Vol. 39 (2008) *ACTA PHYSICA POLONICA B* No 8

ASYMPTOTIC BEHAVIOR OF ANOMALOUS DIFFUSIONS DRIVEN BY α-STABLE NOISE

Zbigniew Michna

Department of Mathematics and Cybernetics
Wrocław University of Economics
Komandorska 118/120, 53-345 Wrocław, Poland

(Received May 7, 2008)

In this paper we discuss decomposition principle for α-stable Lévy processes. We investigate asymptotic properties of components and stochastic integrals driven by such processes providing an important class of anomalous diffusions. We consider two case studies with integrands being fractional Brownian motion and gamma process.

PACS numbers: 05.40.–a, 02.50.Ey, 05.20.–y, 05.45.–a

1. Introduction

Continuous time random walk (CTRW) was introduced in pioneering works by Sher, Montroll and Weiss [1, 2]. Beginning with stochastic formulation in [1] of transport phenomena in terms of CTRW, the physical community showed a steady interest in the anomalous diffusion. In recent years, systems exhibiting anomalous diffusion behavior attracted growing attention in the various fields of physics and related sciences. After the first step made by Einstein and Smoluchowski who explained why the range reached by a Brownian particle is proportional to the square root of the movement duration, there were constructed many physical examples of anomalous diffusion where this kind of behavior is often violated and being replaced by an anomalous power-law scaling, see Metzler and Klafter [3] and references therein. On a mathematical side, Lévy stable motion and fractional Lévy stable motion, see Mercik *et al.* [4], provide the most prominent examples of such anomalous diffusion processes. For example, the stochastic resonance was investigated by Dybiec and Gudowska-Nowak [5], where the generic double-well potential models perturbed by an α-stable noise were applied via the overdamped Langevin equation. The fractional Fokker–Planck equation for the probability distribution of particles whose motion is governed by a subordinated

Langevin equation, which is driven by an α-stable noise rather than a Gaussian was studied by Magdziarz and Weron [6] and Weron, Magdziarz and Weron [7]. A new type of subordinator by modifying the well known strictly increasing α-stable process was introduced by Magdziarz and Weron, [8], where it was demonstrated how the empirical time-domain stretched exponential function can be obtained from the anomalous diffusion model.

The list of systems displaying subdiffusive dynamics is diverse and very extensive. It encompasses charge career transport in amorphous semiconductors, diffusion in percolative and porous materials, stretched exponential relaxation phenomena and protein conformational dynamics. Since the CTRW has found its widespread applications in many scientific fields such as statistical physics, finance and insurance, there are many models which replace CTRW by a certain diffusion. In this paper we investigate processes which can serve as an approximation of CTRW. We consider anomalous diffusions (stable Lévy processes) and stochastic integrals driven by stable noise. We investigate some distributional properties of a stochastic integral driven by an anomalous diffusion. Especially, we study the tail probability of supremum over finite horizon which is related to the distribution of the first passage time of a given process. Similar problems for the generalized Wiener process driven by Lévy stable noises are investigated in Dybiec, Gudowska-Nowak and Hänggi [9,10]. Stable distributions and processes serve as models in statistical physics see *e.g.* Dybiec and Gudowska-Nowak [5,11] or finance modeling and risk theory see *e.g.* Weron [12], Burnecki and Weron [13], Michna [14] and Magdziarz, Miśta and Weron [15] and references therein.

Stable Lévy process plays an important role among stable processes like the Brownian motion among Gaussian processes (see *e.g.* Janicki and Weron [16]). Thus we start with the definition of an α-stable Lévy process. We will consider stochastic processes on the time interval $[0,1]$. A stochastic process $\{Z(t), 0 \le t \le 1\}$ is called an α-stable Lévy process ($0 < \alpha \le 2$) if

1. $Z(0) = 0$ a.s.

2. Z has independent increments.

3. $Z(t) - Z(s)$ has distribution $S_\alpha(\sigma(t-s)^{1/\alpha}, \beta, 0)$ for any $0 \le s < t \le 1$ that is α-stable distribution with scale parameter $\sigma(t-s)^{1/\alpha}$, skewness β and shift parameters equal zero.

For $\alpha = 2$ we get Brownian motion. If the two first assumptions are satisfied and the process has stationary increments we call such a process a Lévy process. In this paper we will consider $0 < \alpha < 2$. By Lévy–Itô integral representation an α-stable Lévy process can be written

$$Z(t) = \int\limits_{|y|<1} y\,(N_t(dy) - t\nu(dy)) + \int\limits_{|y|\geq 1} y\,N_t(dy) + at\,, \tag{1}$$

where N is the point process of jumps of Z: $N = \sum_{s:\Delta Z(s)\neq 0} \delta_{(s,\Delta Z(s))}$ (see *e.g.* Kallenberg [17]) and $N_t(A) = N([0,t]\times A)$ for $A \in \mathbb{R}$ such that $0 \notin \bar{A}$. N is a Poisson point process with the mean measure $ds \times \nu(dy)$ where $\nu(dy)$ is a Lévy measure on $\mathbb{R}\setminus 0$

$$\nu(dy) = \frac{P}{y^{1+\alpha}} \boldsymbol{I}_{(0,\infty)}(y)\,dy + \frac{Q}{|y|^{1+\alpha}} \boldsymbol{I}_{(-\infty,0)}(y)\,dy\,. \tag{2}$$

and

$$P = \frac{1+\beta}{2}\,\alpha\,C_\alpha \sigma^\alpha, \quad Q = \frac{1-\beta}{2}\,\alpha\,C_\alpha \sigma^\alpha\,, \tag{3}$$

where

$$C_\alpha = \left(\int\limits_0^\infty s^{-\alpha} \sin s\,ds\right)^{-1} = \begin{cases} \frac{1-\alpha}{\Gamma(2-\alpha)\cos(\pi\alpha/2)} & \text{if } \alpha \neq 1\,, \\ 2/\pi & \text{if } \alpha = 1\,. \end{cases} \tag{4}$$

In this paper we investigate an α-stable Lévy motion under condition on the length of the largest jump. Then we get a certain decomposition into a simple process and a Lévy process which has finite some exponential moments and these processes are independent. We show that this Lévy process converges to the α-stable Lévy process uniformly on compact sets with probability one. In other limit we can approximate this process by Brownian motion. We use this decomposition to find an exact asymptotic of the tail distribution of supremum for stochastic integrals with respect to an α-stable Lévy motion. In Samorodnitsky and Taqqu [18] the tail distribution of supremum for an integral of deterministic function with respect to independently scattered α-stable measure is investigated. Hult and Lindskog [19] find an exact asymptotic behavior of the supremum of an stochastic integral driven by regularly varying Lévy processes with predictable integrands. The tail distribution of supremum of Lévy processes is treated in Albin [20], Willekens [21], Braverman [22], Braverman [23], Braverman and Samorodnitsky [24], Michna and Weron [25] and Rosiński and Samorodnitsky [26].

2. Decomposition of α-stable Lévy process

An α-stable Lévy process can be represented as the following series (see LePage [27] or Samorodnitsky and Taqqu [18]) for $0 < \alpha < 1$

$$Z(t) = \sigma C_\alpha^{1/\alpha} \sum_{k=1}^{\infty} \Gamma_k^{-1/\alpha} \gamma_k \boldsymbol{I}\{U_k \le t\} \tag{5}$$

for $\alpha = 1$

$$Z(t) = \sigma C_1 \sum_{k=1}^{\infty} \left(\Gamma_k^{-1} \gamma_k \boldsymbol{I}\{U_k \le t\} - \beta t b_k^{(1)} \right) + \beta t \sigma C_1 \ln(\sigma C_1) \tag{6}$$

and for $1 < \alpha < 2$

$$Z(t) = \sigma C_\alpha^{1/\alpha} \sum_{k=1}^{\infty} \left(\Gamma_k^{-1/\alpha} \gamma_k \boldsymbol{I}\{U_k \le t\} - \beta t b_k^{(\alpha)} \right) , \tag{7}$$

where $0 \le t \le 1$ and $\{\Gamma_k\}_{k=1}^{\infty}$ is a sequence of arrival epochs in a Poisson process with unit arrival rate, $\{\gamma_k\}_{k=1}^{\infty}$ is a sequence of iid random variables satisfying

$$\mathbb{P}(\gamma_k = 1) = 1 - \mathbb{P}(\gamma_k = -1) = \frac{1+\beta}{2}$$

and $\{U_k\}_{k=1}^{\infty}$ is a sequence of iid random variables uniformly distributed on $[0, 1]$. These sequences are independent and

$$b_k^{(\alpha)} = \begin{cases} \int\limits_{1/k}^{1/(k-1)} y^{-2} \sin y \, dy & \text{if } \alpha = 1 \, , \\ \frac{\alpha}{\alpha-1} \left(k^{(\alpha-1)/\alpha} - (k-1)^{(\alpha-1)/\alpha} \right) & \text{if } 1 < \alpha < 2 \, , \end{cases} \tag{8}$$

where $k \in \mathbb{N}$.

Consider an α-stable Lévy process under condition $\Gamma_1 = x$, where $x > 0$ (Γ_1 has exponential distribution with parameter equal 1). Then we get the following theorem (for a similar treatment of symmetric α-stable processes see, Michna [28]).

Theorem 1 (decomposition principle) *Under condition $\Gamma_1 = x$ an α-stable Lévy process has the following form*

$$Z_x(t) = A_x(t) + Y_x(t) \, , \tag{9}$$

where

$$A_x(t) = \sigma C_\alpha^{1/\alpha} x^{-1/\alpha} \gamma_1 \boldsymbol{I}\{U_1 \le t\}$$

and

$$Y_x(t) \stackrel{d}{=}$$

$$\begin{cases} \sigma C_\alpha^{1/\alpha} \sum_{k=1}^{\infty} (\Gamma_k + x)^{-1/\alpha} \gamma_k \boldsymbol{I}\{U_k \le t\} & \text{if } \alpha < 1, \\ \sigma C_1 \sum_{k=1}^{\infty} \left[(\Gamma_k + x)^{-1} \gamma_k \boldsymbol{I}\{U_k \le t\} - \beta t b_k^{(1)} \right] + \beta t \sigma C_1 \ln(\sigma C_1) & \text{if } \alpha = 1, \\ \sigma C_\alpha^{1/\alpha} \sum_{k=1}^{\infty} \left[(\Gamma_k + x)^{-1/\alpha} \gamma_k \boldsymbol{I}\{U_k \le t\} - \beta t b_k^{(\alpha)} \right] & \text{if } 1 < \alpha < 2. \end{cases}$$

The process Y_x is well-defined that is the sum converges almost surely and is a Lévy process with finite some exponential moments around zero. Its Lévy measure is the following

$$\nu_x(dy) = \frac{P}{y^{1+\alpha}} \boldsymbol{I}_{(0,\, \sigma C_\alpha^{1/\alpha} x^{-1/\alpha})}(y)\, dy + \frac{Q}{|y|^{1+\alpha}} \boldsymbol{I}_{(-\sigma C_\alpha^{1/\alpha} x^{-1/\alpha},\, 0)}(y)\, dy$$

and its distribution is given by

$$\mathbb{E} e^{isY_x(1)} = \exp(ism_x + \int\limits_{\mathbf{R}\setminus 0} \left(e^{isy} - 1 - isy \boldsymbol{I}\{|y| \le 1\} \right) \nu_x(dy)),$$

where

$$m_x = \begin{cases} -\frac{\beta \sigma C_\alpha^{1/\alpha} \alpha}{\alpha - 1} ((\sigma^\alpha C_\alpha - x)^+ + x)^{(\alpha-1)/\alpha} & \text{if } \alpha \ne 1, \\ -\beta \sigma C_1 \left(\int\limits_1^\infty \frac{\sin y}{y^2}\, dy + \int\limits_0^1 \left(\frac{\sin y}{y^2} - \frac{1}{y} \right) dy + \ln \frac{(\sigma C_1 - x)^+ + x}{\sigma C_1} \right) & \text{if } \alpha = 1. \end{cases}$$

The processes A_x and Y_x are independent.

Proof: In the proof we take advantage of Th. 4.1 and 5.1 of Rosiński [29]. In our cases the function H from Rosiński [29] for the process Y_x has the following form

$$H_x(\Gamma_k, \gamma_k) = \frac{\sigma C_\alpha^{1/\alpha}}{(\Gamma_k + x)^{1/\alpha}}\ \gamma_k\, .$$

Thus

$$\begin{aligned} \sigma_x(r, \cdot) \ &= \ \mathbb{P}(H_x(r, \gamma_k) \in \cdot) = \frac{1+\beta}{2}\, \delta_{(r+x)^{-1/\alpha} \sigma C_\alpha^{1/\alpha}}(\cdot) \\ &\quad + \frac{1-\beta}{2}\, \delta_{-(r+x)^{-1/\alpha} \sigma C_\alpha^{1/\alpha}}(\cdot) \end{aligned}$$

and

$$\begin{aligned} \nu_x(dy) \ &= \ \int\limits_0^\infty \sigma_x(r, dy)\, dr \\ &= \ \frac{P}{y^{1+\alpha}} \boldsymbol{I}_{(0, \sigma C_\alpha^{1/\alpha} x^{-1/\alpha})}(y)\, dy + \frac{Q}{|y|^{1+\alpha}} \boldsymbol{I}_{(-\sigma C_\alpha^{1/\alpha} x^{-1/\alpha}, 0)}(y)\, dy\, . \quad (10) \end{aligned}$$

The function $B_x(s)$ which is responsible for centering terms is the following

$$B_x(s) = \int_0^s \int_{-1}^1 y\sigma_x(r,dy)\,dr \; = \beta\sigma C_\alpha^{1/\alpha} \int_{(\sigma^\alpha C_\alpha - x)^+}^{s\vee(\sigma^\alpha C_\alpha - x)} (r+x)^{-1/\alpha}\,dr\,.$$

- Case $1<\alpha<2$. Since

$$\begin{aligned} B_x(s) \;&=\; \frac{\beta\sigma C_\alpha^{1/\alpha}\alpha}{\alpha-1} \\ &\quad\times\Big[(s\vee(\sigma^\alpha C_\alpha - x)+x)^{(\alpha-1)/\alpha} - ((\sigma^\alpha C_\alpha - x)^+ + x)^{(\alpha-1)/\alpha}\Big] \end{aligned}$$

and $\lim_{s\to\infty} B_x(s)=\infty$ we get that the following series is convergent a.s.

$$Y_x^{(a)}(t) = \sigma C_\alpha^{1/\alpha}\sum_{k=1}^{\infty}\Big[(\Gamma_k+x)^{-1/\alpha}\gamma_k \boldsymbol{I}\{U_k\le t\} - \beta t a_k^{(x)}\Big]\,, \qquad (11)$$

where

$$\begin{aligned} \beta\sigma C_\alpha^{1/\alpha} a_k^{(x)} \;&=\; B_x(k)-B_x(k-1) \\ &=\; \frac{\beta\sigma C_\alpha^{1/\alpha}\alpha}{\alpha-1}\Big[(k+x)^{(\alpha-1)/\alpha} - (k-1+x)^{(\alpha-1)/\alpha}\Big] \end{aligned}$$

for $k\in\mathbb{N}$ and $k\ge 1+\sigma^\alpha C_\alpha - x$. Put

$$Y_{x,n}(t) = \sigma C_\alpha^{1/\alpha}\sum_{k=1}^{n}(\Gamma_k+x)^{-1/\alpha}\gamma_k \boldsymbol{I}\{U_k\le t\} - \frac{\sigma C_\alpha^{1/\alpha}\beta t\alpha}{\alpha-1}\, n^{(\alpha-1)/\alpha} \qquad (12)$$

and

$$Y_{x,n}^{(a)}(t) = \sigma C_\alpha^{1/\alpha}\sum_{k=1}^{n}(\Gamma_k+x)^{-1/\alpha}\gamma_k \boldsymbol{I}\{U_k\le t\} - tB_x(n) \qquad (13)$$

and note that $Y_{x,n}^{(a)}(t)\to Y_x^{(a)}(t)$ as $n\to\infty$ a.s. for all t (the convergence is even uniformly in $t\in[0,1]$, see Th. 5.1 of Rosiński [29]). Since $\lim_{n\to\infty}(n+x)^{(\alpha-1)/\alpha} - n^{(\alpha-1)/\alpha}=0$ we get

$$Y_{x,n}(t) - Y_{x,n}^{(a)}(t) + \frac{\beta\sigma C_\alpha^{1/\alpha} t\alpha}{\alpha-1}((\sigma^\alpha C_\alpha - x)^+ + x)^{(\alpha-1)/\alpha} \to 0$$

as $n \to \infty$ which gives that $Y_{x,n}(t) \to Y_x(t)$ as $n \to \infty$ a.s. for all t. Thus by Th. 4.1 of [29]

$$\mathbb{E}e^{isY_x^{(a)}(1)} = \exp\Big(\int\limits_{\mathbf{R}\setminus 0} (e^{isy} - 1 - isy\boldsymbol{I}\{|y| \le 1\})\, \nu_x(dy)\Big)\,,$$

where measure ν_x is defined in (10) which results in

$$\mathbb{E}e^{isY_x(1)} = \exp\Big(ism_x + \int\limits_{\mathbf{R}\setminus 0} (e^{isy} - 1 - isy\boldsymbol{I}\{|y| \le 1\})\, \nu_x(dy)\Big)\,,$$

where

$$m_x = -\frac{\beta\sigma C_\alpha^{1/\alpha}\alpha}{\alpha - 1}((\sigma^\alpha C_\alpha - x)^+ + x)^{(\alpha-1)/\alpha}$$

- Case $\alpha = 1$. Since

$$B_x(s) = \beta\sigma C_1(\ln|(s \vee (\sigma C_1 - x) + x)| - \ln|(\sigma C_1 - x)^+ + x|)$$

and $\lim_{s\to\infty} B_x(s) = \infty$ we get that the following series is convergent a.s.

$$Y_x^{(a)}(t) = \sigma C_1 \sum_{k=1}^{\infty} \left[(\varGamma_k + x)^{-1}\gamma_k \boldsymbol{I}\{U_k \le t\} - \beta t a_k^{(x)} \right]\,, \tag{14}$$

where

$$\begin{aligned}
\beta\sigma C_1 a_k^{(x)} &= B_x(k) - B_x(k-1)\\
&= \beta\sigma C_1(\ln|(k \vee (\sigma C_1 - x) + x)| - \ln|((k-1) \vee (\sigma C_1 - x) + x)|)
\end{aligned}$$

for $k \in \mathbb{N}$. Put

$$\begin{aligned}
Y_{x,n}(t) &= \sigma C_1 \sum_{k=1}^{n} (\varGamma_k + x)^{-1/\alpha}\gamma_k \boldsymbol{I}\{U_k \le t\}\\
&\quad -t\beta\sigma C_1 \left(\int\limits_{1/n}^{\infty} \frac{\sin y}{y^2}\, dy - \ln(\sigma C_1) \right)
\end{aligned} \tag{15}$$

and

$$Y_{x,n}^{(a)}(t) = \sigma C_1 \sum_{k=1}^{n} (\varGamma_k + x)^{-1/\alpha}\gamma_k \boldsymbol{I}\{U_k \le t\} - tB_x(n) \tag{16}$$

and note that $Y^{(a)}_{x,n}(t) \to Y^{(a)}_x(t)$ as $n \to \infty$ a.s. for all t (the convergence is even uniformly in $t \in [0,1]$, see Th. 5.1 of Rosiński [29]). Since

$$\int_{1/n}^{\infty} \frac{\sin y}{y^2}\, dy - \ln n = \int_{1}^{\infty} \frac{\sin y}{y^2}\, dy + \int_{1/n}^{1} \left(\frac{\sin y}{y^2} - \frac{1}{y} \right) dy$$

and by Taylor expansion

$$\left| \left(\frac{\sin y}{y^2} - \frac{1}{y} \right) \boldsymbol{I}_{(1/n,1)}(y) \right| \le \frac{|y|}{6}\,,$$

where the right-hand side is integrable on $(0,1)$ which by Lebesgue dominated convergence theorem gives that

$$\int_{1/n}^{\infty} \frac{\sin y}{y^2}\, dy - \ln n \to \int_{1}^{\infty} \frac{\sin y}{y^2}\, dy + \int_{0}^{1} \left(\frac{\sin y}{y^2} - \frac{1}{y} \right) dy < \infty\,.$$

Using the fact $\lim_{n\to\infty} \ln(n+x) - \ln n = 0$ we get

$$\begin{aligned} &Y_{x,n}(t) - Y^{(a)}_{x,n}(t) \\ &+ t\beta\sigma C_1 \left(\int_{1}^{\infty} \frac{\sin y}{y^2}\, dy + \int_{0}^{1} \left(\frac{\sin y}{y^2} - \frac{1}{y} \right) dy + \ln \frac{(\sigma C_1 - x)^+ + x}{\sigma C_1} \right) \to 0 \end{aligned}$$

as $n \to \infty$ which gives that $Y_{x,n}(t) \to Y_x(t)$ as $n \to \infty$ a.s. for all t. Thus by Th. 4.1 of [29]

$$\mathbb{E} e^{isY^{(a)}_x(1)} = \exp\left(\int_{\mathbf{R}\setminus 0} \left(e^{isy} - 1 - isy\boldsymbol{I}\{|y| \le 1\}\right) \nu_x(dy) \right),$$

where measure ν_x is defined in (10) which results in

$$\mathbb{E} e^{isY_x(1)} = \exp(ism_x + \int_{\mathbf{R}\setminus 0} \left(e^{isy} - 1 - isy\boldsymbol{I}\{|y| \le 1\}\right) \nu_x(dy))\,,$$

where

$$m_x = -\beta\sigma C_1 \left(\int_{1}^{\infty} \frac{\sin y}{y^2}\, dy + \int_{0}^{1} \left(\frac{\sin y}{y^2} - \frac{1}{y} \right) dy + \ln \frac{(\sigma C_1 - x)^+ + x}{\sigma C_1} \right).$$

- Case $\alpha < 1$. Since

$$B_x(s) = \frac{\beta\sigma C_\alpha^{1/\alpha}\alpha}{\alpha-1}\left[(s \vee (\sigma^\alpha C_\alpha - x) + x)^{(\alpha-1)/\alpha} - ((\sigma^\alpha C_\alpha - x)^+ + x)^{(\alpha-1)/\alpha}\right]$$

and $\lim_{s\to\infty} B_x(s) = -\frac{\beta\sigma C_\alpha^{1/\alpha}\alpha}{\alpha-1}((\sigma^\alpha C_\alpha - x)^+ + x)^{(\alpha-1)/\alpha}$ by Th. 4.1 of Rosiński [29] no centering is needed and

$$\mathbb{E}e^{isY_x(1)} = \exp\left(ism_x + \int\limits_{\mathbf{R}\setminus 0} (e^{isy} - 1 - isy\boldsymbol{I}\{|y| \le 1\})\,\nu_x(dy)\right),$$

where

$$m_x = -\frac{\beta\sigma C_\alpha^{1/\alpha}\alpha}{\alpha-1}\left((\sigma^\alpha C_\alpha - x)^+ + x\right)^{(\alpha-1)/\alpha}.$$

Since the jumps of the process Y_x are bounded some exponential moments around zero of Y_x are finite (see *e.g.* Protter [30] Th. 34).

□

By the existence of some exponential moments around zero Y_x has finite moments of all orders and it is well known that for Lévy processes

$$\mathbb{E}Y_x(1) = m_x + \int\limits_{|y|>1} y\,\nu_x(dy)$$

and

$$\mathbf{Var}Y_x(1) = \int\limits_{\mathbf{R}\setminus 0} y^2\,\nu_x(dy).$$

Thus integrating we get

$$\mathbb{E}Y_x(1) = \begin{cases} -\beta\sigma C_\alpha^{1/\alpha}\frac{\alpha}{\alpha-1}x^{(\alpha-1)/\alpha} & \text{if } \alpha \ne 1, \\ -\beta\sigma C_1\left(\int\limits_1^\infty \frac{\sin y}{y^2}\,dy + \int\limits_0^1 \left(\frac{\sin y}{y^2} - \frac{1}{y}\right)dy + \ln\left(\frac{x}{\sigma C_1}\right)\right) & \text{if } \alpha = 1, \end{cases}$$

$$\mathbf{Var}Y_x(1) = \frac{\sigma^2 C_\alpha^{2/\alpha}\alpha}{2-\alpha}\,x^{(\alpha-2)/\alpha}. \tag{17}$$

Note that $\mathbf{Var}Y_x(1) = \frac{\sigma^2 C_\alpha^{2/\alpha}\alpha}{2-\alpha}\,x^{(\alpha-2)/\alpha} \to 2\sigma^2$ as $\alpha \uparrow 2$ (since $a\Gamma(a) = \Gamma(a+1) \to 1$ as $a \downarrow 0$, see the form of C_α in (4)) which is variance of $S_2(\sigma,\beta,0)$!

Let us notice that $Y_x(t) - \mathbb{E}Y_x(1)t$ is a square integrable martingle thus we have the following decomposition

$$Z_x(t) = M_x(t) + N_x(t)\,,$$

where $M_x(t) = Y_x(t) - \mathbb{E}Y_x(1)t$ is a square integrable martingle and $N_x(t) = A_x(t) + \mathbb{E}Y_x(1)t$ is a finite variation process which gives that Z_x is the so-called decomposable process (see *e.g.* Protter [30]).

3. Asymptotic behavior of Y_x component

In this section we investigate asymptotic behavior of the process Y_x which appears in the decomposition of an α-stable Lévy process under condition of the length of the largest jump. We consider two domains when $x \downarrow 0$ and $x \to \infty$. The process Y_x can serve as an approximation of the α-stable Lévy process Z.

Theorem 2 *The process Y_x converges uniformly on compact sets to the process Z with probability one as $x \downarrow 0$ more precisely*

$$\sup_{0\le t\le 1} |Y_x(t) - Z(t)| \le x\alpha^{-1}\sigma C_\alpha^{1/\alpha} Z_{\frac{\alpha}{\alpha+1}} \tag{18}$$

a.s. where $x \ge 0$ and

$$Z_{\frac{\alpha}{\alpha+1}} = \sum_{k=1}^{\infty} \frac{1}{{\Gamma_k}^{(\alpha+1)/\alpha}} \tag{19}$$

is an $\frac{\alpha}{\alpha+1}$-stable random variable with skewness parameter one and shift parameter zero.

Proof: Let us consider the function $g(x) = a^{-1/\alpha} - (a+x)^{-1/\alpha}$ for a fixed $a > 0$ and $x \ge 0$. Since

$$\frac{d}{dx}\left[a^{-1/\alpha} - (a+x)^{-1/\alpha}\right] = \frac{1}{\alpha(a+x)^{1+1/\alpha}} \le \frac{1}{\alpha a^{1+1/\alpha}}$$

it is easy to notice that

$$\left| a^{-1/\alpha} - (a+x)^{-1/\alpha} \right| \le \frac{1}{\alpha a^{1+1/\alpha}} x\,. \tag{20}$$

Thus we obtain

$$\sup_{0\le t\le 1} \sigma C_\alpha^{1/\alpha} \left| \sum_{k=1}^{\infty} \left[\Gamma_k^{-1/\alpha} - (\Gamma_k + x)^{-1/\alpha} \right] \gamma_k \boldsymbol{I}\{U_k \le t\} \right|$$
$$\le \sigma C_\alpha^{1/\alpha} \sum_{k=1}^{\infty} \left| \Gamma_k^{-1/\alpha} - (\Gamma_k + x)^{-1/\alpha} \right|$$
$$\le \frac{x}{\alpha} \sigma C_\alpha^{1/\alpha} \sum_{k=1}^{\infty} \frac{1}{\Gamma_k{}^{1+1/\alpha}}$$
$$= \frac{x}{\alpha} \sigma C_\alpha^{1/\alpha} \sum_{k=1}^{\infty} \frac{1}{\Gamma_k{}^{(\alpha+1)/\alpha}}$$
$$= \frac{x}{\alpha} \sigma C_\alpha^{1/\alpha} Z_{\frac{\alpha}{\alpha+1}} ,$$

where $Z_{\frac{\alpha}{\alpha+1}}$ is an $\frac{\alpha}{\alpha+1}$-stable random variable with skewness parameter one and shift parameter zero (the series is a.s. convergent, see Samorodnitsky and Taqqu [18]).

□

A different approximation we get as $x \to \infty$.

Theorem 3

$$\sqrt{\frac{2-\alpha}{\alpha\sigma^2 C_\alpha^{2/\alpha}}}\, x^{\frac{1}{\alpha}-\frac{1}{2}} \left(Y_x(t) - \mathbb{E}Y_x(1)t\right) \Rightarrow B(t) \tag{21}$$

as $x \to \infty$ weakly in the Skorokhod space D equipped with the uniform metric where B is a standard Brownian motion.

Proof: For $x > \sigma^\alpha C_\alpha$ processes $Y_x(t) - \mathbb{E}Y_x(1)t$ is a Lévy process with the following characteristic function

$$\mathbb{E}\exp(is(Y_x(t) - \mathbb{E}Y_x(1)t)) = \exp\left(t \int_{\mathbf{R}\setminus 0} (e^{isy} - 1 - isy\boldsymbol{I}\{|y| \le 1\})\nu_x(dy) \right)$$
$$= \exp\left(t \int_{|y| < \sigma C_\alpha^{\frac{1}{\alpha}} x^{-\frac{1}{\alpha}}} (e^{isy} - 1 - isy\boldsymbol{I}\{|y| \le 1\})\nu(dy) \right) .$$

Put $\epsilon = \sigma C_\alpha^{\frac{1}{\alpha}} x^{-\frac{1}{\alpha}}$ then

$$\mathbb{E}(Y_x(t) - \mathbb{E}Y_x(1)t) = \mathbf{Var}Y_x(t) = \sigma^2(\epsilon) = \frac{\sigma^\alpha C_\alpha \alpha}{2-\alpha}\, \epsilon^{2-\alpha} .$$

Since $\lim_{\epsilon\to 0}\frac{\sigma(\epsilon)}{\epsilon}=\infty$ and by Prop. 2.2 and Th. 2.1 of Asmussen and Rosiński [31] we have that $\sigma^{-1}(\epsilon)(Y_x(t)-\mathbb{E}Y_x(1)t)\Rightarrow B(t)$ as $\epsilon\to 0$ weakly in the Skorokhod space D equipped with the uniform metric where B is a standard Brownian motion. This finishes the proof.

□

4. Asymptotic behavior of the supremum tail distribution of stochastic integrals

We will write $g(u)\cong h(u)$ if $\lim_{u\to\infty}\frac{g(u)}{h(u)}=1$. Let us notice that

$$Z(t)=A(t)+Y(t)\,, \tag{22}$$

where

$$A(t)=\sigma C_\alpha^{1/\alpha}\Gamma_1^{-1/\alpha}\gamma_1\boldsymbol{I}\{U_1\le t\} \tag{23}$$

and

$$Y(t)=\begin{cases}\sigma C_\alpha^{1/\alpha}\sum_{k=1}^\infty \Gamma_{k+1}^{-1/\alpha}\gamma_{k+1}\boldsymbol{I}\{U_{k+1}\le t\} & \text{if } \alpha<1\,,\\ \sigma C_1\sum_{k=1}^\infty\left[\Gamma_{k+1}^{-1}\gamma_{k+1}\boldsymbol{I}\{U_{k+1}\le t\}-\beta t b_k^{(1)}\right] & \\ +\beta t\sigma C_1\ln(\sigma C_1) & \text{if } \alpha=1\,,\\ \sigma C_\alpha^{1/\alpha}\sum_{k=1}^\infty\left[\Gamma_{k+1}^{-1/\alpha}\gamma_{k+1}\boldsymbol{I}\{U_{k+1}\le t\}-\beta t b_k^{(\alpha)}\right] & \text{if } 1<\alpha<2\,.\end{cases} \tag{24}$$

Let us consider the following process

$$X(t)=\int_0^t V(s)\,dZ(s)\,, \tag{25}$$

where V is a measurable stochastic process independent of the process Z and

$$\int_0^1 \mathbb{E}V^2(s)\,ds<\infty\,. \tag{26}$$

Moreover, we assume that

$$\sup_{t\le 1}X(t)<\infty \qquad \text{a.s.} \tag{27}$$

Note that

$$dX(t)=V(s)\,dA(s)+V(s)\,dY(s)\,.$$

Thus we are able to state the main result of this section which generalizes the results of Samorodnitsky and Taqqu for the case V being a deterministic function to the case V being a stochastic process.

Theorem 4 *If* $\alpha > 1$*, (26) and (27) are satisfied and*

$$\frac{1+\beta}{2}\int_0^1 \mathbb{E}[V^+(s)]^\alpha\,ds + \frac{1-\beta}{2}\int_0^1 \mathbb{E}[V^-(s)]^\alpha\,ds > 0$$

then

$$\mathbb{P}\left(\sup_{t\le 1}\int_0^t V(s)\,dZ(s) > u\right)$$
$$\cong u^{-\alpha}\,\sigma^\alpha C_\alpha\left[\frac{1+\beta}{2}\int_0^1 \mathbb{E}[V^+(s)]^\alpha\,ds + \frac{1-\beta}{2}\int_0^1 \mathbb{E}[V^-(s)]^\alpha\,ds\right].$$

Proof: Using Lemma A.1 and Lemma A.2 we have that

$$\lim_{u\to\infty}\frac{\mathbb{P}\left(\sup_{t\le 1}\int_0^t V(s)\,dY(s) > u\right)}{\mathbb{P}\left(\sup_{t\le 1}\int_0^t V(s)\,dA(s) > u\right)} = 0\,. \tag{28}$$

It is easy to see that

$$\mathbb{P}\left(\sup_{t\le 1}\int_0^t V(s)\,dZ(s) > u\right) \;\le\; \mathbb{P}\left(\sup_{t\le 1}\int_0^t V(s)\,dA(s) + \sup_{t\le 1}\int_0^t V(s)\,dY(s) > u\right)$$

which by (28), Lemma A.2 and Lemma A.3 yields

$$\limsup_{u\to\infty}\frac{\mathbb{P}\left(\sup_{t\le 1}\int_0^t V(s)\,dZ(s) > u\right)}{\mathbb{P}\left(\sup_{t\le 1}\int_0^t V(s)\,dA(s) > u\right)} \le 1\,.$$

Similarly

$$\mathbb{P}\left(\sup_{t\le 1}\int_0^t V(s)\,dZ(s) > u\right) \;\ge\; \mathbb{P}\left(\sup_{t\le 1}\int_0^t V(s)\,dA(s) + \inf_{t\le 1}\int_0^t V(s)\,dY(s) > u\right)$$

$$= \mathbb{P}\left(\sup_{t\le 1}\int_0^t V(s)\,dA(s) - \sup_{t\le 1}\left[-\int_0^t V(s)\,dY(s)\right] > u\right)$$

and since the process $-Y$ is of the form in (24) with $\beta := -\beta$ it satisfies the thesis of Lemma A.1 which yields

$$\lim_{u\to\infty} \frac{\mathbb{P}\left(\sup_{t\le 1}\left[-\int_0^t V(s)\,dY(s)\right] > u\right)}{\mathbb{P}\left(\sup_{t\le 1}\int_0^t V(s)\,dA(s) > u\right)} = 0\,.$$

Hence using Lemma A.2 and Lemma A.3 we get

$$\liminf_{u\to\infty} \frac{\mathbb{P}\left(\sup_{t\le 1}\int_0^t V(s)\,dZ(s) > u\right)}{\mathbb{P}\left(\sup_{t\le 1}\int_0^t V(s)\,dA(s) > u\right)} \ge 1\,.$$

□

Corollary 1 *Let the assumption of Th. 4 be satisfied and let d be a real bounded function defined on $[0,1]$. Then*

$$\mathbb{P}\left(\sup_{t\le 1}\int_0^t V(s)\,dZ(s) + d(t) > u\right)$$

$$\cong u^{-\alpha}\,\sigma^\alpha C_\alpha\left[\frac{1+\beta}{2}\int_0^1 \mathbb{E}[V^+(s)]^\alpha\,ds + \frac{1-\beta}{2}\int_0^1 \mathbb{E}[V^-(s)]^\alpha\,ds\right]\,.$$

Proof: Let $M = \sup_{t\le 1} d(t)$ and $m = \inf_{t\le 1} d(t)$. Thus

$$\mathbb{P}\left(\sup_{t\le 1}\int_0^t V(s)\,dZ(s) + d(t) > u\right) \le \mathbb{P}\left(\sup_{t\le 1}\int_0^t V(s)\,dZ(s) + M > u\right)$$

and

$$\mathbb{P}\left(\sup_{t\le 1}\int_0^t V(s)\,dZ(s) + d(t) > u\right) \ge \mathbb{P}\left(\sup_{t\le 1}\int_0^t V(s)\,dZ(s) + m > u\right).$$

Since by Th. 4

$$\begin{aligned}
&\mathbb{P}\left(\sup_{t\le 1}\int_0^t V(s)\,dZ(s) + M > u\right)\\
&\cong [u-M]^{-\alpha}\,\sigma^\alpha C_\alpha\left[\frac{1+\beta}{2}\int_0^1 \mathbb{E}[V^+(s)]^\alpha\,ds + \frac{1-\beta}{2}\int_0^1 \mathbb{E}[V^-(s)]^\alpha\,ds\right]\\
&\cong u^{-\alpha}\,\sigma^\alpha C_\alpha\left[\frac{1+\beta}{2}\int_0^1 \mathbb{E}[V^+(s)]^\alpha\,ds + \frac{1-\beta}{2}\int_0^1 \mathbb{E}[V^-(s)]^\alpha\,ds\right].
\end{aligned}$$

Similarly we get for the lower bound which gives the thesis.

□

Case study 4.1 We can investigate the following process

$$X(t) = \int_0^t B_H(s)\,dZ(s) - ct\,,$$

where $c \in \mathbb{R}$, B_H is a standard fractional Brownian motion with $0 < H \le 1$ and Z is an α-stable Lévy motion with $1 < \alpha < 2$ and B_H is independent of Z. Using Corollary 1 we get the exact asymptotic behavior of the so-called finite time ruin probability of the process X

$$\begin{aligned}
\mathbb{P}\left(\sup_{t\le 1}\int_0^t B_H(s)\,dZ(s) - ct > u\right) &\cong u^{-\alpha}\sigma^\alpha C_\alpha\left[\frac{1+\beta}{2}\int_0^1 \mathbb{E}[B_H^+(s)]^\alpha ds\right.\\
\left. + \frac{1-\beta}{2}\int_0^1 \mathbb{E}[B_H^-(s)]^\alpha ds\right] &= u^{-\alpha}\sigma^\alpha C_\alpha\left[\frac{1+\beta}{2}\int_0^1 \mathbb{E}[B_H^+(s)]^\alpha\,ds\right.\\
\left. + \frac{1-\beta}{2}\int_0^1 \mathbb{E}[B_H^+(s)]^\alpha\,ds\right] &= \frac{1}{2}\,u^{-\alpha}\,\sigma^\alpha C_\alpha \int_0^1 \mathbb{E}|B_H(s)|^\alpha\,ds\\
&= \frac{1}{2}\,u^{-\alpha}\,\sigma^\alpha C_\alpha\,\mathbb{E}|B_H(1)|^\alpha \int_0^1 s^{\alpha H}\,ds\\
&= \frac{1}{2}\,u^{-\alpha}\,\sigma^\alpha C_\alpha\,\frac{2^{\frac{\alpha}{2}}\,\Gamma(\frac{\alpha+1}{2})}{(1+\alpha H)\sqrt{\pi}}\,.
\end{aligned}$$

Notice that this asymptotic probability does not depend on β because the process B_H is symmetric.

Case study 4.2 Let us consider the following process

$$X(t) = \int_0^t S(s)\, dZ(s) + d(t)\,,$$

where d is a bounded function defined on $[0,1]$, S is a gamma Lévy process with shape parameter $a > 0$ and scale parameter $b > 0$ that is $S(1)$ has the following density distribution function

$$f(y) = \begin{cases} 0 & \text{if } y \le 0\,, \\ \frac{1}{b^a \Gamma(a)}\, y^{a-1} \exp\left(-\frac{y}{b}\right) & \text{if } y > 0\,. \end{cases}$$

and Z is an α-stable Lévy motion with $1 < \alpha < 2$, $\beta > -1$ and S is independent of Z. Using Corollary 1 we get

$$\begin{aligned} &\mathbb{P}\left(\sup_{t\le 1} \int_0^t S(s)\, dZ(s) + d(t) > u\right) \\ &\cong u^{-\alpha}\sigma^\alpha C_\alpha \left[\frac{1+\beta}{2}\int_0^1 \mathbb{E}[S^+(s)]^\alpha\, ds + \frac{1-\beta}{2}\int_0^1 \mathbb{E}[S^-(s)]^\alpha\, ds\right] \\ &= u^{-\alpha}\,\sigma^\alpha C_\alpha \frac{1+\beta}{2}\int_0^1 \mathbb{E}[S(s)]^\alpha\, ds \\ &= u^{-\alpha}\,\sigma^\alpha C_\alpha \frac{1+\beta}{2}\, b^\alpha \int_0^1 \frac{\Gamma(as+\alpha)}{\Gamma(as)}\, ds \\ &= u^{-\alpha}\,\sigma^\alpha\, b^\alpha\, C_\alpha\, \Gamma(\alpha)\, \frac{1+\beta}{2} \int_0^1 \frac{1}{B(as,\alpha)}\, ds\,, \end{aligned}$$

where $B(x,y)$ ($x > 0$, $y > 0$) is beta function.

5. Conclusions

In this paper we consider an α-stable Lévy process under condition on the length of the largest jump. We show that under this condition α-stable Lévy process can be decomposed into a Lévy process and a certain simple process. This Lévy process can serve as an approximation of the α-stable Lévy process and in other domain approximates Brownian motion. Using this decomposition we investigate the distributional properties of a stochastic integral. We show that the asymptotic distributional properties of supremum of certain stochastic integrals are not far away from those of the α-stable Lévy process that is the tails are also regularly varying. The integrand in the stochastic integrals driven by α-stable noise affects the value of the pre-factor of the asymptotic function and the α-stable Lévy process is responsible for the speed of the asymptotic behavior.

Appendix A

Here we present auxiliary results.

Lemma A.1 *Let $\alpha > 1$ and (26) and (27) be satisfied then*

$$\mathbb{P}\left(\sup_{t\leq 1}\int_0^t V(s)\,dY(s) > u\right) \leq u^{-2}\,C\,,$$

where C is a positive constant.

Proof: First let us construct two stochastic processes. Let $\{T_k\}_{k=1}^{\infty}$ and $T_1, \{T'_k\}_{k=2}^{\infty}$ be two iid sequences of exponential random variables with parameter equal 1 such that the sequences $\{T_k\}_{k=2}^{\infty}$ and $\{T'_k\}_{k=2}^{\infty}$ are independent. Let $\{\gamma_k\}_{k=1}^{\infty}$ and $\{U_k\}_{k=1}^{\infty}$ be the sequences defined in Section 2 and $\{\gamma'_k\}_{k=1}^{\infty}$ and $\{U'_k\}_{k=1}^{\infty}$ their independent copies and the sequences $\{T_k\}_{k=1}^{\infty}$, $\{T'_k\}_{k=2}^{\infty}$, $\{\gamma_k\}_{k=1}^{\infty}$, $\{U_k\}_{k=1}^{\infty}$, $\{\gamma'_k\}_{k=1}^{\infty}$ and $\{U'_k\}_{k=1}^{\infty}$ are independent. Put $\Gamma_1 = \Gamma'_1 = T_1$, $\Gamma_n = \sum_{k=1}^{n} T_k$ and $\Gamma'_n = T_1 + \sum_{k=2}^{n} T'_k$. Let us define process Y as in (24) using the sequences $\{\Gamma_k\}_{k=1}^{\infty}$, $\{\gamma_k\}_{k=1}^{\infty}$ and $\{U_k\}_{k=1}^{\infty}$ and similarly process Y' using the sequences $\{\Gamma'_k\}_{k=1}^{\infty}$, $\{\gamma'_k\}_{k=1}^{\infty}$ and $\{U'_k\}_{k=1}^{\infty}$. Notice that the processes Y and Y' are not independent but under condition $\Gamma_1 = \Gamma'_1 = x$ these two processes are independent. Thus the process $Y_x - Y'_x$ is a symmetric Lévy process which implies by Theorem 1 that this process is a square integrable martingale.

By independence V of Z note that (we can assume that the process V is independent of (Y, Y'))

$$\begin{aligned}
&\mathbb{P}\left(\sup_{t\le 1}\int_0^t V(s)\,d(Y(s)-Y'(s))>u\right)\\
&=\int_0^\infty \mathbb{P}(\sup_{t\le 1}\int_0^t V(s)\,d(Y_x(s)-Y'_x(s))>u)\,e^{-x}\,dx\\
&=\mathbb{E}_V\int_0^\infty \mathbb{P}(\sup_{t\le 1}\int_0^t v(s)\,d(Y_x(s)-Y'_x(s))>u)\,e^{-x}\,dx\,,
\end{aligned}$$

where $\mathbb{E}_V$ is expectation with respect to V and v is a trajectory of V. By (26) $\int_0^1 v^2(s)\,ds<\infty$ a.s. Hence we get that $\int_0^t v(s)\,d(Y_x(s)-Y'_x(s))$ is also a martingale and

$$\begin{aligned}
\mathbb{E}\left[\int_0^1 v(s)\,d(Y_x(s)-Y'_x(s))\right]^2 &= \mathbb{E}[Y_x(1)-Y'_x(1)]^2\int_0^1 v^2(s)\,ds\\
&= 2\,\mathbf{Var}Y_x(1)\int_0^1 v^2(s)\,ds\,. \qquad \text{(A.1)}
\end{aligned}$$

Since the process $\int_0^t v(s)\,d(Y_x(s)-Y'_x(s)$ is a martingale $[\int_0^t v(s)\,d(Y_x(s)-Y'_x(s)]^2$ is a submartingale thus by Doob inequality for submartingales (see *e.g.* Ethier and Kurtz [32] Prop. 2.16) we get

$$\begin{aligned}
&\mathbb{P}\left(\sup_{t\le 1}\int_0^t v(s)\,d(Y_x(s)-Y'_x(s))>u\right)\\
&\le \mathbb{P}\left(\sup_{t\le 1}\left[\int_0^t v(s)\,d(Y_x(s)-Y'_x(s))\right]^2>u^2\right)
\end{aligned}$$

$$\leq u^{-2}\,\mathbb{E}\left[\int_0^1 v(s)\,d(Y_x(s)-Y'_x(s))\right]^2$$

$$= u^{-2}\,2\,\mathbf{Var}Y_x(1)\int_0^1 v^2(s)\,ds\,, \tag{A.2}$$

where in the last line we used (A.1).

Now we consider the general case. Using (A.2) for $M>0$ we obtain

$$\begin{aligned}
&2\,\mathbf{Var}Y_x(1)\int_0^1 v^2(s)\,ds \\
&\geq \limsup_{u\to\infty}(u-M)^2\mathbb{P}\left(\sup_{t\leq 1}\int_0^t v(s)d(Y_x(s)-Y'_x(s))>u-M\right) \\
&= \limsup_{u\to\infty}(u-M)^2\mathbb{P}\left(\sup_{t\leq 1}\left(\int_0^t v(s)dY_x(s)-\int_0^t v(s)dY'_x(s)\right)>u-M\right) \\
&\geq \limsup_{u\to\infty}(u-M)^2\mathbb{P}\left(\sup_{t\leq 1}\int_0^t v(s)dY_x(s)>u,\sup_{t\leq 1}\int_0^t v(s)dY'_x(s)\leq M\right) \\
&= \mathbb{P}\left(\sup_{t\leq 1}\int_0^t v(s)dY'_x(s)\leq M\right) \\
&\times \limsup_{u\to\infty}\,(u-M)^2\mathbb{P}\left(\sup_{t\leq 1}\int_0^t v(s)dY_x(s)>u\right),
\end{aligned}$$

where in the last line we used independence of the processes Y_x and Y'_x. By (27) $\sup_{t\leq 1}\int_0^t V(s)\,dY'(s)<\infty$ a.s. thus using the last inequalities we get

$$\limsup_{u\to\infty} u^2\,\mathbb{P}\left(\sup_{t\leq 1}\int_0^t v(s)\,dY_x(s)>u\right)\leq 2\,\mathbf{Var}Y_x(1)\int_0^1 v^2(s)\,ds$$

for $x > 0$ a.e. and almost all v. Hence for $\epsilon > 0$ and sufficiently large u we have

$$\mathbb{P}\left(\sup_{t\le 1}\int_0^t V(s)\,dY_x(s) > u\right) \le u^{-2}(2\,\mathbf{Var}Y_x(1)\int_0^1 \mathbb{E}V^2(s)\,ds + \epsilon)\,.$$

Since $\int_0^\infty \mathbf{Var}Y_x(1)\,e^{-x}\,dx < \infty$ by (17) for $\alpha > 1$ thus we obtain

$$\mathbb{P}\left(\sup_{t\le 1}\int_0^t V(s)dY(s) > u\right) \le u^{-2}\left(2\int_0^\infty \mathbf{Var}Y_x(1)e^{-x}dx\int_0^1 \mathbb{E}V^2(s)ds + \epsilon\right)$$

for sufficiently large u which gives

$$\mathbb{P}\left(\sup_{t\le 1}\int_0^t V(s)\,dY(s) > u\right) \le u^{-2}\,C$$

for all $u > 0$ and C is a positive constant. This finishes the proof.

□

Now let us consider the first term of the integral (25).

Lemma A.2 *Let A be the stochastic process defined in (23) and*

$$\frac{1+\beta}{2}\int_0^1 \mathbb{E}[V^+(s)]^\alpha\,ds + \frac{1-\beta}{2}\int_0^1 \mathbb{E}[V^-(s)]^\alpha\,ds > 0\,.$$

Then

$$\mathbb{P}\left(\sup_{t\le 1}\int_0^t V(s)\,dA(s) > u\right)$$
$$\cong u^{-\alpha}\,\sigma^\alpha C_\alpha\left[\frac{1+\beta}{2}\int_0^1 \mathbb{E}[V^+(s)]^\alpha\,ds + \frac{1-\beta}{2}\int_0^1 \mathbb{E}[V^-(s)]^\alpha\,ds\right]$$

Proof: Note that

$$\int_0^t V(s)dA(s) = \sigma C_\alpha^{1/\alpha}\Gamma_1^{-1/\alpha}\gamma_1 V(U_1)\boldsymbol{I}\{U_1 \le t\}.$$

Hence

$$\mathbb{P}\left(\sup_{t\le 1}\int_0^t V(s)dA(s) > u\right) = \mathbb{P}\left(\sup_{t\le 1}\sigma C_\alpha^{1/\alpha}\Gamma_1^{-1/\alpha}\gamma_1 V(U_1)\boldsymbol{I}\{U_1\le t\} > u\right)$$
$$= \mathbb{P}(\sigma C_\alpha^{1/\alpha}\Gamma_1^{-1/\alpha}\gamma_1 V(U_1) > u)$$
$$= \int_0^1 \mathbb{P}(\sigma C_\alpha^{1/\alpha}\Gamma_1^{-1/\alpha}\gamma_1 V(s) > u)ds$$
$$= \frac{1+\beta}{2}\int_0^1 \mathbb{P}(\sigma C_\alpha^{1/\alpha}\Gamma_1^{-1/\alpha}V^+(s) > u)ds$$
$$+\frac{1-\beta}{2}\int_0^1 \mathbb{P}(\sigma C_\alpha^{1/\alpha}\Gamma_1^{-1/\alpha}V^-(s) > u)ds$$
$$= \frac{1+\beta}{2}\mathbb{E}_V\int_0^1 \mathbb{P}(\sigma C_\alpha^{1/\alpha}\Gamma_1^{-1/\alpha}v^+(s) > u)ds$$
$$+\frac{1-\beta}{2}\mathbb{E}_V\int_0^1 \mathbb{P}(\sigma C_\alpha^{1/\alpha}\Gamma_1^{-1/\alpha}v^-(s) > u)ds$$
$$= \frac{1+\beta}{2}\mathbb{E}_V\int_0^1 [1-\exp(-u^{-\alpha}\sigma^\alpha C_\alpha[v^+(s)]^\alpha)]ds$$
$$+\frac{1-\beta}{2}\mathbb{E}_V\int_0^1 [1-\exp(-u^{-\alpha}\sigma^\alpha C_\alpha[v^-(s)]^\alpha)]ds\,. \tag{A.3}$$

Let us notice that for $x \ge 0$ $|1-e^{-x}-x| \le x^2/2$ and $|1-e^{-x}-x| \le |1-e^{-x}|+|x| = 1-e^{-x}+x \le 2x$ which yields $|1-e^{-x}-x| \le 2x^{1+\delta}$ where $0\le\delta\le 1$. Thus

$$\left|\mathbb{E}_V\int_0^1 \left[1-\exp\left(-u^{-\alpha}\sigma^\alpha C_\alpha[v^+(s)]^\alpha\right)\right]\,ds - \mathbb{E}_V\int_0^1 u^{-\alpha}\sigma^\alpha C_\alpha[v^+(s)]^\alpha\,ds\right|$$
$$= \left|\mathbb{E}_V\int_0^1 [1-\exp(-u^{-\alpha}\sigma^\alpha C_\alpha[v^+(s)]^\alpha) - u^{-\alpha}\sigma^\alpha C_\alpha[v^+(s)]^\alpha]\,ds\right|$$
$$\le 2(u^{-\alpha}\sigma^\alpha C_\alpha)^{1+\delta}\mathbb{E}_V\int_0^1 [v^+(s)]^{\alpha(1+\delta)}\,ds$$

$$= 2(u^{-\alpha}\sigma^{\alpha}C_{\alpha})^{1+\delta}\int_0^1 \mathbb{E}[V^{+}(s)]^{\alpha(1+\delta)}\,ds\,,$$

where we take $\alpha(1+\delta) \le 2$ and hence $\int_0^1 \mathbb{E}[V^{+}(s)]^{\alpha(1+\delta)}\,ds < \infty$ by (26). Thus from the last computations it follows that

$$\mathbb{E}_V \int_0^1 [1-\exp(-u^{-\alpha}\sigma^{\alpha}C_{\alpha}[v^{+}(s)]^{\alpha})]\,ds \cong u^{-\alpha}\sigma^{\alpha}C_{\alpha}\int_0^1 \mathbb{E}[V^{+}(s)]^{\alpha}\,ds\,.$$

Similarly we obtain for the second term in (A.3) that

$$\mathbb{E}_V \int_0^1 [1-\exp(-u^{-\alpha}\sigma^{\alpha}C_{\alpha}[v^{-}(s)]^{\alpha})]\,ds \cong u^{-\alpha}\sigma^{\alpha}C_{\alpha}\int_0^1 \mathbb{E}[V^{-}(s)]^{\alpha}\,ds$$

which finishes the proof.

□

It is easy to prove the following lemma.

Lemma A.3 *Suppose that X is a random variable with a regularly varying tail with index $\theta > 0$ and T is a non-negative random variable such that*

$$\lim_{u\to\infty}\frac{\mathbb{P}(T>u)}{\mathbb{P}(X>u)} = 0\,.$$

Then

$$\lim_{u\to\infty}\frac{\mathbb{P}(X+T>u)}{\mathbb{P}(X>u)} = \lim_{u\to\infty}\frac{\mathbb{P}(X-T>u)}{\mathbb{P}(X>u)} = 1\,.$$

Proof: See *e.g.* Samorodnitsky and Taqqu [18] Lemma 4.4.2.

□

REFERENCES

[1] E.W. Montroll, G.H. Weiss, *J. Math. Phys.* **6**, 167 (1965).

[2] H. Scher, E.W. Montroll, *Phys. Rev.* **B12**, 2455 (1975).

[3] R. Metzler, J. Klafter, *Phys. Rep.* **339**, 1 (2000).

[4] S. Mercik, K. Weron, K. Burnecki, A. Weron, *Acta Phys. Pol. B* **34**, 3773 (2003).

[5] B. Dybiec, E. Gudowska-Nowak, *Acta Phys. Pol. B* **37**, 1479 (2006).
[6] M. Magdziarz, A. Weron, *Phys. Rev.* **E75**, 056702 (2007).
[7] A. Weron, M. Magdziarz, K. Weron, *Phys. Rev.* **E77**, 036704 (2008).
[8] M. Magdziarz, K. Weron, *Acta Phys. Pol. B* **37**, 1617 (2006).
[9] B. Dybiec, E. Gudowska-Nowak, P. Hänggi, *Phys. Rev.* **E73**, 046104 (2006).
[10] B. Dybiec, E. Gudowska-Nowak, P. Hänggi, *Phys. Rev.* **E75**, 021109 (2007).
[11] B. Dybiec, E. Gudowska-Nowak, *Fluct. Noise Lett.* **4**, L273 (2004).
[12] R. Weron, *J. Modern Phys.* **C12**, 200 (2001).
[13] K. Burnecki, A. Weron, *Acta Phys. Pol. B* **35**, 1443 (2004).
[14] Z. Michna, *Probab. Math. Statist.* **25**, 173 (2005).
[15] M. Magdziarz, P. Miśta, A. Weron, *Acta Phys. Pol. B* **38**, 1647 (2007).
[16] A. Janicki, A. Weron, *Simulation and Chaotic Behavior of α–Stable Stochastic Processes*, Marcel Dekker, New York 1994.
[17] O. Kallenberg, *Foundations of Modern Probability*, Springer, New York 1997.
[18] G. Samorodnitsky, M. Taqqu, *Non–Gaussian Stable Processes: Stochastic Models with Infinite Variance*, Chapman and Hall, London 1994.
[19] H. Hult, F. Lindskog, *Ann. Probab.* **35**, 309 (2007).
[20] J.M.P. Albin, *Stat. Probab. Lett.* **16**, 219 (1993).
[21] E. Willekens, *Stoch. Proc. Appl.* **26**, 173 (1987).
[22] M. Braverman, *Stoch. Proc. Appl.* **68**, 265 (1997).
[23] M. Braverman, *Stat. Prob. Lett.* **43**, 41 (1999).
[24] M. Braverman, G. Samorodnitsky, *Stoch. Proc. Appl.* **56**, 207 (1995).
[25] Z. Michna, A. Weron, *Acta Phys. Pol. B* **38**, 1881 (2007).
[26] J. Rosiński, G. Samorodnitsky, *Ann. Prob.* **21**, 996 (1993).
[27] R. LePage, *Lect. Notes Math.* **860**, 279 (1980).
[28] Z. Michna, *Studia Math.* **180**, 1 (2007).
[29] J. Rosiński, *Lévy Processes — Theory and Applications*, Eds O.E. Barndorff-Nielsen, T. Mikosch, S. Resnick, Birkhäuser, Boston 2001, p. 401.
[30] Ph. Protter, *Stochastic Integration and Differential Equations*, Springer, New York 1990.
[31] S. Asmussen, J. Rosiński, *J. Appl. Probab.* **38**, 482 (2001).
[32] S. Ethier, T. Kurtz, *Markov Processes. Characterization and Convergence*, John Wiley, New York 1986.