sigma_k^2 - \tilde{n}^2(t) dt^2. \quad (23)$$

Now, in higher-dimensional theories there is the question of which metric frame is the correct representation of our four-dimensional spacetime. In

many braneworld theories, the physical metric in 4D is identified with the induced one, while in other approaches the physical metric either is assumed to be conformally related to the induced one or is determined by the condition of classical confinement in the absence of non-gravitational forces [22]. For the sake of simplicity, here we choose to identify the metric (23) with the induced one, so we have $\tilde{a}(t) = a(t, 0) \equiv a_0(t)$ and $\tilde{n}(t) = n(t, 0) \equiv n_0(t)$. It follows that Eqs. (12) and (13) are identically satisfied by the Einstein tensor ${}^{(4)}G_\nu^\mu$ constructed from (23). From the knowledge $a_0(t)$ and $n_0(t)$ one can also obtain other cosmological quantities such as the Hubble parameter $H = \dot{a}_0/(n_0 a_0)$ or the deceleration parameter $q = -(a_0 \ddot{a}_0)/\dot{a}_0^2 + (a_0 \dot{n}_0)/(\dot{a}_0 n_0)$. Difficulties may instead arise from the exact evaluation of the 4D proper time τ when dealing with the integral $\tau = \int n_0 dt$ and a generic value of $n_0(t)$. Finally, if we define

$$\kappa_4^2 p_\phi = \kappa_4^2 p_{\text{eff}} + \frac{k}{a_0^2} \quad \text{and} \quad \kappa_4^2 \rho_\phi = \kappa_4^2 \rho_{\text{eff}} - \frac{3k}{a_0^2} \quad (24)$$

we can model the fluid in terms of a scalar field ϕ , minimally coupled to Einstein gravity and self-interacting through a potential $V(\phi)$, with pressure and energy density given by

$$p_\phi = \frac{\pm \dot{\phi}_0^2}{(2n_0^2)} - V, \quad (25)$$

$$\rho_\phi = \frac{\pm \dot{\phi}_0^2}{(2n_0^2)} + V, \quad (26)$$

where the upper (lower) sign corresponds to a standard (phantom) scalar field.

4. Some possible brane scenarios

In this section we describe two of the possible brane scenarios consistent with our bulk solution. We shall choose simple values for the scale factor $a(t, y)$ and then determine $\Lambda_{4, \text{eff}}$ and $({}^{(4)}T_\nu^\mu)_{\text{eff}}$ together with the parameters q and H . In the following we shall consider a space-like fifth dimension, so $\epsilon = 1$ and give relevant 4D quantities as a function of the 4D proper time τ .

(A) Let us first consider the case $a(t, y) = (\gamma\lambda/\kappa) \sin \kappa(t - \lambda|y|)$.

This choice gives $n_0(t) = 1$ so the coordinate time t now coincides with the 4D proper time τ and therefore

$$a_0(\tau) = (\gamma\lambda/\kappa) \sin \kappa \tau, \quad q(\tau) = \tan^2 \kappa \tau, \quad H(\tau) = \kappa \cot \kappa \tau. \quad (27)$$

The evolution of the universe begins with a big bang at $\tau = 0$, reaches a maximum $(a_0)_{\max} = (\gamma \lambda)/\kappa$ and terminates with a big rip at $\kappa \tau = \pi$. We do not give numerical values for γ and λ while κ depends on the scale factor of the fifth dimension. The cosmological constant $\Lambda_{4,\text{eff}}$ and the tensor $^{(4)}T_{\nu}^{\mu}$ are given by

$$\Lambda_{4,\text{eff}} = -3\kappa^2, \quad p_{\text{eff}} = -\frac{(\gamma^2 \lambda^2 + k)}{\kappa_4^2 a_0^2(\tau)}, \quad \rho_{\text{eff}} = \frac{3(\gamma^2 \lambda^2 + k)}{\kappa_4^2 a_0^2(\tau)}. \quad (28)$$

As to the standard scalar field, we have:

$$\phi = \frac{\sqrt{2}}{\kappa_4} \ln \tan \frac{\kappa \tau}{2}, \quad V = \frac{2\kappa^2}{\kappa_4^2 \sin^2 \kappa \tau}, \quad V(\phi) = \frac{2\kappa^2}{\kappa_4^2} \cosh^2 \frac{\kappa_4 \phi}{\sqrt{2}}. \quad (29)$$

The same results, starting from different points of view, were obtained in [23].

(B) Now let us consider the case $a(t, y) = (\gamma \lambda / \kappa) (1 - \kappa(t - \lambda|y|))^{-1}$.

This choice gives $n_0(t) = \sqrt{2/(\sqrt{5} - 1)} (1 - \kappa t)^{-1}$ so the relation between the coordinate time t and the 4D proper time τ is

$$(1 - \kappa t) = \exp \left[-\sqrt{(\sqrt{5} - 1)/2} \kappa \tau \right].$$

Therefore

$$a_0(\tau) = \frac{\gamma \lambda}{\kappa} \exp \left(\sqrt{\frac{\sqrt{5} - 1}{2}} \kappa \tau \right), \quad q = -1, \quad H = \sqrt{\frac{\sqrt{5} - 1}{2}} \kappa. \quad (30)$$

The cosmological constant $\Lambda_{4,\text{eff}}$ and the tensor $^{(4)}T_{\nu}^{\mu}$ are given by

$$\Lambda_{4,\text{eff}} = \frac{3(\sqrt{5} - 1)}{2} \kappa^2, \quad p_{\text{eff}} = -\frac{k}{\kappa_4^2 a_0^2(\tau)}, \quad \rho_{\text{eff}} = \frac{3k}{\kappa_4^2 a_0^2(\tau)}. \quad (31)$$

The values of $a(t, y)$ chosen in the previous illustrative examples reproduce results already known in the literature. Less simple choices for the 5D scale factor may describe new brane scenarios but also may require a more involved treatment. In conclusion, this paper investigates the influence of the $^{(4)}R$ term included in the bulk action on the spherically symmetric braneworld solutions. The brane dynamics is made closed by using the modified junction conditions as the boundary conditions for the embedding of the brane in the bulk, so it is possible to obtain brane cosmological solutions consistent with the bulk geometry. We started from a particularly simple time-dependent solution in the bulk away from the brane, but other physically acceptable solutions in the bulk can be considered provided that the related brane dynamics is in accordance to the observations on the brane.

Appendix A

Transforming static bulk solutions into dynamical ones

The Randall–Sundrum (A)dS₅ model is the simple braneworld with curved extra dimension that allows for a meaningful approach to cosmology, therefore we start from this model but, at this point, we do not yet require Z₂ symmetry on reflection around the value $Y = 0$ so we write

$$dS^2 = e^{-2\kappa Y} (A^2 d\sigma_k^2 - dT^2) + \varepsilon dY^2. \quad (\text{A.1})$$

Here κ and A are, respectively, the constant scale factors for the extra dimension Y and for the ordinary three-space and $d\sigma_k^2$ is the line element of maximally symmetric three-spaces with curvature index $k = +1, 0, -1$:

$$d\sigma_k^2 = \frac{dr^2}{1 - kr^2} + r^2(d\vartheta^2 + \sin^2 \vartheta d\varphi^2). \quad (\text{A.2})$$

Since our purpose is to describe the time evolution on the braneworld, we need to transform the static bulk solution (A.1) into a dynamical one. This goal was already achieved in literature where dynamical solutions are derived from the static Randall–Sundrum (A)dS₅ metric by considering boosts along the fifth dimension [16]. Applied to the actual case, we generalize those transformations as

$$\begin{cases} T = \frac{1 - F(t, y)}{\chi} - \varepsilon \frac{\chi}{\kappa^2} G(t, y), \\ e^{\kappa Y} = \frac{F(t, y) + G(t, y) - 1}{\sqrt{1 - \frac{\chi^2}{\kappa^2}}}, \end{cases} \quad (\text{A.3})$$

where $F(t, y)$ and $G(t, y)$ are dimensionless functions and χ is a constant, with the dimensions of κ , related to the boost along the fifth dimension. The coordinate y is chosen so that the hypersurface $y = 0$ coincides with the brane.

As a result the metric (A.1) becomes:

$$dS^2 = \frac{1}{(F + G - 1)^2} \left\{ \left(\frac{\kappa^2 - \chi^2}{\kappa^2} \right) A^2 d\sigma_k^2 + \left(\frac{\kappa^2 - \varepsilon \chi^2}{\kappa^4 \chi^2} \right) [-\kappa^2 (dF)^2 + \varepsilon \chi^2 (dG)^2] \right\}. \quad (\text{A.4})$$

Note that the static line element (A.1) can be recovered from the above equations on condition that as $\chi \rightarrow 0$ it results $F \approx 1 - \chi t - \varepsilon(\chi^2/\kappa^2)e^{\kappa y}$ and $G \approx \chi t + e^{\kappa y}$.

The line element (A.4) is in the form commonly used in cosmological applications

$$dS^2 = a^2(t, y) d\sigma_k^2 - n^2(t, y) dt^2 + \varepsilon \Phi^2(t, y) dy^2. \tag{A.5}$$

Now we can choose suitable functions F and G to obtain explicit expressions for the metric coefficients a , n and Φ . Comparing Eqs. (A.4) and (A.5) we get

$$\frac{A^2(\kappa^2 - \chi^2)}{\kappa^2(F + G - 1)^2} = a^2, \tag{A.6a}$$

$$\frac{a^2(\kappa^2 - \varepsilon\chi^2)}{A^2\kappa^2\chi^2(\kappa^2 - \chi^2)} \left[\kappa^2 \left(\frac{\partial F}{\partial t} \right)^2 - \varepsilon\chi^2 \left(\frac{\partial G}{\partial t} \right)^2 \right] = n^2, \tag{A.6b}$$

$$-\frac{a^2(\kappa^2 - \varepsilon\chi^2)}{A^2\kappa^2\chi^2(\kappa^2 - \chi^2)} \left[\kappa^2 \left(\frac{\partial F}{\partial y} \right)^2 - \varepsilon\chi^2 \left(\frac{\partial G}{\partial y} \right)^2 \right] = \varepsilon\Phi^2, \tag{A.6c}$$

$$\kappa^2 \left(\frac{\partial F}{\partial t} \right) \left(\frac{\partial F}{\partial y} \right) - \varepsilon\chi^2 \left(\frac{\partial G}{\partial t} \right) \left(\frac{\partial G}{\partial y} \right) = 0. \tag{A.6d}$$

Once the new metric coefficients are known it is possible from the Einstein equations (3) in the bulk, that is, away from the brane at $y = 0$, to obtain the energy-momentum tensor ⁽⁵⁾T_A^B. This can be achieved by recalling that in the coordinate system (A.5) the non-vanishing components of the Einstein tensor G_A^B are

$$G_r^r = G_\theta^\theta = G_\varphi^\varphi = -\frac{1}{n^2} \left[\frac{\ddot{\Phi}}{\Phi} + \frac{2\ddot{a}}{a} + \frac{\dot{\Phi}}{\Phi} \left(\frac{2\dot{a}}{a} - \frac{\dot{n}}{n} \right) + \frac{\dot{a}}{a} \left(\frac{\dot{a}}{a} - \frac{2\dot{n}}{n} \right) \right] + \frac{\varepsilon}{\Phi^2} \left[\frac{2a''}{a} + \frac{n''}{n} + \frac{a'}{a} \left(\frac{a'}{a} + \frac{2n'}{n} \right) - \frac{\Phi'}{\Phi} \left(\frac{2a'}{a} + \frac{n'}{n} \right) \right] - \frac{k}{a^2}, \tag{A.7a}$$

$$G_t^t = -\frac{3}{n^2} \left(\frac{\dot{a}^2}{a^2} + \frac{\dot{a}\dot{\Phi}}{a\Phi} \right) + \frac{3\varepsilon}{\Phi^2} \left(\frac{a''}{a} + \frac{a'^2}{a^2} - \frac{a'\Phi'}{a\Phi} \right) - \frac{3k}{a^2}, \tag{A.7b}$$

$$G_y^y = -\frac{3}{n^2} \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} - \frac{\dot{a}\dot{n}}{an} \right) + \frac{3\varepsilon}{\Phi^2} \left(\frac{a'^2}{a^2} + \frac{a'n'}{an} \right) - \frac{3k}{a^2}, \tag{A.7c}$$

$$G_t^y = -\frac{3\varepsilon}{\Phi^2} \left(\frac{\dot{a}'}{a} - \frac{\dot{a}n'}{an} - \frac{a'\dot{\Phi}}{a\Phi} \right). \tag{A.7d}$$

Here a dot and a prime denote partial derivatives with respect to t and y , respectively. In this work we require that there is no energy flow from the brane towards the bulk and *vice versa* so it must be ${}^{(5)}T_t^y = 0$, therefore the choice of the functions F and G must give accordingly ${}^{(5)}G_t^y = 0$. However, as Eq. (A.7d) shows, there is no energy flow only for suitable values of the metric coefficients. A particularly simple choice which makes ${}^{(5)}G_t^y = 0$ is to assume that the metric coefficients in the bulk have the form of plane waves propagating in the fifth dimension, so they become functions either of the argument $u = t - \lambda y$ or of the argument $v = t + \lambda y$. Of course, the particular metric which we finally obtain is dependent on this choice. In detail, from ${}^{(5)}G_t^y = 0$ we can derive

$$\frac{1}{n(u)\Phi(u)} \frac{da(u)}{du} = \frac{1}{n(v)\Phi(v)} \frac{da(v)}{dv} = \gamma, \quad (\text{A.8})$$

where γ is a dimensionless constant. Now we shall assume the Z_2 symmetry $y \rightarrow -y$ and construct a solution of Eqs. (A.6) by matching a solution depending only on u (for $y > 0$) to a solution depending only on v (for $y < 0$).

The result in (A.8) suggests to multiply (A.6b) by (A.6c) so, taking into account (A.6d), we have

$$\left[\frac{a^2(\kappa^2 - \varepsilon\chi^2)}{A^2(\kappa^2 - \chi^2)} \right]^2 \left[\left(\frac{\partial F}{\partial t} \right) \left(\frac{\partial G}{\partial y} \right) - \left(\frac{\partial F}{\partial y} \right) \left(\frac{\partial G}{\partial t} \right) \right]^2 = n^2 \Phi^2. \quad (\text{A.9})$$

Eliminating G by (A.6a) and taking the square root one finally obtains

$$\frac{\kappa^2 - \varepsilon\chi^2}{A\kappa^2\chi\sqrt{\kappa^2 - \chi^2}} \left| - \left(\frac{\partial F}{\partial t} \right) \left(\frac{\partial a}{\partial y} \right) + \left(\frac{\partial F}{\partial y} \right) \left(\frac{\partial a}{\partial t} \right) \right| = n\Phi. \quad (\text{A.10})$$

Let us first begin working on the $y > 0$ side. We put $a(t, y) = a(t - \lambda y)$ into (A.10) and recalling (A.8) we obtain the following partial differential equation for F

$$\lambda \frac{\partial F}{\partial t} + \frac{\partial F}{\partial y} = \frac{A\kappa^2\chi\sqrt{\kappa^2 - \chi^2}}{\gamma(\kappa^2 - \varepsilon\chi^2)}. \quad (\text{A.11})$$

The general solution is

$$F(t, y) = \frac{A\kappa^2\chi\sqrt{\kappa^2 - \chi^2}}{2\gamma\lambda(\kappa^2 - \varepsilon\chi^2)}(t + \lambda y) + f_{(-)}(t - \lambda y). \quad (\text{A.12})$$

The function $f_{(-)}(u)$ can be determined by (A.6d) after eliminating G by (A.6a). The result is

$$\frac{df_{(-)}}{du} = \frac{A\chi\sqrt{\kappa^2 - \chi^2}}{a^2(\kappa^2 - \varepsilon\chi^2)} \left\{ \varepsilon \frac{\chi}{\kappa} \left(\frac{da}{du} \right) + \frac{1}{\gamma\lambda} \sqrt{\kappa^4 a^4 + 4\varepsilon\gamma^2\lambda^2 \left(\frac{da}{du} \right)^2} \right\} \tag{A.13}$$

which can be integrated, once the scale factor a has been fixed, reminding that in the limit $\chi \rightarrow 0$ it must be $F(t, y) \rightarrow 1$ and so also $f_{(-)}(u) \rightarrow 1$. Of course, if one is only interested in determining n and Φ from (A.6b) and (A.6c) it is sufficient the simple knowledge of the derivative of $f_{(-)}(u)$. Obviously, n and Φ will be a function of the scale factor a . Proceeding in an analogous manner when working on the $y < 0$ side, we obtain

$$F(t, y) = \frac{A\kappa^2\chi\sqrt{\kappa^2 - \chi^2}}{2\gamma\lambda(\kappa^2 - \varepsilon\chi^2)}(t - \lambda y) + f_{(+)}(t + \lambda y), \tag{A.14}$$

where $f_{(+)}(v)$ and $f_{(-)}(u)$ are the same function f of the two different arguments v and u . As a consequence we can write the function $F(t, y)$ on both sides of the brane at $y = 0$ simply as

$$F(t, y) = \frac{A\kappa^2\chi\sqrt{\kappa^2 - \chi^2}}{2\gamma\lambda(\kappa^2 - \varepsilon\chi^2)}(t + \lambda|y|) + f(t - \lambda|y|). \tag{A.15}$$

The function $G(t, y)$ can then be easily derived from Eq. (A.6a). Finally, we can obtain from Eqs. (A.6b) and (A.6c) the metric coefficients $n(t - \lambda|y|)$ and $\Phi(t - \lambda|y|)$ which are given as a function of $a(t - \lambda|y|)$ by

$$n^2 = \frac{1}{2\gamma^2\lambda^2} \left(\kappa^2 a^2 + \sqrt{\kappa^4 a^4 + 4\varepsilon\gamma^2\lambda^2 (a^*)^2} \right), \tag{A.16}$$

$$\Phi^2 = \frac{\varepsilon}{2\gamma^2} \left(-\kappa^2 a^2 + \sqrt{\kappa^4 a^4 + 4\varepsilon\gamma^2\lambda^2 (a^*)^2} \right), \tag{A.17}$$

where the superscribed asterisk ^{*} denote derivative with respect to $w = (t - \lambda|y|)$. The bulk line element away from the brane is therefore

$$dS^2 = a^2 d\sigma_k^2 - \frac{1}{2\gamma^2\lambda^2} \left(\kappa^2 a^2 + \sqrt{\kappa^4 a^4 + 4\varepsilon\gamma^2\lambda^2 (a^*)^2} \right) dt^2 + \frac{1}{2\gamma^2} \left(-\kappa^2 a^2 + \sqrt{\kappa^4 a^4 + 4\varepsilon\gamma^2\lambda^2 (a^*)^2} \right) dy^2 \tag{A.18}$$

as given previously in Eq. (16). From Eqs. (A.7) we get

$$G_r^r = G_\vartheta^\vartheta = G_\varphi^\varphi = 6\varepsilon\kappa^2 - \frac{k}{a^2(t - \lambda|y|)}, \quad (\text{A.19a})$$

$$G_t^t = G_y^y = 6\varepsilon\kappa^2 - \frac{3k}{a^2(t - \lambda|y|)}, \quad (\text{A.19b})$$

in accordance with the 5D cosmological constant and the 5D energy-momentum tensor given previously in Eqs. (19) and (20).

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